Universal quantum groups acting on classical and quantum spaces

Lecture 1 - Compact quantum groups and their actions

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1. Classical groups as collections of symmetries
2. Compact quantum groups
3. Actions of compact quantum groups
4. Further properties of actions
5. Non-compact (not neccessarily compact) case

Classical symmetry groups

Groups entered mathematics as collections of symmetries of a given object (a finite set, a figure on the plane, a manifold, a space of solutions of an equation).

How do we define a symmetry group of a given object X? We look at the family of 'all possible transformations' of X, usually preserving some structure of X.

In modern language: we search for a *universal* object in the category of all groups acting on X.

Compact quantum groups

We will be interested in the cases when the classical symmetry groups are compact/finite.
The idea of defining a compact quantum group is based on 'quantising' the algebra of continuous functions on a compact group.

Definition ([Wor_2])

A compact quantum group is a unital C^*-algebra A with a unital $*$-algebra homomorphism $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ such that

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta \qquad \text{(coassociativity)}$$

and

$$\overline{\Delta(\mathsf{A})(\mathsf{A} \otimes 1_{\mathsf{A}})} = \mathsf{A} \otimes \mathsf{A} = \overline{\Delta(\mathsf{A})(1_{\mathsf{A}} \otimes \mathsf{A})} \qquad \text{(quantum cancellation rules).}$$

The tensor products here are minimal/spatial tensor products of C^*-algebras.

Let us list some basic properties of and notions related to compact quantum groups we will need:

- there exists a unique bi-invariant state, a so called *Haar state* $h \in \mathsf{A}^*$:

$$(h \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta = (\mathrm{id}_{\mathsf{A}} \otimes h) \circ \Delta = h(\cdot) 1_{\mathsf{A}}.$$

- a *finite-dimensional unitary corepresentation* of A is a unitary matrix $U \in M_n \otimes \mathsf{A}$ such that (in the leg notation)

$$(\mathrm{id}_{M_n} \otimes \Delta)(U) = U_{12} U_{13}.$$

  In terms of the entries:

$$\Delta(U_{ij}) = \sum_{k=1}^{n} U_{ik} \otimes U_{kj};$$

- A admits a unique dense Hopf*-algebra $(\mathcal{A}, \Delta, \epsilon, S)$, spanned by the *coefficients* of the finite-dimensional unitary corepresentations of A; in particular $\Delta : \mathcal{A} \to \mathcal{A} \odot \mathcal{A}$. Hopf*-algebras arising in this way admit an intrinsic characterisation, which makes the theory of compact quantum groups in a sense completely algebraic.

- h is faithful on $\mathcal{A}$; if it is faithful on A, we say that A is in the reduced form. This can be always achieved by considering the C^*-completion of the GNS-representation π_h of $\mathcal{A}$. Conversely, one can consider the universal C^*-completion of $\mathcal{A}$ - if the reduced and universal object coincide, we say that A is *coamenable*.
- the coproduct has a unique normal extension to a coassociative unital *-homomorphism $\tilde{\Delta} : \mathsf{M} \to \mathsf{M}\overline{\otimes}\mathsf{M}$, where $\mathsf{M} = \pi_h(\mathsf{A})''$;
- if h is tracial, then (A, Δ) is a *Kac algebra of compact type*; in particular $S^2 = \mathrm{id}_{\mathcal{A}}$.

Examples of compact quantum groups

- commutative CQGs - algebras $C(G)$, where G-compact group. The coproduct is determined by (recall that $C(G) \otimes C(G) \approx C(G \times G)$)

$$\Delta(f)(s,t) = f(st), \;\; s,t \in G, f \in C(G).$$

- cocommutative CQGs - essentially (subtlety is related to a possible non-coamenability) algebras $C^*(\Gamma)$ or $C_r^*(\Gamma)$, where Γ - discrete group. The coproduct is given by the (continuous linear extension of)

$$\Delta(\pi_\gamma) = \pi_\gamma \otimes \pi_\gamma, \quad \gamma \in \Gamma.$$

Examples continued: deformations of classical compact Lie groups

- Note that $C(SU(2))$ is a commutative unital C^*-algebra generated by the functions $\alpha, \gamma : SU(2) \to \mathbb{C}$ such that

$$\alpha^*\alpha + \gamma^*\gamma = 1.$$

Group multiplication on $SU(2)$ induces on $C(SU(2))$ the coproduct

$$\Delta(\alpha) = \alpha \otimes \alpha - \gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Let $q \in [-1, 1) \setminus \{0\}$. Define $SU_q(2)$ - unital C^*-algebra generated by operators α, γ such that:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma^*\gamma = 1,$$

$$\gamma^*\gamma = \gamma\gamma^*, \quad q\gamma\alpha = \alpha\gamma, \quad q\gamma^*\alpha = \alpha\gamma^*.$$

The coproduct making $SU_q(2)$ a compact quantum group is given by the formulas

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Compact matrix quantum groups ([Wor_1])

If a compact quantum group A admits $n \in \mathbb{N}$ and a unitary matrix $U = (u_{ij})_{i,j=1}^n \in M_n(\mathrm{A})$ such that

- $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$;
- $\{u_{ij} : i, j = 1, \ldots, n\}$ generates A as a C^*-algebra,

then A is called a *compact matrix quantum group* and U a *fundamental unitary corepresentation* of A. The counit of $\mathcal{A}$ is determined by the formula $\epsilon(u_{ij}) = \delta_{ij}$, the antipode by the formula $S(u_{ij}) = u_{ji}^*$.

Actions

The action of a group G on a set X can be described as a map $\alpha : G \times X \to X$ satisfying certain natural conditions. In the quantum case we (as usual) 'invert the arrows':

Definition ([Pod], [Wan$_2$])

Let A be a compact quantum group and let B be a unital C^*-algebra. A map

$$\alpha : \mathsf{B} \to \mathsf{A} \otimes \mathsf{B}$$

is called a (left, continuous) action of A on B if α is a unital *-homomorphism,

$$(\Delta \otimes \mathrm{id}_{\mathsf{B}}) \circ \alpha = (\mathrm{id}_{\mathsf{A}} \otimes \alpha) \circ \alpha$$

and additionally $\alpha(\mathsf{B})(\mathsf{A} \otimes 1_{\mathsf{B}})$ is dense in $\mathsf{A} \otimes \mathsf{B}$ (*Podleś/continuity condition*).

Unless stated otherwise when we say that $\alpha : \mathsf{B} \to \mathsf{A} \otimes \mathsf{B}$ is an action we mean that all the above are satisfied. Note that each compact quantum group acts on itself via the coproduct.

On *continuity* of actions and its consequences

If $\mathsf{A} = C(G)$ for a compact group G, X is a compact space and $G \curvearrowright X$ is a continuous action, then in the quantum picture we define

$$\alpha : C(X) \to C(G) \otimes C(X) \approx C(G \times X)$$

via

$$\alpha(f)(g,x) = f(gx), \quad g \in G, x \in X, f \in C(X).$$

Conversely given an action of $\alpha : C(X) \to C(X) \otimes C(G)$ we define for each $g \in G$ first

$$\alpha_g = (\mathrm{ev}_g \otimes \mathrm{id}_{C(X)}) \circ \alpha$$

and then note that the 'action relation' implies that $\alpha_g \circ \alpha_h = \alpha_{gh}$ for $g, h \in G$.

The condition $\alpha(\mathsf{B})(\mathsf{A} \otimes 1_{\mathsf{B}})$ is dense in $\mathsf{A} \otimes \mathsf{B}$ is a kind of a nondegeneracy or continuity property for the action, in the case above guaranteeing that each α_g is a surjection. It excludes for example $\alpha = \rho(\cdot)1_{\mathsf{A}} \otimes 1_{\mathsf{B}}$, where ρ is a character on B.

The continuity condition implies that there exists $\mathcal{B}$, a dense *-subalgebra of B, such that $\alpha|_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} \odot \mathcal{B}$ is a Hopf*-algebraic action ([Wan$_2$], [Boc]); in particular we have

$$(\epsilon \otimes \mathrm{id}_{\mathcal{B}})\alpha|_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}}.$$

The algebra $\mathcal{B}$ is a linear span of *spectral subspaces* $\mathcal{B}_u$ corresponding to *irreducible* unitary corepresentations of A.

The action $\alpha : \mathsf{B} \to \mathsf{A} \otimes \mathsf{B}$ is called *faithful* if the set $\mathrm{Lin}\{(\mathrm{id}_{\mathsf{A}} \otimes \phi) \circ \alpha(b) : b \in \mathsf{B}, \phi \in \mathsf{B}^*\}$ is dense in A.

Exercises:

- Check that if $U \in M_n(\mathsf{A}) \approx \mathsf{A} \otimes M_n$ is a unitary corepresentation, then the map
$$x \to U^*(1_{\mathsf{A}} \otimes x)U, \quad x \in M_n,$$
defines an action of A on M_n.
- Arbitrary (not necessarily finite-dimensional) unitary corepresentations are defined as unitaries in the multiplier algebra $M(\mathsf{A} \otimes K(\mathsf{H}))$, where H is a Hilbert space, again satisfying the suitably understood relation
$$(\Delta \otimes \mathrm{id}_{K(\mathsf{H})})(U) = U_{13}U_{23};$$
do they define actions of A in our sense by the formula from the first part of the exercise?

Lifting of actions to von Neumann algebras

Definition

We say that the action $\alpha : \mathsf{B} \to \mathsf{A} \otimes \mathsf{B}$ preserves a functional $\omega \in \mathsf{B}^*$ if

$$\forall_{b \in \mathsf{B}} \quad (\mathrm{id}_{\mathsf{A}} \otimes \omega)(\alpha(b)) = \omega(b) 1_{\mathsf{A}}.$$

Theorem ([Wan$_2$])

Let $\alpha : \mathsf{B} \to \mathsf{A} \otimes \mathsf{B}$ be an action preserving a state $\tau \in \mathsf{B}^*$. Then α lifts to a normal *-homomorphism $\tilde{\alpha} : \pi_\tau(\mathsf{B})'' \to \pi_h(\mathsf{A})'' \overline{\otimes} \pi_\tau(\mathsf{B})''$ defined by the formula

$$\tilde{\alpha}(\pi_\tau(b)) = (\pi_h \otimes \pi_\tau)(\alpha(b)), \quad b \in \mathsf{B},$$

which is an action in the Hopf-von Neumann algebra sense.

Ergodic actions

Definition

We say that the action $\alpha : \mathsf{B} \to \mathsf{A} \otimes \mathsf{B}$ is ergodic if the fixed point space $\mathrm{Fix}(\alpha) = \{b \in \mathsf{B} : \alpha(b) = 1_{\mathsf{A}} \otimes b\}$ is one-dimensional.

For arbitrary action α the map $E_\alpha : \mathsf{B} \to \mathsf{B}$ defined by

$$E_\alpha = (h \otimes \mathrm{id}_{\mathsf{B}})\alpha$$

is a norm-one projection from B onto $\mathrm{Fix}(\alpha)$. Moreover $\{b \in \mathcal{B} : \alpha(b) = 1_{\mathsf{A}} \otimes b\}$ is dense in $\mathrm{Fix}(\alpha)$.

Theorem ([Boc], [BDRV])

Spectral subspaces of ergodic actions are finite-dimensional, with their dimensions dominated by the (quantum) dimensions of the corresponding corepresentations.

Some ergodic actions can be described in abstract categorical terms. The theory becomes much richer then the classical one.

Category of CQGs acting on a given C^*-algebra

Consider the category $\mathfrak{C}(\mathsf{B}) := \{(\mathsf{A}, \alpha)\}$ of compact quantum groups acting on a given C^*-algebra B. A morphism in the category $\mathfrak{C}(\mathsf{B})$:

$$\gamma : (\mathsf{A}_1, \alpha_1) \to (\mathsf{A}_2, \alpha_2)$$

is a unital *-homomorphism $\gamma : \mathsf{A}_1 \to \mathsf{A}_2$ such that

$$(\gamma \otimes \gamma) \circ \Delta_{\mathsf{A}_1} = \Delta_{\mathsf{A}_2} \circ \gamma, \quad \alpha_2 = (\gamma \otimes \mathrm{id}_{\mathsf{B}}) \circ \alpha_1.$$

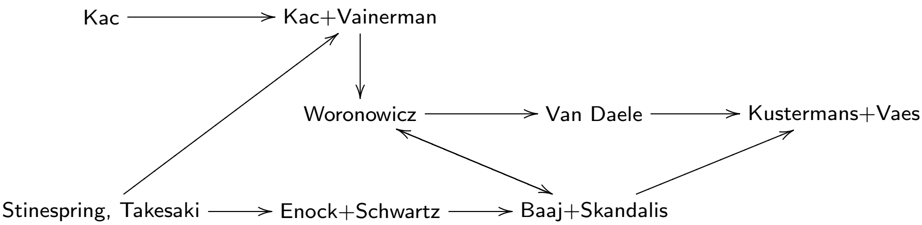
We say that the category $\mathfrak{C}(\mathsf{B})$ admits a universal object, if there is (A_u, α_u) in $\mathfrak{C}(\mathsf{B})$ such that for all (A, α) in $\mathfrak{C}(\mathsf{B})$ there exists a unique morphism $\gamma : (\mathsf{A}_u, \alpha_u) \to (\mathsf{A}, \alpha)$. Abstract categorical nonsense implies that if such a universal object exists, it is unique.

$$(\mathsf{A}_u, \alpha_u) - \text{quantum symmetry group of B}$$

Further we will also consider categories of actions preserving some 'extra' structure of B.

Locally compact quantum semigroups

The story of development of locally compact quantum groups is long and complicated - we will just offer a diagram:



Locally compact quantum semigroups

For now it suffices for us to work in a simplified context of locally compact quantum semigroups:

Definition

A locally compact quantum semigroup is a C^*-algebra A equipped with a nondegenerate *-homomorphism $\Delta : \mathsf{A} \to M(\mathsf{A} \otimes \mathsf{A})$ which is coassociative:

$$(\mathrm{id}_{\mathsf{A}} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta.$$

Some comments on the definition:

- The multiplier algebra of a C^*-algebra C is the largest C^*-algebra containing C as an essential ideal. It is equipped with the *strict* topology: locally convex topology determined by the seminorms $\{l_c, r_c : c \in \mathsf{C}\}$, where $l_c(x) = \|cx\|$, $r_c(x) = \|xc\|$ $(x \in M(\mathsf{C}))$.
- A bounded linear map $\phi : \mathsf{C} \to M(\mathsf{D})$ is called *strict* ([Kus]) if it is strictly continuous on bounded subsets of C. Such a map admits a unique strict extension to $\tilde{\phi} : M(\mathsf{C}) \to M(\mathsf{D})$.
- A *-homomorphism $\pi : \mathsf{C} \to M(\mathsf{D})$ is called nondegenerate if $\pi(\mathsf{C})\mathsf{D}$ is dense in D. Nondegenerate *-homomorphisms are strict.
- When we compose strict maps we always have in mind respective strict extensions: so for example it makes sense to write $(\mathrm{id}_{\mathsf{A}} \otimes \Delta) \circ \Delta$ and understand it as a map from A (or $M(\mathsf{A})$) to $M(\mathsf{A} \otimes \mathsf{A} \otimes \mathsf{A})$.

'Non-compact' actions

Definition

Let (A, Δ) be a locally compact quantum semigroup and let B be a C^*-algebra. An action of A on B is a nondegenerate *-homomorphism $\alpha : \mathsf{B} \to M(\mathsf{A} \otimes \mathsf{B})$ such that

$$(\Delta \otimes \mathrm{id}_{\mathsf{B}}) \circ \alpha = (\mathrm{id}_{\mathsf{A}} \otimes \alpha) \circ \alpha,$$

$$\mathrm{Lin}\{(a \otimes 1_{M(\mathsf{B})})\alpha(b) : a \in \mathsf{A}, b \in \mathsf{B}\} \text{ is (contained in and) dense in } \mathsf{A} \otimes \mathsf{B}.$$

Exercises ([Soł$_2$]):

- Let G be a locally compact group and B a C^*-algebra. Put $\mathsf{A} := C_0(G)$. Assume $\alpha : \mathsf{B} \to M(\mathsf{A} \to \mathsf{B})$ is a nondegenerate *-homomorphism such that $(\Delta \otimes \mathrm{id}_{\mathsf{B}}) \circ \alpha = (\mathrm{id}_{\mathsf{A}} \otimes \alpha) \circ \alpha$. Prove that the following are equivalent:
  1. $(\mathrm{ev}_e \otimes \mathrm{id}_{\mathsf{B}}) \circ \alpha = \mathrm{id}_{\mathsf{B}}$;
  2. $\mathrm{Ker}(\alpha) = \{0\}$;
  3. $\mathrm{Lin}\{(a \otimes 1_{M(\mathsf{B})})\alpha(b) : a \in \mathsf{A}, b \in \mathsf{B}\}$ is dense in $\mathsf{A} \otimes \mathsf{B}$;
  4. $\mathrm{Lin}\{(\omega_f \otimes \mathrm{id}_{\mathsf{B}})\alpha(b) : f \in L^1(G), b \in \mathsf{B}\}$ is dense in B ($\omega_f \in C_0(G)^*$ is given by the integration with respect to the Haar measure on G).
- Show that if the above conditions hold, α induces a continuous action of G on B by automorphisms.

Even when we pass to locally compact quantum *groups* (i.e. assume the existence of suitably well-behaved left- and right- invariant weights), there is still no 'nice' underlying algebraic object $\mathcal{A}$ behind our C^*-algebra (as it was in the compact case). Although in many cases we may have a 'presentation' of our locally compact quantum group in terms of generators and relations, the generators will usually be unbounded and will have to be understood as operators affiliated with a given operator algebra. This makes the attempts of developing the theory we will discuss in the next two lectures much more difficult.

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