# Universal quantum groups acting on classical and quantum spaces

# Lecture 2 - Quantum symmetry groups of finite structures and their limits

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# Quantum permutation groups

Both classically and in the quantum framework the simplest symmetry groups are (quantum) permutation groups, which can be viewed as the universal (quantum) groups acting on a given finite set.

Let $n \in \mathbb{N}$ and let $X_n = \{1, \ldots, n\}$, so that $C(X_n) \approx \mathbb{C}^n$. Write $\chi_i$ for the indicator function of $\{i\}$.

**Theorem ([Wan$_1$])**

*The category $\mathfrak{C}(C(X_n))$ of quantum groups acting on the n-point set admits the universal object. It is denoted $\mathrm{A}_s(n)$ and called the quantum permutation group of an $n-$point set.*

$\mathrm{A}_s(n)$ is a compact matrix quantum group, its fundamental unitary is the $n$ by $n$ matrix whose entries are orthogonal projections (magic unitary):

$$U = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}, \qquad (q_{ij})^2 = q_{ij}^* = q_{ij}, \quad i,j = 1, \ldots, n.$$

In particular $\mathrm{A}_s(n)$ is the universal $C^*$-algebra generated by $n^2$-projections satisfying the orthogonality relations making $U$ above a unitary. The coproduct, counit and the antipode are determined by:

$$\Delta(q_{ij}) = \sum_{k=1}^{n} q_{ik} \otimes q_{kj},$$

$$\epsilon(q_{ij}) = \delta_{ij},$$

$$S(q_{ij}) = q_{ji}^*.$$

The action of $\mathrm{A}_s(n)$ on $C(X_n)$ is given by

$$\alpha(\chi_i) = \sum_{j=1}^{n} q_{ij} \otimes \chi_j.$$

It is easy to check that the universal $C^*$-algebra generated by $n^2$ *commuting* projections such that $U$ from the last slide is a unitary is isomorphic (also as a compact quantum group) with $C(S_n)$. In other words $C(S_n)$ is a quantum subgroup of $\mathrm{A}_s(n)$.

Exercises:

- prove Wang's theorem.
- check the statement from the top of the slide and identify the corresponding $q_{ij}$ with an element of $C(S_n)$.
- show that $\mathrm{A}_s(2) \approx C(S_2)$, $\mathrm{A}_s(3) \approx C(S_3)$, but $\mathrm{A}_s(4)$ is infinite-dimensional.

Analysing generators and relations between them one can show that $\mathrm{A}_s(4)$ can be viewed as a $SO_{-1}(3)$, a deformation of $C(SO(3))$ ([BB]). For $n \geq 5$ the quantum group $\mathrm{A}_s(n)$ is not even coamenable.

# Some interesting properties of quantum permutation groups

- the irreducible corepresentations of $\mathrm{A}_s(n)$ satisfy the ‘fusion rules’ identical to these of irreducible representations of $SO(3)$;
- there is an intimate relation between the last result and the Temperley-Lieb algebra, offering further connections to planar algebras and subfactor theory;
- the formulas for the Haar state integration on $\mathrm{A}_s(n)$ are related to ‘free’ combinatorics and random matrix models.

More information can be found in the survey paper [$\mathrm{BBC}_1$].

Recently with Goswami we studied different possible definitions of quantum group of all finite permutations (quantum versions of $S_\infty$).

# Subgroups of quantum permutation groups

## Definition ([Pod])

Let $A_1$, $A_2$ be compact quantum groups. We will say that $A_2$ is a quantum subgroup of $A_1$ if there exists a surjective morphism $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ (recall that $\mathcal{A}_i$ denote the respective dense Hopf $^*$-algebras) such that

$$(\pi \otimes \pi)\Delta_2 = \Delta_1 \pi.$$

The existence of the morphism as above is equivalent to the existence of a surjective morphism between the universal versions of $A_1$ and $A_2$. If $A_1$ and $A_2$ are coamenable we can just think about $\pi : A_1 \to A_2$.

## Definition ([BBS])

A compact quantum group A is called a quantum group of permutations if it is a quantum subgroup of some $A_s(n)$.

Earlier arguments imply that $C(S_n)$ is a quantum subgroup of $A_s(n)$. All quantum subgroups of $A_4$ are known ([BB]); when $n \geq 5$ the classification problem is not tractable.

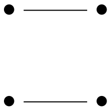
# Quantum isometry groups of graphs

Let $(V, E)$ be a finite (possibly directed) graph. Define the quantum symmetry group of $(V, E)$ as the universal quantum group acting on $C(V)$ and 'preserving' the edges; in other words acting simultaneously on $V$ and on $E$ in a compatible way ([Bic]).

The quantum symmetry group of $(V, \emptyset)$ coincides with $\mathrm{A}_s(\mathrm{card}(V))$. On the other hand the quantum symmetry group of the $n$-cycle is commutative, so it coincides with $C(\mathbb{Z}/_{n\mathbb{Z}})$. Similarly the quantum symmetry graph of the complete graph is commutative.

# Quantum isometry groups of graphs

The quantum symmetry group of the graph:



is infinite-dimensional. It is called the quantum dihedral group or the quantum hyperoctahedral group $A_h(2)$.

Banica and Bichon studied many finite graphs and gave some criteria for graphs to have 'genuinely quantum symmetries'.

## Problem

Does there exist a finite graph such that its classical symmetry group does not act on it ergodically (transitively), but the quantum one does?

# Quantum isometry groups of finite metric spaces

Let $(X, d)$ be a finite metric space ($\mathrm{card}(X) = n$). We say that an action $\alpha : C(X) \to \mathsf{A} \otimes C(X)$ preserves the metric if the matrix $(a_{ij})_{i,j=1}^n \in M_n(\mathsf{A})$ determined by the condition $\alpha(\chi_i) = \sum_{j=1}^n a_{ij} \otimes \chi_j$ ($i = 1, \ldots, n$) commutes with the scalar matrix $(d_{ij} := d(i,j))_{i,j=1}^n$.

## Definition ([Ban])

The quantum isometry group of $(X, d)$ is the universal compact quantum group acting on $C(X)$ in a metric preserving fashion. It is a quantum subgroup of $\mathsf{A}_s(n)$.

Finite metric spaces can be interpreted as multi-coloured (complete) graphs.

# Quantum symmetry group of $M_2$-negative result

Consider now actions on finite-dimensional, but not necessary commutative $C^*$-algebras.

**Theorem**

*The category $\mathfrak{C}(M_n)$ does not admit a universal object if $n > 1$.*

The problem is related to the fact that there is a universal object in the category of compact quantum *semigroups* acting on $M_n$, but it is not a compact quantum *group*.

# Quantum symmetry groups of a finite $C^*$-algebra with a faithful state

**Theorem ([$\mathrm{Wan}_1$])**

*Let $\mathsf{D}$ be a finite-dimensional $C^*$-algebra with a faithful state $\omega$. The category $\mathfrak{C}(\mathsf{D}, \omega)$ of quantum groups acting on $\mathsf{D}$ and preserving the state $\omega$ admits a universal object. We denote it $\mathsf{A}_{aut}(\mathsf{D}, \omega)$.*

For example if $\mathsf{D} = M_n$ and $\omega = \mathrm{tr}$, the corresponding universal object, denoted usually simply by $\mathsf{A}_{aut}(M_n)$ is the universal $C^*$-algebra generated by the operators $\{a_{ij}^{kl} : i, j, k, l = 1, \ldots, n\}$ satisfying the relations:

$$\sum_{v=1}^{n} a_{ij}^{kv} a_{rs}^{vl} = \delta_j^r a_{is}^{kl},$$

$$\sum_{v=1}^{n} a_{kv}^{ij} a_{vl}^{rs} = \delta_j^r a_{kl}^{is},$$

$$(a_{ij}^{kl})^* = a_{ji}^{kl},$$

$$\sum_{r=1}^{n} a_{rr}^{kl} = \delta_{kl} = \sum_{r=1}^{n} a_{kl}^{rr}.$$

The action of $\mathsf{A}_{aut}(M_n)$ on $M_n$ is given by the formula

$$\alpha(e_{ij}) = \sum_{k,l=1}^{n} e_{kl} \otimes a_{ij}^{kl}, \quad i, j = 1, \ldots, n$$

($e_{ij}$ denote matrix units in $M_n$).

Exercises:

- prove the statements on $\mathrm{A}_{aut}(M_n)$.
- compute the universal object in the category $\mathfrak{C}(M_2, \omega)$, where $\omega$ is a non-tracial faithful state on $M_2$.
- use the last computation to argue that $\mathfrak{C}(M_2)$ does not admit a universal object.

Sołtan proved recently in [Sol₁] that the universal compact quantum group in $\mathfrak{C}(M_2, \omega)$ is isomorphic to $SO_q(3)$ (with $q$ dependent on the choice of $\omega$).

Each finite (i.e. finite dimensional) quantum group acts on itself, hence is a quantum subgroup of some $\mathrm{A}_{aut}(\mathrm{D}, \omega)$.

### Problem

Is each finite quantum group a quantum group of permutations?

# Inductive limits

## Theorem ([BGS])

*Suppose that* $(\mathsf{A}_n)_{n\in\mathbb{N}}$ *is a sequence of compact quantum groups and for each* $n, m \in \mathbb{N}$, $n \leq m$ *there is a compact quantum group morphism* $\pi_{n,m} : \mathsf{A}_n \to \mathsf{A}_m$ *with the compatibility property*

$$\pi_{m,k} \circ \pi_{n,m} = \pi_{n,k}, \quad n \leq m \leq k.$$

*Then the inductive limit of* $C^*$*-algebras* $(\mathsf{A}_n)_{n\in\mathbb{N}}$*, denoted further* $\mathsf{A}_\infty$*, has a canonical structure of a compact quantum group. It has the following universality property: for any compact quantum group* $(\mathsf{A}, \Delta)$ *such that there are morphisms* $\pi_n : \mathsf{A}_n \to \mathsf{A}$ *satisfying for all* $m, n \in \mathbb{N}$, $m \geq n$ *the equality* $\pi_m \circ \pi_{n,m} = \pi_n$*, there exists a unique morphism* $\pi_\infty : \mathsf{A}_\infty \to \mathsf{A}$ *such that* $\pi_n = \pi_\infty \circ \pi_{n,\infty}$ *for all* $n \in \mathbb{N}$ *(here* $\pi_{n,\infty}$ *is the canonical unital* $C^*$*-homomorphism from* $\mathsf{A}_n$ *into* $\mathsf{A}_\infty$*).*

The theorem from the last slide can be used to define the quantum symmetry group of an $AF$-algebra (a $C^*$-algebra arising as an inductive limit of a sequence of finite dimensional $C^*$-algebras) with a fixed faithful state. At each stage we consider a quantum symmetry group of a given finite dimensional $C^*$-algebra, at the same time making sure that the corresponding actions preserve structures present at all previous steps.

The resulting quantum symmetry groups can be interpreted as quantum symmetry groups of Bratteli diagrams. We will revisit this in the next lecture; here we just give one example:

# Quantum symmetry group of the Cantor set

## Theorem ([BGS])

*Let* B *be the AF algebra arising as a limit of the unital embeddings*

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \cdots$$

*with a tree-like Bratteli diagram. It is isomorphic to the algebra* $C(\mathfrak{C})$ *of continuous functions on the middle-third Cantor set. Suppose that the fixed faithful state* $\omega$ *is the canonical trace on* B*. Then the corresponding sequence of compact quantum groups is given by* $\mathsf{A}_1 = C(\mathbb{Z}/_{2\mathbb{Z}})$ *and*

$$\mathsf{A}_{n+1} = (\mathsf{A}_n \star \mathsf{A}_n) \oplus (\mathsf{A}_n \star \mathsf{A}_n), \quad n \in \mathbb{N}.$$

$\mathsf{A}_2$ - quantum hyperoctahedral group $\mathsf{A}_h(2)$ of [BBC$_2$].

## Corollary

The *quantum symmetry group of the Cantor set*, the inductive limit of the sequence of compact quantum groups listed in the last theorem, is the universal $C^*$-algebra generated by the family of selfadjoint projections

$$\{p\} \cup \bigcup_{n\in\mathbb{N}} \{p_{m_1,\ldots m_n} : m_1, \ldots m_n \in \{1,2,3,4\}\}$$

subjected to the following relations:

$$p_1, p_2 \leq p, \quad p_3, p_4 \leq p^{\perp},$$

$$p_{m_1,\ldots,m_n,1}, p_{m_1,\ldots,m_n,2} \leq p_{m_1,\ldots,m_n}, \quad p_{m_1,\ldots,m_n,3}, p_{m_1,\ldots,m_n,4} \leq p_{m_1,\ldots,m_n}^{\perp}$$

$(n \in \mathbb{N}, m_1, \ldots m_n \in \{1,2,3,4\})$.

Exercises ([BGS]):

- Suppose that $(X, d_X), (Y, d_Y)$ are compact metric spaces and $T : X \times Y \to X \times Y$ is an isometry satisfying the following condition: $\alpha_T(C(X) \otimes 1_Y) \subset C(X) \otimes 1_Y$, where $\alpha_T : C(X \times Y) \to C(X \times Y)$ is given by the composition with $T$. Show that $T$ is a product isometry.
- Deduce from the last theorem that the fact above fails to hold for 'quantum symmetries' (in particular decide what should be meant by that statement).

# Free product of compact quantum groups ([Wan$_3$])

**Theorem**

*Let $\mathsf{A}_1$, $\mathsf{A}_2$ be compact quantum groups. The free product (identifying the units) $\mathsf{A} := \mathsf{A}_1 \star \mathsf{A}_2$ has a natural compact quantum group structure (with the canonical embeddings of $\mathsf{A}_i$ being CQG morphisms). The Haar state of $\mathsf{A}$ is the free product of the Haar states on $\mathsf{A}_1$ and $\mathsf{A}_2$.*
*If $U_1 \in M_{n_1}(\mathsf{A}_1)$, $U_2 \in M_{n_2}(\mathsf{A}_2)$ are fundamental unitary corepresentations, then $U_1 \oplus U_2 \in M_{n_1+n_2}(\mathsf{A})$ is a fundamental unitary corepresentation of $\mathsf{A}$.*

One can similarly consider free product of compact quantum groups with amalgamation over a common quantum subgroup.

# Quantum universal unitary groups ([Wan$_3$], [VDW])

**Theorem**

*Let $n \in \mathbb{N}$ and let $Q \in M_n$ be invertible. Let $\mathrm{A}_u(Q)$ be the universal $C^*$-algebra generated by elements $u_{ij}$ $(i, j = 1, \ldots, n)$ such that if $U = (u_{ij})_{i,j=1}^n$, $U^t = (u_{ji})_{i,j=1}^n$, $\overline{U} = (u_{ij}^*)_{i,j=1}^n$, $U^* = (u_{ji}^*)_{i,j=1}^n$, then*

$$UU^* = I_n = U^* U,$$

$$U^t Q \overline{U} Q^{-1} = I_n = Q \overline{U} Q^{-1} U^t.$$

*The algebra $\mathrm{A}_u(Q)$ has a unique compact quantum group structure such that $U$ is a (fundamental) irreducible corepresentation. Any compact matrix quantum group is a quantum subgroup of some $\mathrm{A}_u(Q)$.*

# Quantum universal orthogonal groups ([VDW])

**Theorem**

*Let $n \in \mathbb{N}$ and let $Q \in M_n$ be invertible. Let $\mathrm{A}_o(Q)$ be the universal $C^*$-algebra generated by elements $u_{ij}$ $(i, j = 1, \ldots, n)$ such that*

$$U = \overline{U},$$

$$UU^t = I_n = U^t U,$$

$$U^t Q \overline{U} Q^{-1} = I_n = Q \overline{U} Q^{-1} U^t.$$

*The algebra $\mathrm{A}_o(Q)$ has a unique compact quantum group structure such that $U$ is a (fundamental) irreducible corepresentation.*

For $n = 2$ each $\mathrm{A}_0(Q)$ is isomorphic to a certain $SU_q(2)$.

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