Universal quantum groups acting on classical and quantum spaces

Lecture 3 - Quantum isometry groups of noncommutative manifolds

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Dirac operator in classical Riemannian geometry

Let M be a compact connected Riemannian manifold with no boundary. If M is orientable (for example $M = \mathbb{T}$) or is even-dimensional and 'very nice', i.e. has a *spin* or *spin*^{c} structure (for example $M = S^2$), then using the constructions with Clifford algebras and spinor bundles one can encode a lot of the structure of M in a (pseudo)differential selfadjoint operator D acting on a Hilbert space, on which we have a natural representation of $C^\infty(M)$ (by multiplication). Variants of this operator are called a Dirac or Hodge-Dirac operator. It is unbounded, but has very good spectral properties.

Connes suggested that as C^*-algebras can be thought of as noncommutative counterparts of topological spaces, we should think of noncommutative manifolds by fixing a natural dense *-subalgebra of a given C^*-algebra (to be thought of as the algebra of smooth functions), and a Hilbert space operator playing the role of the Dirac operator.

Spectral triples on C^*-algebras ([Con])

Definition

Let B be a C^*-algebra. A triple $(\mathcal{B}, \mathrm{H}, D)$ is called a spectral triple on B if $\mathcal{B}$ is a dense *-subalgebra of B, B is faithfully and nondegenerately represented on H and D is a (usually unbounded) self-adjoint operator with compact resolvents on the Hilbert space H such that for all $b \in \mathcal{B}$ the commutator $[D, b]$ is densely defined and bounded.

$(\mathcal{B}, \mathrm{H}, D)$ is *of compact type* if $[D, b]$ is compact for each $b \in \mathcal{B}$. Given a spectral triple we can define a distance on the state space of B by the formula

$$d(\phi, \psi) = \sup\{|\phi(b) - \psi(b)| : b \in \mathcal{B}, \|[D, b]\| \leq 1\}.$$

We say that a spectral triple is *Rieffel regular* if the topology on the state space determined by the distance above coincides with the weak*-topology..

Construction of a noncommutative Laplacian ([Con], [Gos])

In many examples, under natural *summability conditions* (of the type $|D|^{-p}$ being trace class for some $p > 0$) a manipulation with the Dixmier trace yields a natural positive normalised functional τ on $\mathcal{B}$, leading via the GNS type construction to a Hilbert space H_D^0 (and via another abstract construction of the space of 1-forms to another Hilbert space H_D^1).
Consider the map $d_D(\cdot) = [D, \cdot]$. Assume that the corresponding Hilbert space operator

$$d_D : \mathsf{H}_D^0 \to \mathsf{H}_D^1$$

is densely defined and closable.

Definition

The noncommutative Laplacian is the self-adjoint operator on H_D^0 defined by

$$\mathcal{L} = -d_D^* d_D.$$

Admissibility ([Gos])

We call a spectral triple *admissible* if

1. the construction described above yields a well defined Laplacian $\mathcal{L}$;
2. $\mathcal{B} \subset \mathrm{Dom}(\mathcal{L})$, $\mathcal{L}(\mathcal{B}) \subset \mathcal{B}$;
3. $\mathcal{L}$ has compact resolvents, its eigenvectors belong to $\mathcal{B}$;
4. $\mathcal{L}$ has a one-dimensional kernel;
5. the complex linear span of eigenvectors of $\mathcal{L}$, denoted by $\mathcal{B}_0$, is norm-dense in $\mathcal{B}$.

Note that we view above $\mathcal{B}$ as a subspace of H^0_D. Spectral triples associated with 'good' connected compact Riemannian manifolds are admissible.

Classical case

The following observation is a starting point of the theory of quantum isometry groups associated with spectral triples.

Lemma ([Gos])

Let M be a compact Riemannian manifold. Then a smooth map $\gamma : M \to M$ is a Riemannian isometry if and only if γ commutes with the Laplacian:

$$\mathcal{L}(f \circ \gamma) = (\mathcal{L}(f)) \circ \gamma, \quad f \in C^{\infty}(M).$$

Quantum definition

Let $(\mathcal{B}, \mathsf{H}, D)$ be an admissible spectral triple (on a C^*-algebra B).

Definition

We say that a compact quantum group A acts on B by smooth isometries if the action $\alpha : \mathsf{B} \to \mathsf{A} \otimes \mathsf{B}$ satisfies the following condition: for each $\phi \in \mathsf{A}^*$ the map

$$\alpha_\phi := (\phi \otimes \mathrm{id}_{\mathsf{B}}) \circ \alpha$$

maps $\mathcal{B}_0$ into $\mathcal{B}_0$ and commutes with $\mathcal{L}$ on $\mathcal{B}_0$. The category of compact quantum groups acting on B by smooth isometries (with morphisms understood as before) will be denoted $\mathfrak{C}_{\mathrm{iso}}(\mathsf{B})$.

The last lemma and a simple argument based on the dominated convergence theorem imply that the above definition coincides with the standard one in the classical case.

Theorem ([Gos])

Let $(\mathcal{B}, \mathsf{H}, D)$ *be an admissible spectral triple (on a* C^**-algebra* B*). The category* $\mathfrak{C}_{\text{iso}}(\mathsf{B})$ *admits a universal object. It will be denoted* $\text{QISO}(\mathcal{B}, \mathsf{H}, D)$ *and called the quantum isometry group of the triple* $(\mathcal{B}, \mathsf{H}, D)$*.*

Orientation preservation isometries - case study

Consider $\mathbb{T}$ with the standard differentiation operator $i\frac{\partial}{\partial t}$ playing the role of the Dirac operator. The spectral decomposition of D in $L^2(\mathbb{T})$ is given by the formula

$$D = i \sum_{n\in\mathbb{Z}} n z^n;$$

the Laplacian is given on the other hand by

$$\mathcal{L} = -\sum_{n\in\mathbb{Z}} n^2 z^n.$$

It is then clear that the condition on preserving the eigenspaces of the Dirac operator is stronger than this corresponding to the eigenspaces of Laplacian. It turns out that the extra restriction is the requirement that the action preserves the *orientation* of the manifold.

Definition of quantum groups of orientation preserving isometries

For quantum groups of orientation preserving isometries we use the approach based on the unitary corepresentations.
Let $(\mathcal{B}, \mathsf{H}, D)$ be a spectral triple.

Definition ([BG])

We say that a compact quantum group A acts on $(\mathcal{B}, \mathsf{H}, D)$ by orientation preserving isometries if there exists a unitary corepresentation $U \in M(\mathsf{A} \otimes K(\mathsf{H}))$ such that

1. for every $\phi \in \mathsf{A}^*$ the map $U_\phi := (\phi \otimes \mathrm{id}_{K(\mathsf{H})})(U)$ commutes with D;
2. for every $\phi \in \mathsf{A}^*$ the map

$$x \longrightarrow (\phi \otimes \mathrm{id}_{K(\mathsf{H})})(U(1_{\mathsf{A}} \otimes x)U^*)$$

maps $\mathcal{B}$ into $\mathcal{B}''$.

We consider the category $\mathfrak{C}^+_{\mathrm{iso}}(\mathsf{B})$ of actions as above (note that now morphisms intertwine respective unitary corepresentations). Under some more assumptions on the spectral triple the universal object in this category exists and is denoted by $\mathrm{QISO}^+(\mathcal{B}, \mathsf{H}, D)$ (actually we also need to take care of possibly non-faithful actions). Note it need not have an action on B in the sense considered in these lectures.

In 'good' situations we actually have a more satisfactory and transparent approach.

Cyclic and separating vector

Let $(\mathcal{B}, \mathsf{H}, D)$ - a spectral triple of compact type.

Assume that

- D has a one-dimensional kernel spanned by $\xi \in \mathsf{H}$;
- ξ is cyclic and separating for $\mathcal{B}$;
- each eigenvector of D belongs to $\mathcal{B}\xi$;
- $\mathcal{B}_{00} := \mathrm{Lin}\{b \in \mathcal{B} : b\xi \text{ is an eigenvector of } D\}$ is norm dense in $\mathcal{B}$.

Let $\hat{D} : \mathcal{B}_{00} \to \mathcal{B}_{00}$,

$$\hat{D}(b)\xi = D(b\xi), \quad b \in \mathcal{B}_{00}.$$

Let $\tau(b) := \langle \xi, b\xi \rangle$.

Definition

Let $\mathfrak{C}_+(\mathcal{B}, \mathsf{H}, D)$ be the category with objects (A, α) such that A is a compact quantum group acting on B such that the action $\alpha : \mathsf{B} \to \mathsf{A} \otimes \mathsf{B}$ satisfies the following conditions:

1. α is τ preserving;
2. α maps $\mathcal{B}_{00}$ into $\mathsf{A} \odot \mathcal{B}_{00}$;
3. $\alpha \hat{D} = (\mathrm{id}_{\mathsf{A}} \odot \hat{D})\alpha$.

Theorem ([BG])

The category $\mathfrak{C}_+(\mathcal{B}, \mathsf{H}, D)$ *admits a universal object, denoted further* $\mathrm{QISO}^+(\mathcal{B}, \mathsf{H}, D)$.

Idea of the proof

The proof of the existence result above (and actually in all earlier cases) follows the following pattern:

- begin by considering the analogous category using only the C^*-algebraic language (no coproducts) and let A be an object in it, satisfying a version of the faithfulness condition;
- use the fact that $\hat{D}$ has a compact resolvent to consider separately its eigenspaces. Fix such an eigenspace, say V_i, and exploit the fact that the commutation relation implies that $\alpha : V_i \to V_i \odot \mathsf{A}$;
- the τ-preserving condition and a calculation with the (orthonormal) basis in V_i implies that we can construct a morphism from $\mathsf{A}_u(Q_i)$ to A for some invertible matrix $Q_i \in M_{\dim(V_i)}$;
- obtain a surjective morphism from $\star_{i\in\mathbb{N}}\mathsf{A}_u(Q_i)$ to A;
- quotient out all the intersection of all ideals of $\star_{i\in\mathbb{N}}\mathsf{A}_u(Q_i)$ which arise as kernels of maps as above to obtain the C^*-algebra $\mathrm{QISO}^+(\mathcal{B}, \mathsf{H}, D)$;
- deduce the existence of the coproduct on $\mathrm{QISO}^+(\mathcal{B}, \mathsf{H}, D)$ from its universal properties.

Quantum symmetry groups of AF-algebras revisited

Let B be a unital AF C^*-algebra, the norm closure of an increasing sequence (unital embeddings) $(\mathrm{B}_n)_{n\in\mathbb{N}}$ of finite dimensional C^*-algebras. Define

$$\mathrm{B}_0 := \mathbb{C}1_{\mathrm{B}}, \mathcal{B} = \bigcup_{n=1}^{\infty} \mathrm{B}_n.$$

Suppose that B is acting on a Hilbert space H and that $\xi \in \mathrm{H}$ is a separating and cyclic unit vector for B. Let P_n denote the orthogonal projection onto the subspace $\mathrm{H}_n := \mathrm{B}_n\xi$ of H and write

$$Q_0 = P_0 = P_{\mathbb{C}\xi}, Q_n = P_n - P_{n-1}, \quad n \in \mathbb{N}.$$

Christensen and Ivan showed that there exists a (strictly increasing) sequence of real numbers $(\alpha_n)_{n=1}^{\infty}$ such that the selfadjoint operator

$$D = \sum_{n\in\mathbb{N}} \alpha_n Q_n$$

yields a spectral triple $(\mathcal{B}, \mathrm{H}, D)$ (which is *Rieffel regular*).

Theorem ([BGS])

The quantum group of the orientation preserving isometries associated with a Christensen-Ivan spectral triple $(\mathcal{B}, \mathsf{H}, D)$ *on an AF* C^**-algebra* B *coincides with the inductive limit considered in the previous lecture (the faithful state on* B *we work with is the vector state given by* $\xi \in \mathsf{H}$*).*

Spectral triples on group algebras

Γ - finitely generated discrete group with (minimal, symmetric) generating set $S := \{\gamma_1, \cdots \gamma_n\}$

$l : \Gamma \to \mathbb{N}_0$ - word-length function

Define the Dirac operator D on $\ell^2(\Gamma)$ by

$$D(\delta_\gamma) = l(\gamma)\delta_\gamma.$$

$C_r^*(\Gamma)$ - C^*-algebra generated in $B(\ell^2(\Gamma))$ by the left regular representation λ

$\mathbb{C}[\Gamma] := \mathrm{Lin}\{\lambda_\gamma : \gamma \in \Gamma\}$.

The vector $\delta_e \in \ell^2(\Gamma)$ is cyclic and separating for $C_r^*(\Gamma)$. The spectral triple $(\mathbb{C}[\Gamma], \ell^2(\Gamma), D)$ satisfies all the assumptions of the last slides.

The triple $(\mathbb{C}[\Gamma], \ell^2(\Gamma), D)$ carries all the geometric information about Γ (in the sense of Gromov). Ozawa and Rieffel showed for example that if Γ is hyperbolic then the triple above is Rieffel regular.

Translation of the conditions discussed above leads to the following:

Theorem ([BS])

The quantum isometry group (of orientation preserving isometries) $\mathrm{QISO}^+(\mathbb{C}[\Gamma], \ell^2(\Gamma), D)$ *(denoted further by* $\mathrm{QISO}^+(\Gamma, S)$*) exists. It is a compact matrix quantum group with the fundamental unitary* $[q_{t,s}]_{t,s\in S}$*. The elements* $\{q_{t,s} : t, s \in S\}$ *must satisfy the commutation relations implying that the prescription*

$$\alpha(\lambda_\gamma) = \sum_{\gamma' \in S: l(\gamma)=l(\gamma')} q_{\gamma',\gamma} \otimes \lambda_{\gamma'}, \quad \gamma \in \Gamma$$

defines (inductively) a unital **-homomorphism from* $C^*(\Gamma)$ *to* $\mathrm{QISO}^+(\Gamma, S) \otimes C^*(\Gamma)$*.*

The result above gives a concrete recipe for for computing $\mathrm{QISO}^+(\Gamma, S)$ in concrete examples.

Finite groups

Theorem

Let $n \in \mathbb{N} \setminus \{1, 2, 4\}$, $S = \{1, n-1\}$. *Then* $\mathrm{QISO}^+(\mathbb{Z}_n, S) \approx C^*(\mathbb{Z}_n) \oplus C^*(\mathbb{Z}_n)$ *(as a* C^**-algebra). Its action on* $C^*(\mathbb{Z}_n)$ *is given by the formula*

$$\alpha(\lambda_1) = \lambda_1 \otimes A + \lambda_{n-1} \otimes B,$$

where A *and* B *are respectively identified with* $\lambda_1 \oplus 0, 0 \oplus \lambda_1 \in C^*(\mathbb{Z}_n) \oplus C^*(\mathbb{Z}_n)$. *The fundamental corepresentation is given by* $\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}$.

Proposition

$\mathrm{QISO}^+(\mathbb{Z}_4, S)$ is the universal C^*-algebra generated by two normal elements A, B satisfying the following relations:

$$AB + BA = AB^* + BA^* = A^*B + BA^* = 0,\ A^2 + B^2 = (A^*)^2 + (B^*)^2,$$

$$A^2B + B^3 = B^*,\ B^2A + A^3 = A^*, A^4 + B^4 + 2A^2B^2 = 1, AA^* + BB^* = 1.$$

It is not commutative.

Permutation/dihedral group

One can compute $\mathrm{QISO}^+(S_3, S)$ for different generating sets S. In both cases the resulting C^*-algebra is isomorphic to $C^*(S_3) \oplus C^*(S_3)$, but its actions on $C^*(S_3)$ are different (similar thing happens for the infinite dihedral group D_∞). We do not know if

$$\mathrm{QISO}^+(S_3, S) \approx \mathrm{QISO}^+(S_3, S') \text{ as compact quantum groups?}$$

Free abelian groups

Theorem

$\mathrm{QISO}^+(\mathbb{Z}, \{1, -1\})$ *is isomorphic (as a compact quantum group) to the (commutative) compact quantum group* $C(\mathbb{T} \rtimes \mathbb{Z}_2)$*. Its action on* $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ *is given by the standard (isometric) action of the group* $\mathbb{T} \rtimes \mathbb{Z}_2$ *on* $\mathbb{T}$*.*

Note that this is a classical isometry group (not the orientation preserving isometry group) - can you see what is the reason for that?
We do not know if $\mathrm{QISO}^+(\mathbb{Z}^2, \{\pm e_1, \pm e_2\})$ is commutative.

Free groups

Theorem

Consider the free group on two generators with the usual generating set S. $\mathrm{QISO}^+(\mathbb{F}_2, S)$, is the universal C^*-algebra generated by partial isometries A, B, C, D, E, F, G, H such that if $P_A, P_B, \ldots$ denote respectively the range projections of $A, B, \ldots$ and $Q_A, Q_B, \ldots$ denote the initial projections of $A, B, \ldots$ then the matrix

$$\begin{bmatrix} P_A & P_B & P_C & P_D \\ P_E & P_F & P_G & P_H \\ Q_B & Q_A & Q_D & Q_C \\ Q_F & Q_E & Q_H & Q_G \end{bmatrix} \tag{1}$$

is a magic unitary (all entries are orthogonal projections, the sum of each row/column is equal to 1).

Theorem (continued)

The coproduct in $\mathrm{QISO}^+(\mathbb{F}_2, S)$ *is determined by the condition that the (unitary) matrix*

$$U = \begin{bmatrix} A & B & C & D \\ B^* & A^* & D^* & C^* \\ E & F & G & H \\ F^* & E^* & H^* & G^* \end{bmatrix}$$

is a fundamental corepresentation. In particular the restriction of the coproduct of $\mathrm{QISO}^+(\mathbb{F}_2, S)$*to the* C^**-algebra generated by the entries of the matrix in* (1) *coincides with the coproduct on Wang's* $A_s(4)$ *– the universal compact quantum group acting on 4 points.*

In the quantum group language one could say $A_s(4)$ is a quantum group quotient of $\mathrm{QISO}^+(\mathbb{F}_2, S)$ (more generally A_{2n} is a C^*-subalgebra of $\mathrm{QISO}^+(\mathbb{F}_n, S)$). There is a close connection between these quantum groups and quantum hyperoctahedral groups.

The theory of quantum isometry and symmetry groups, even in the compact case, is still very young, so that there are many examples of questions to be looked at:

- find out more about the already computed quantum groups: their corepresentations, distributions of their matrix elements with respect to the Haar states, etc.;
- investigate what properties of quantum isometry groups $\mathrm{QISO}^+(\Gamma, S)$ related to group C^*-algebras do not depend on the choice of the generating set S;
- find sufficient conditions for the $\mathrm{QISO}(\Gamma, S)$ to be commutative or finite;
- answer the question: are all quantum isometry groups of classical manifolds commutative?
- investigate the actions of quantum isometry groups – are they ever ergodic? Are they ergodic without the spaces in question being 'quantum homogenous spaces'?
- ask which compact/finite quantum groups arise as quantum isometry/symmetry groups?

Locally compact situation

Technical aspects of the locally compact case are much more involved. We just state some relevant questions/remarks:

- we do not know a single case of a locally compact quantum group which was constructed as (or at least can be interpreted as) a quantum isometry group of some classical quantum space (natural candidates for infinite quantum permutation groups are not locally compact quantum groups);
- It is not clear if one can consider any 'universal' locally compact quantum groups (the Wang-Van Daele universal compact quantum groups were crucial for the existence results)?
- the theory of 'noncompact' quantum manifolds is much less developed, remaining mainly on a purely algebraic level;
- natural limit constructions usually take us out of the locally compact category.

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