Compact quantum subgroups of a co-amenable quantum group are co-amenable

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Quantum group

Let $\mathbb{G} = (M, \Gamma, \phi, \psi)$ be a (locally compact) quantum group:

- M is a von Neumann algebra
- $\Gamma\colon M \to M \overline{\otimes} M$ is a co-multiplication on M, ie, a normal, unital $*$-homomorphism such that

$$(\Gamma \otimes \mathrm{id})\Gamma = (\mathrm{id} \otimes \Gamma)\Gamma$$

- ϕ and ψ are left and right Haar weights.

We denote

$$\mathrm{L}^\infty(\mathbb{G}) = M \qquad \text{and} \qquad \mathrm{L}^1(\mathbb{G}) = M_*.$$

GNS-construction applied to (M, ϕ) gives a Hilbert space, which we denote by $\mathrm{L}^2(\mathbb{G})$. We identify $\mathrm{L}^\infty(\mathbb{G})$ with its image under the natural representation on $\mathrm{L}^2(\mathbb{G})$ and consider $\mathrm{L}^\infty(\mathbb{G}) \subseteq \mathrm{B}(\mathrm{L}^2(\mathbb{G}))$.

Multiplicative unitary

There is a unitary operator on $\mathrm{L}^2(\mathbb{G}) \otimes \mathrm{L}^2(\mathbb{G})$ such that

$$\Gamma(x) = W^*(1 \otimes x)W \qquad (x \in \mathrm{L}^\infty(\mathbb{G}))$$

and satisfies the pentagonal relation

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

Define

$$\mathrm{C}_0(\mathbb{G}) = \overline{\{\,(\mathrm{id} \otimes \omega)W;\ \omega \in \mathrm{B}(\mathrm{L}^2(\mathbb{G}))_*\,\}}^{\|\cdot\|}$$

and $\mathrm{C}(\mathbb{G}) = \mathrm{M}(\mathrm{C}_0(\mathbb{G}))$ (the multiplier algebra of $\mathrm{C}_0(\mathbb{G})$).
$\mathrm{C}_0(\mathbb{G})$ gives the reduced C*-algebraic version of $\mathbb{G}$.

Comparision to yesterday's notation

- $A = \mathsf{C}_0(\mathbb{G})$.
- $B = \mathsf{C}_0(\mathbb{H}) = \mathsf{C}(\mathbb{H})$ is a compact quantum subgroup.
- $\pi\colon \mathsf{C}_0(\mathbb{G}) \to \mathsf{C}(\mathbb{H})$ a surjective $*$-homomorphism such that

$$(\pi \otimes \pi)\Gamma_{\mathbb{G}} = \Gamma_{\mathbb{H}}\pi.$$

Dual quantum group

The underlying C*-algebra of the dual quantum group $\widehat{\mathbb{G}}$ of $\mathbb{G}$ is

$$C_0(\widehat{\mathbb{G}}) = \overline{\{\,(\omega \otimes \mathrm{id})W;\ \omega \in B(L^2(\mathbb{G}))_*\,\}}^{\|\cdot\|}.$$

$L^\infty(\widehat{\mathbb{G}})$ is the weak* closure of $C_0(\widehat{\mathbb{G}})$. Put $\widehat{W} = \Sigma W^* \Sigma$ so that the co-multiplication of $\widehat{\mathbb{G}}$ is given by

$$\widehat{\Gamma}\colon L^\infty(\widehat{\mathbb{G}}) \to L^\infty(\widehat{\mathbb{G}}) \,\overline{\otimes}\, L^\infty(\widehat{\mathbb{G}})$$
$$\widehat{\Gamma}(x) = \widehat{W}^*(1 \otimes x)\widehat{W}.$$

It can be shown that there exist Haar weights for $\widehat{\mathbb{G}}$ and so $\widehat{\mathbb{G}}$ is a quatum group.

Pontryagin's duality holds: $\widehat{\widehat{\mathbb{G}}} = \mathbb{G}$.

Duality between commutative and co-commutative

Let G be a locally compact group. Now

$$\begin{aligned} W\xi(s,t) &= \xi(s, s^{-1}t) \\ \widehat{W}\xi(s,t) &= \xi(ts, t) \end{aligned} \qquad (\xi \in \mathrm{L}^2(G \times G),\ s,t \in G).$$

Define

$$\omega_{\xi,\zeta}(x) = (x\xi \mid \zeta), \qquad (x \in \mathrm{B}(\mathrm{L}^2(G)),\ \xi,\zeta \in \mathrm{L}^2(G)).$$

Then

$$(\mathrm{id} \otimes \omega_{\xi,\zeta})W = M_{\overline{\zeta} * \check{\xi}}$$

where $M_{\overline{\zeta} * \check{\xi}}$ is pointwise multiplication by $\overline{\zeta} * \check{\xi} \in \mathrm{A}(G) \subseteq \mathrm{C}_0(G)$.
On the other hand

$$(\omega_{\xi,\zeta} \otimes \mathrm{id})W = \lambda(\xi\overline{\zeta})$$

where $\lambda(\xi\overline{\zeta})$ is the convolution by $\xi\overline{\zeta} \in \mathrm{L}^1(G)$.

Compact and discrete quantum groups

A quantum group $\mathbb{G}$ is compact if the C*-algebra $C_0(\mathbb{G})$ is unital.

A quantum group $\mathbb{G}$ is discrete if its dual $\widehat{\mathbb{G}}$ is compact.

A commutative quantum groups $\mathbb{G} = G$ is compact iff G is a compact group; $\mathbb{G}$ is discrete iff G is discrete.

A co-commutative quantum group $\mathbb{G} = \widehat{G}$ is compact iff G is discrete, and $\mathbb{G}$ is discrete iff G is compact.

Co-amenability

A quantum group $\mathbb{G}$ is co-amenable if there is a bounded approximate identity in $\mathrm{L}^1(\mathbb{G})$. This is equivalent to $\mathrm{C}_0(\mathbb{G})^*$ being unital and to the existence of a non-zero multiplicative functional on $\mathrm{C}_0(\mathbb{G})$.

All commutative quantum groups and all discrete quantum groups are co-amenable.

A co-commutative quantum group $\mathbb{G} = \widehat{G}$ is co-amenable iff G is amenable.

Involution on $\mathrm{L}^1(\mathbb{G})$

The antipode $S\colon \mathrm{C}_0(\mathbb{G}) \subseteq \mathrm{dom}(S) \to \mathrm{C}_0(\mathbb{G})$ is a closed, densely defined operator. For every $\sigma \in B(\mathrm{L}^2(\mathbb{G}))_*$, $(\mathrm{id} \otimes \sigma)W \in \mathrm{dom}\, S$ and

$$S\big((\mathrm{id} \otimes \sigma)W\big) = (\mathrm{id} \otimes \sigma)W^*.$$

For every $\omega \in \mathrm{L}^1(\mathbb{G})$, define $\overline{\omega}$ by $\overline{\omega}(a) = \overline{\omega(a^*)}$, $a \in \mathrm{C}_0(\mathbb{G})$. Following Kustermans (01), define

$$\mathrm{L}^1_*(\mathbb{G}) = \{\, \omega \in \mathrm{L}^1(\mathbb{G}); \text{there is } \eta \in \mathrm{L}^1(\mathbb{G}) \text{ such that } \overline{\omega}(S(x)) = \eta(x) \text{ for every } x \in \mathrm{dom}(S) \,\}.$$

We obtain an involution $\omega \mapsto \omega^*\colon \mathrm{L}^1_*(\mathbb{G}) \to \mathrm{L}^1_*(\mathbb{G})$ by $\omega^* = \overline{\omega} S$. $\mathrm{L}^1_*(\mathbb{G})$ is a Banach $*$-algebra with respect to the norm $\|\omega\|_* = \max\{\|\omega\|, \|\omega^*\|\}$.

First lemma

Recall that $W \in \mathrm{M}(\mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\widehat{\mathbb{G}}))$, and define

$$\tau\colon \mathrm{L}^1_*(\mathbb{H}) \to \mathrm{C}(\widehat{\mathbb{G}}), \qquad \tau(\omega) = (\omega\pi \otimes \mathrm{id})W.$$

When $\mathbb{G} = G$ is a commutative quantum group and $\mathbb{H} = H$ is its subgroup,

$$\tau(f)\xi(s) = \int_H f(h)\xi(h^{-1}s)\,dh \qquad (f \in \mathrm{L}^1(H), \xi \in \mathrm{L}^2(G)).$$

Lemma

The map τ is a $*$-isomorphism from $\mathrm{L}^1_*(\mathbb{H})$ into $\mathrm{C}(\widehat{\mathbb{G}})$.

Extension to $C_0(\widehat{\mathbb{H}})$

Let $U \in B(L^2(\mathbb{H}) \otimes L^2(\mathbb{H}))$ be the left multiplicative unitary associated with $\mathbb{H}$.
Since $\widehat{\mathbb{H}}$ is discrete (and hence co-amenable), the universal C*-completion of $L^1_*(\mathbb{H})$ is just $C_0(\widehat{\mathbb{H}})$. Therefore there is a $*$-homomorphism $\rho\colon C_0(\widehat{\mathbb{H}}) \to C(\widehat{\mathbb{G}})$ such that

$$\rho\big((\omega \otimes \mathrm{id})U\big) = \tau(\omega) = (\omega\pi \otimes \mathrm{id})W \qquad (\omega \in L^1_*(\mathbb{H})).$$

It follows that

$$(\mathrm{id} \otimes \rho)U = (\pi \otimes \mathrm{id})W$$

in $M(C(\mathbb{H}) \otimes C_0(\widehat{\mathbb{G}}))$.

Embedding $C_0(\widehat{\mathbb{H}})$ to $C(\widehat{\mathbb{G}})$

Theorem

The map ρ *is a non-degenerate* $*$*-isomorphism from* $C_0(\widehat{\mathbb{H}})$ *to* $C(\widehat{\mathbb{G}})$ *and*

$$(\rho \otimes \rho)\widehat{\Gamma}_B = \widehat{\Gamma}\rho.$$

Proof.

Matrix coefficients of the irreducible unitary representations of $\mathbb{H}$:

$$u^{\alpha}_{pq} \qquad (\alpha \in I, 1 \leq p, q \leq n(\alpha)).$$

$B_0 := \operatorname{span}\{u^{\alpha}_{pq}\}$ is a Hopf $*$-algebra, which is dense in $C(\mathbb{H})$.
For every α in I and $0 \leq p, q \leq n(\alpha)$, there is ω^{α}_{pq} in $L^1(\mathbb{H})$ such that

$$\omega^{\alpha}_{pq}(u^{\beta}_{rs}) = \delta^{\beta}_{\alpha}\delta^{r}_{p}\delta^{s}_{q}.$$

The functional ω^{α}_{pq} is in fact in $L^1_*(\mathbb{H})$ because $(\omega^{\alpha}_{pq})^* = \omega^{\alpha}_{qp}$.

Proof continued

Let

$$\widehat{B}_0 = \operatorname{span}\{\, \omega^{\alpha}_{pq};\ \alpha, p, q \,\} \subseteq \mathrm{L}^1_*(\mathbb{H}).$$

Then

$$\widehat{B}_0 \cong \text{alg-}\bigoplus M_{n(\alpha)}.$$

Therefore $\widehat{B}_0$ has a unique C*-completion:

$$\mathrm{C}_0(\widehat{\mathbb{H}}) \cong c_0\text{-}\bigoplus M_{n(\alpha)}.$$

Since $\tau\colon \widehat{B}_0 \to \mathrm{C}(\widehat{\mathbb{G}})$ is a $*$-isomorphism, it follows from the uniqueness of the C*-completion of $\widehat{B}_0$ that $\rho\colon \mathrm{C}_0(\widehat{\mathbb{H}}) \to \mathrm{C}(\widehat{\mathbb{G}})$ is a $*$-isomorphism.

$$\begin{array}{ccc} & \widehat{B}_0 & \\ {\scriptstyle\lambda_u}\swarrow & & \searrow{\scriptstyle\tau} \\ \mathrm{C}_0(\widehat{\mathbb{H}}) & \xrightarrow[\rho]{} & \mathrm{C}(\widehat{\mathbb{G}}) \end{array}$$

Proof continued

Claim: $(\rho \otimes \rho)\widehat{\Gamma}_{\mathbb{H}} = \widehat{\Gamma}\rho$.

For every $\omega \in \mathrm{L}^1_*(\mathbb{H})$,

$$\begin{aligned}
\widehat{\Gamma}\rho\big((\omega \otimes \mathrm{id})U\big) &= \widehat{\Gamma}\big((\omega\pi \otimes \mathrm{id})W\big) = \widehat{W}^*\big(1 \otimes (\omega\pi \otimes \mathrm{id})W\big)\widehat{W} \\
&= \Sigma W\big((\omega\pi \otimes \mathrm{id})W \otimes 1\big)W^*\Sigma \\
&= \Sigma\big((\omega\pi \otimes \mathrm{id} \otimes \mathrm{id})W_{23}W_{12}W_{23}^*\big)\Sigma \\
&= \Sigma\big((\omega\pi \otimes \mathrm{id} \otimes \mathrm{id})W_{12}W_{13}\big)\Sigma
\end{aligned}$$

by the pentagonal equation. Since $(\pi \otimes \mathrm{id})W = (\mathrm{id} \otimes \rho)U$,

$$\begin{aligned}
\widehat{\Gamma}\rho\big((\omega \otimes \mathrm{id})U\big) &= (\omega\pi \otimes \mathrm{id} \otimes \mathrm{id})W_{13}W_{12} \\
&= (\omega \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \rho \otimes \rho)U_{13}U_{12} \\
&= (\rho \otimes \rho)\widehat{\Gamma}_{\mathbb{H}}\big((\omega \otimes \mathrm{id})U\big).
\end{aligned}$$

The claim follows because the elements $(\omega \otimes \mathrm{id})U$ with ω in $\mathrm{L}^1_*(\mathbb{H})$ are dense in $\mathrm{C}_0(\widehat{\mathbb{H}})$. $\square$

Weak* = strict on $C(\widehat{\mathbb{H}})$

Since $\widehat{\mathbb{H}}$ is discrete, $C(\widehat{\mathbb{H}}) = L^\infty(\widehat{\mathbb{H}})$.

Lemma

The weak topology and the strict topology agree on bounded sets of* $C(\widehat{\mathbb{H}})$.

Amenability

The quantum group $\mathbb{G}$ is said to be *amenable* if there exists a state m of $\mathrm{L}^\infty(\mathbb{G})$ such that

$$m(\omega \otimes \mathrm{id})\Gamma(x) = \omega(1)m(x) \qquad (\omega \in \mathrm{L}^1(\mathbb{G}),\ x \in \mathrm{L}^\infty(\mathbb{G})).$$

Such a state is called a *left invariant mean* on $\mathrm{L}^\infty(\mathbb{G})$.

$$\mathbb{G} \text{ co-amenable} \implies \widehat{\mathbb{G}} \text{ amenable}$$

[Bedos–Tuset].

$$\mathbb{G} \text{ amenable} \overset{?}{\implies} \widehat{\mathbb{G}} \text{ co-amenable}$$

The implication is true when $\mathbb{G}$ is commutative [Leptin] or discrete [Tomatsu].

Co-amenability is inherited by compact quantum subgroups

Theorem

Suppose that $\mathbb{H}$ is a compact quantum subgroup of a co-amenable quantum group $\mathbb{G}$. Then also $\mathbb{H}$ is co-amenable.

Proof.

By [Tomatsu] it suffices to show that $\widehat{\mathbb{H}}$ is amenable. Since $\mathbb{G}$ is co-amenable, $\widehat{\mathbb{G}}$ is amenable and so there is a state m of $\mathrm{C}(\widehat{\mathbb{G}})$ such that

$$m(\sigma \otimes \mathrm{id})\widehat{\Gamma}(x) = \sigma(1)m(x) \qquad (\sigma \in \mathrm{L}^1(\widehat{\mathbb{G}}), x \in \mathrm{C}(\widehat{\mathbb{G}})).$$

Proof continued

Claim: $m\rho$ is a left invariant mean on $\mathrm{C}(\widehat{\mathbb{H}}) = \mathrm{L}^\infty(\widehat{\mathbb{H}})$.

Let $\omega \in \mathrm{L}^1(\widehat{\mathbb{H}})$. Since "weak* = strictly", there is $\sigma \in \mathrm{L}^1(\widehat{\mathbb{G}}) = \mathrm{L}^\infty(\widehat{\mathbb{G}})_*$ such that $\omega = \sigma\rho$. Now for every $x \in \mathrm{C}(\widehat{\mathbb{H}})$

$$
\begin{aligned}
m\rho(\omega \otimes \mathrm{id})\widehat{\Gamma}_{\mathbb{H}}(x) &= m(\sigma \otimes \mathrm{id})(\rho \otimes \rho)\widehat{\Gamma}_{\mathbb{H}}(x) \\
&= m(\sigma \otimes \mathrm{id})\widehat{\Gamma}\rho(x) = \sigma(1)m\rho(x) = \omega(1)m\rho(x)
\end{aligned}
$$

because m is left invariant. Therefore $m\rho$ is a left invariant mean on $\mathrm{C}(\widehat{\mathbb{H}}) = \mathrm{L}^\infty(\widehat{\mathbb{H}})$ and so $\widehat{\mathbb{H}}$ is amenable. □