Amenability of operator algebras on Banach spaces, II

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The finite-dimensional case

Example

Let E be a Banach space with $n := \dim E < \infty$ so that

$$\mathcal{B}(E) = \mathcal{K}(E) \cong M_n.$$

Let G be a finite subgroup of invertible elements of M_n such that span $G = M_n$.
Set

$$\mathbf{d} := \frac{1}{|G|} \sum_{g \in G} g \otimes g^{-1}.$$

Then

$$a \cdot \mathbf{d} = \mathbf{d} \cdot a \qquad (a \in M_n)$$

and $\Delta \mathbf{d} = I_n$.
Hence, $\mathcal{K}(E) = \mathcal{B}(E)$ is amenable.

Some more results

Theorem (B. E. Johnson, 1972)

$\mathcal{K}(E)$ is amenable if $E = \ell^p$ with $1 < p < \infty$ or $E = \mathcal{C}(\mathbb{T})$.

Amenable Banach algebras must have bounded approximate identities. . .

Theorem (N. Grønbæk & G. A. Willis, 1994)

Suppose that E has the approximation property. Then $\mathcal{K}(E)$ has a bounded approximate identity if and only if E^* has the bounded approximation property.

Example

Let $E = \ell^2 \hat{\otimes} \ell^2$. Then E has the approximation property, but $E^* = \mathcal{B}(\ell^2)$ doesn't. Hence, $\mathcal{K}(E)$ does not have a bounded approximate identity and is thus not amenable.

Finite, biorthogonal systems

Definition

A finite, biorthogonal system is a set $\{(x_j, \phi_k) : j, k = 1, \ldots, n\} \subset E \times E^*$ such that

$$\langle x_j, \phi_k \rangle = \delta_{j,k} \qquad (j, k = 1, \ldots, n).$$

Remark

If $\{(x_j, \phi_k) : j, k = 1, \ldots, n\}$ is a finite, biorthogonal system, then

$$\theta : M_n \to \mathcal{F}(E), \quad [\alpha_{j,k}] \mapsto \sum_{j,k=1}^{n} \alpha_{j,k} x_j \otimes \phi_k$$

is an algebra homomorphism.

Property ($\mathbb{A}$)

Definition (N. Grønbæk, BEJ, & G. A. Willis, 1994)

We say that E has property ($\mathbb{A}$) if there is a net $(\{(x_{j,\lambda}, \phi_{k,\lambda}) : j, k = 1, \ldots, n_\lambda\})_\lambda$ of finite biorthogonal systems with corresponding homomorphisms $\theta_\lambda : M_{n_\lambda} \to \mathcal{F}(E)$ with the following properties:

1. $\theta_\lambda(I_{n_\lambda}) \to \mathrm{id}_E$ uniformly on compacts;
2. $\theta_\lambda(I_{n_\lambda})^* \to \mathrm{id}_{E^*}$ uniformly on compacts;
3. for each λ, there is a finite group G_λ of invertible elements of M_{n_λ} spanning M_{n_λ} such that

$$\sup_\lambda \max_{g \in G_\lambda} \|\theta_\lambda(g)\| < \infty.$$

Property ($\mathbb{A}$) and the amenability of $\mathcal{K}(E)$

The idea behind ($\mathbb{A}$)

Use the diagonals of the M_{n_λ}'s to construct an approximate diagonal for $\mathcal{K}(E)$.

Theorem (N. Grønbæk, BEJ, & G. A. Willis, 1994)

Suppose that E has property ($\mathbb{A}$). Then $\mathcal{K}(E)$ is amenable.

Examples

1. $L^p(\mu)$ has property ($\mathbb{A}$) for all $1 \leq p < \infty$ and all μ.
2. $\mathcal{C}(K)$ has property ($\mathbb{A}$) for each compact K, as does therefore $L^\infty(\mu)$ for each μ.

The "scalar plus compact" problem

Question

Is there an infinite-dimensional Banach space E such that $\mathcal{B}(E) = \mathcal{K}(E) + \mathbb{C}\,\mathrm{id}_E$?

Theorem (S. A. Argyros & R. G. Haydon, 2009)

There is a Banach space E such that $\mathcal{B}(E) = \mathcal{K}(E) + \mathbb{C}\,\mathrm{id}_E$ and $E^* = \ell^1$.

Theorem (N. Grønbæk, BEJ, & G. A. Willis, 1994)

Suppose that E^* has property $(\mathbb{A})$. Then so has E.

Corollary

There is an infinite-dimensional Banach space E such that $\mathcal{B}(E)$ is amenable.

Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, I

Theorem (G. A. Willis, unpublished)

Let $p, q \in (1, \infty)$ be such that $p \neq q$. Then $\mathcal{B}(\ell^p \oplus \ell^q)$ is not amenable.

Ingredients

1. A quotient of an amenable Banach algebra is again amenable.
2. Every complemented closed ideal of an amenable Banach algebra is amenable.
3. Every amenable Banach algebra has a bounded approximate identity.
4. **Pitt's Theorem.** *If $p > q$, then $\mathcal{B}(\ell^p, \ell^q) = \mathcal{K}(\ell^p, \ell^q)$.*

Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, II

Proof.

Suppose that $p > q$. Note that

$$\mathcal{B}(\ell^p \oplus \ell^q) = \begin{bmatrix} \mathcal{B}(\ell^p) & \mathcal{B}(\ell^q, \ell^p) \\ \mathcal{B}(\ell^p, \ell^q)\mathcal{K}(\ell^p, \ell^q) & \mathcal{B}(\ell^q) \end{bmatrix}$$

and

$$\mathcal{K}(\ell^p \oplus \ell^q) = \begin{bmatrix} \mathcal{K}(\ell^p) & \mathcal{K}(\ell^q, \ell^p) \\ \mathcal{K}(\ell^p, \ell^q) & \mathcal{K}(\ell^q) \end{bmatrix},$$

so that

$$\mathcal{C}(\ell^p \oplus \ell^q) = \begin{bmatrix} \mathcal{C}(\ell^p) & * \\ 0 & \mathcal{C}(\ell^q) \end{bmatrix}.$$

Then $I := \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \neq \{0\}$ is a complemented ideal of $\mathcal{C}(\ell^p \oplus \ell^q)$, thus is amenable, and thus has a BAI. But $I^2 = \{0\}\ldots$ □

Non-amenability of $\mathcal{B}(\ell^p)$ for $p = 1, 2, \infty$

Theorem (C. J. Read, <2006)

$\mathcal{B}(\ell^1)$ *is not amenable.*

Progress since

1. Simplification of Read's proof by G. Pisier, 2004.
2. Simultaneous proof for the non-amenability of $\mathcal{B}(\ell^p)$ for $p = 1, 2, \infty$ by N. Ozawa, 2006.

Question

Is $\mathcal{B}(\ell^p)$ amenable for any $p \in (1, \infty) \setminus \{2\}$?

What if $\mathcal{B}(\ell^p)$ were amenable?

Theorem (M. Daws & VR, 2008)

The following are equivalent for a Banach space E and $p \in [1, \infty)$:

1. $\mathcal{B}(\ell^p(E))$ *is amenable;*
2. $\ell^\infty(\mathcal{B}(\ell^p(E)))$ *is amenable.*

Idea

- $\ell^p(\ell^p(E)) \cong \ell^p(E)$
- $\ell^\infty(\mathcal{B}(\ell^p(E))) \cong$ block diagonal matrices in $\mathcal{B}(\ell^p(\ell^p(E)))$

Corollary

Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. Then so are the Banach algebras $\ell^\infty(\mathcal{B}(\ell^p))$ and $\ell^\infty(\mathcal{K}(\ell^p))$.

$\mathcal{L}^p$-spaces, I

Definition (J. Lindenstrauss & A. Pełczyński, 1968)

Let $p \in [1, \infty]$ and let $\lambda \geq 1$. A Banach space E is called a $\mathcal{L}^p_\lambda$-space if, for every finite-dimensional subspace X of E, there is a finite-dimensional subspace $Y \supset X$ of E with $d(Y, \ell^p_{\dim Y}) \leq \lambda$. We call E an $\mathcal{L}^p$-space if it is an $\mathcal{L}^p_\lambda$-space for some $\lambda \geq 1$.

Examples

1. All Banach spaces isomorphic to an L^p-space are $\mathcal{L}^p$-spaces.
2. Let $p \in (1, \infty) \setminus \{2\}$. Then $\ell^p(\ell^2)$ and $\ell^2 \oplus \ell^p$ are $\mathcal{L}^p$-spaces, but not isomorphic to L^p-spaces.

$\mathcal{L}^p$-spaces, II

Theorem (M. Daws & VR, 2008)

Let $p \in [1, \infty]$. Then one of the following is true:

1. $\ell^\infty(\mathcal{K}(E))$ *is amenable for every $\mathcal{L}^p$-space E with* $\dim E = \infty$;
2. $\ell^\infty(\mathcal{K}(E))$ *is not amenable for any $\mathcal{L}^p$-space E with* $\dim E = \infty$.

Corollary

Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. Then $\ell^\infty(\mathcal{K}(E))$ is amenable for every $\mathcal{L}^p$-space E with $\dim E = \infty$.

Question

Is $\ell^\infty(\mathcal{K}(\ell^2 \oplus \ell^p))$ amenable for any $p \in (1, \infty) \setminus \{2\}$?

Ozawa's proof revisited, I

Definition

A locally compact group G has Kazhdan's property (T) if there are $\epsilon > 0$ and a compact set $K \subset G$ with the following property: for every irreducible, unitary representation π of G on $\mathfrak{H}$ and for every unit vector $\xi \in \mathfrak{H}$, there is $k \in K$ such that

$$\|\pi(k)\xi - \xi\| > \epsilon.$$

Examples

1. All compact groups have property (T), as does $\mathrm{SL}(3, \mathbb{Z})$.
2. Amenable groups have property (T) if and only if they are compact.
3. $\mathbb{F}_2$ and $\mathrm{SL}(2, \mathbb{R})$ are not amenable, but lack property (T).

Ozawa's proof revisited, II

The setup

Since $\mathrm{SL}(3,\mathbb{Z})$ has property (T), it is finitely generated by $g_1, \ldots, g_m$, say.

Write $\mathbb{P}$ for the set of prime numbers.

Let $p \in \mathbb{P}$, and let Λ_p be the projective plane over $\mathbb{Z}/p\mathbb{Z}$.

Then $\mathrm{SL}(3,\mathbb{Z})$ acts on Λ_p through matrix multiplication.

This group action induces a unitary representation $\pi_p \colon \mathrm{SL}(3,\mathbb{Z}) \to \mathcal{B}(\ell^2(\Lambda_p))$.

Choose $S_p \subset \Lambda_p$ with $|S_p| = \frac{|\Lambda_p|-1}{2}$ and define a unitary $\pi_p(g_{m+1}) \in \mathcal{B}(\ell^2(\Lambda_p))$ via

$$\pi_p(g_{m+1})e_\lambda = \begin{cases} e_\lambda, & \lambda \in S_p, \\ -e_\lambda, & \lambda \notin S_p. \end{cases}$$

Ozawa's proof revisited, III

Ozawa's Lemma

It is impossible to find, for each $\epsilon > 0$, *a number* $r \in \mathbb{N}$ *with the following property: for each* $p \in \mathbb{P}$ *there are* $\xi_{1,p}, \eta_{1,p}, \ldots, \xi_{r,p}, \eta_{r,p} \in \ell^2(\Lambda_p)$ *such that* $\sum_{k=1}^{r} \xi_{k,p} \otimes \eta_{k,p} \neq 0$ *and*

$$\left\| \sum_{k=1}^{r} \xi_{j,p} \otimes \eta_{k,p} - (\pi_p(g_j) \otimes \pi_p(g_j))(\xi_{k,p} \otimes \eta_{k,p}) \right\|_{\ell^2(\Lambda_p)\hat{\otimes}\ell^2(\Lambda_p)} \leq \epsilon \left\| \sum_{k=1}^{r} \xi_{k,p} \otimes \eta_{k,p} \right\|_{\ell^2(\Lambda_p)\hat{\otimes}\ell^2(\Lambda_p)} \qquad (j = 1, \ldots, m+1).$$

Ozawa's proof revisited, IV

Ingredients

1. $\mathrm{SL}(3,\mathbb{Z})$ has Kazhdan's property (T).
2. The non-commutative Mazur map is uniformly continuous.
3. A key inequality. For $p = 1, 2, \infty$, $N \in \mathbb{N}$, $S \in \mathcal{B}(\ell^p, \ell^p_N)$, and $T \in \mathcal{B}(\ell^{p'}, \ell^{p'}_N)$:

$$\sum_{n=1}^{\infty} \|Se_n\|_{\ell^2_N} \|Te_n^*\|_{\ell^2_N} \leq N\|S\|\|T\|.$$

(This estimate is no longer true for $p \in (1,\infty) \setminus \{2\}$.)

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, I

Theorem (VR, 2009)

Let E be a Banach space with a basis $(x_n)_{n=1}^\infty$ such that there is $C > 0$ with

$$\sum_{n=1}^{\infty} \|Sx_n\| \|Tx_n^*\| \leq C\, N \|S\| \|T\|$$

$$(N \in \mathbb{N},\ S \in \mathcal{B}(E, \ell^2_N),\ T \in \mathcal{B}(E^*, \ell^2_N)).$$

Then $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$ is not amenable.

Example

It is easy to see that the following spaces satisfy the hypotheses of the theorem: c_0, ℓ^1, and ℓ^2.

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, II

Lemma

Let $\mathfrak{A}$ be an amenable Banach algebra, and let $e \in \mathfrak{A}$ be an idempotent. Then, for any $\epsilon > 0$ and any finite subset F of $e\mathfrak{A}e$, there are $a_1, b_1, \ldots, a_r, b_r \in \mathfrak{A}$ such that

$$\sum_{k=1}^{r} a_k b_k = e$$

and

$$\left\| \sum_{k=1}^{r} x a_k \otimes b_k - a_k \otimes b_k x \right\|_{\mathfrak{A}\hat{\otimes}\mathfrak{A}} < \epsilon \qquad (x \in F).$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, III

Sketched proof of the Theorem

Embed

$$\ell^\infty\text{-}\bigoplus_{p\in\mathbb{P}} \mathcal{B}(\ell^2(\Lambda_p)) \subset \ell^\infty\text{-}\bigoplus_{p\in\mathbb{P}} \mathcal{K}(\ell^2 \oplus E) =: \mathfrak{A}$$

as "upper left corners". Let $\mathfrak{A}$ act on

$$\ell^2(\mathbb{P}, \ell^2 \oplus E) \cong \ell^2(\mathbb{P}, \ell^2) \oplus \ell^2(\mathbb{P}, E).$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates of the p^{th} ℓ^2-summand of

$$\ell^2(\mathbb{P}, \ell^2) \oplus \ell^2(\mathbb{P}, E).$$

Set $e = (P_p)_{p \in \mathbb{P}}$. Then e is an idempotent in $\mathfrak{A}$ with

$$e\mathfrak{A}e = \ell^\infty\text{-}\bigoplus_{p \in \mathbb{P}} \mathcal{B}(\ell^2(\Lambda_p)).$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, V

Sketched proof of the Theorem (continued)

Assume towards a contradiction that $\ell^\infty(\mathbb{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.

Let $\epsilon > 0$ be arbitrary. By the previous Lemma there are thus $a_1, b_1, \ldots, a_r, b_r \in \mathfrak{A}$ such that $\sum_{k=1}^r a_k b_k = e$ and

$$\left\| \sum_{k=1}^r x a_k \otimes b_k - a_k \otimes b_k x \right\| < \frac{\epsilon}{(C+1)(m+1)} \qquad (x \in F),$$

where

$$F := \{(\pi_p(g_j))_{p \in \mathbb{P}} : j = 1, \ldots, m+1\}.$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VI

Sketched proof of the Theorem (continued)

For $p, q \in \mathbb{P}$ and $n \in \mathbb{N}$, define

$$T_p(q,n) := \sum_{k=1}^{r} P_p a_k(e_q \otimes e_n) \otimes P_p^* b_k^*(e_q^* \otimes e_n^*) + P_p a_k(e_q \otimes x_n) \otimes P_p^* b_k^*(e_q^* \otimes x_n^*)$$

Note that

$$T_p(q,n) \in \ell^2(\Lambda_p)\hat{\otimes}\ell^2(\Lambda_p).$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VII

Sketched proof of the Theorem (continued)

It follows that

$$\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \| T_p(q,n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q,n) \| \leq \frac{\epsilon}{m+1} |\Lambda_p|$$

for $j = 1, \ldots, m+1$ and $p \in \mathbb{P}$ and thus

$$\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \sum_{j=1}^{m+1} \| T_p(q,n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q,n) \| \leq \epsilon |\Lambda_p|.$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VIII

Sketched proof of the Theorem (continued)

On the other hand:

$$
\begin{aligned}
&\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \| T_p(q, n) \| \\
&\geq \sum_{n=1}^{\infty} \left| \sum_{k=1}^{r} \langle P_p a_{k,p} e_n, P_p^* b_{k,p}^* e_n^* \rangle + \sum_{k=1}^{r} \langle P_p a_{k,p} x_n, P_p^* b_{k,p}^* x_n^* \rangle \right| \\
&= \operatorname{Tr} \sum_{k=1}^{r} b_{k,p} P_p a_{k,p} \\
&= \operatorname{Tr} \sum_{k=1}^{r} P_p a_{k,p} b_{k,p} \\
&= \operatorname{Tr} P_p = |\Lambda_p|.
\end{aligned}
$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

Sketched proof of the Theorem (conclusion)

It follows that, for each $p \in \mathbb{P}$, there are $q \in \mathbb{P}$ and $n \in \mathbb{N}$ with $T_p(q,n) \neq 0$ and

$$\|T_p(q,n) - ((\pi_p(g_j) \otimes \pi_p(g_j))T_p(q,n)\| \leq \epsilon \|T_p(q,n)\|$$

for $j = 1, \ldots, m+1$, which violates Ozawa's Lemma. $\square$

p-summing operators

Definition

Let $p \in [1, \infty)$, and E and F be Banach spaces. A linear map $T : E \to F$ is called p-summing if the amplification $\mathrm{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F$ extends to a bounded map from $\ell^p \check{\otimes} E$ to $\ell^p(F)$. The operator norm of $\mathrm{id}_{\ell^p \otimes T} : \ell^p \check{\otimes} E \to \ell^p(F)$ is called the p-summing norm of T and denoted by $\pi_p(T)$.

Theorem (Y. Gordon, 1969)

$$\pi_p(\mathrm{id}_{\ell^2_N}) \sim N^{\frac{1}{2}}$$

for all $p \in [1, \infty)$.

A Lemma

Lemma

Let $p \in (1, \infty)$. Then there is $C > 0$ such that

$$\sum_{n=1}^{\infty} \|Se_n\|_{\ell^2_N} \|Te_n^*\|_{\ell^2_N} \leq C\,N\|S\|\|T\|$$

$$(N \in \mathbb{N},\ S \in \mathcal{B}(\ell^p, \ell^2_N),\ T \in \mathcal{B}(\ell^{p'}, \ell^2_N)).$$

Proof of the Lemma

Proof.

Identify algebraically

$$\mathcal{B}(\ell^p, \ell^2_N) = \ell^{p'} \check{\otimes} \ell^2_N = \ell^{p'} \otimes \ell^2_N = \ell^{p'}(\ell^2_N), \quad \text{and}$$
$$\mathcal{B}(\ell^{p'}, \ell^2_N) = \ell^{p} \check{\otimes} \ell^2_N = \ell^{p} \otimes \ell^2_N = \ell^{p}(\ell^2_N).$$

Note that

$$\begin{aligned} \sum_{n=1}^{\infty} \|Se_n\|_{\ell^2_N} \|Te_n^*\|_{\ell^2_N} &\leq \|S\|_{\ell^{p'}(\ell^2_N)} \|T\|_{\ell^p(\ell^2_N)}, \qquad \text{by Hölder,} \\ &\leq \pi_{p'}(\mathrm{id}_{\ell^2_N}) \pi_p(\mathrm{id}_{\ell^2_N}) \|S\| \|T\| \\ &\leq C\, N \|S\| \|T\|, \qquad \text{by Gordon.} \end{aligned}$$

□

Non-amenability of $\mathcal{B}(\ell^p)$ for $p \in (1, \infty)$

Corollary

Let $p \in (1, \infty)$ and let E be an $\mathcal{L}^p$-space with $\dim E = \infty$. Then $\ell^\infty(\mathcal{K}(E))$ is not amenable.

Theorem (VR, 2009)

Let $p \in (1, \infty)$, and let E be an $\mathcal{L}^p$-space. Then $\mathcal{B}(\ell^p(E))$ is not amenable.

Proof.

If $\mathcal{B}(\ell^p(E))$ is amenable, then so is $\ell^\infty(\mathcal{B}(\ell^p(E)))$ as is $\ell^\infty(\mathcal{K}(\ell^p(E)))$. Impossible! □

Corollary

Let $p \in (1, \infty)$. Then $\mathcal{B}(\ell^p)$ and $\mathcal{B}(L^p[0,1])$ are not amenable.