The homomorphism problem: Fourier and L^1-group algebras

Lecture 1: Fourier algebras

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The Fourier-Stieltjes and Fourier algebras

G – loc. comp. grp.

$\mathrm{B}(G) = \{\langle \pi(\cdot)\xi|\eta\rangle : \pi : G \to \mathcal{U}(\mathcal{H}_\pi) \text{ w*-cts rep'n}\}$

$u \in \mathrm{B}(G) \Leftrightarrow u \in \mathcal{CB}(G)$ with

$$\|u\|_{\mathrm{B}} = \sup\left\{\left|\int_G uf\right| : f \in \mathrm{L}^1(G), \sup_\pi \left\|\int_G f\pi\right\|_{\mathcal{B}(\mathcal{H}_\pi)} \leq 1\right\}$$

$< \infty$, Banach algebra [Eymard]

$\mathrm{A}(G) = \{\langle \lambda(\cdot)f|g\rangle : f, g \in \mathrm{L}^2(G), \lambda \text{ left reg. rep'n}\}$
– closed ideal in $\mathrm{B}(G)$, spectrum $\Phi_{\mathrm{A}(G)} \cong G$

$\mathrm{A}(G)^* \cong \mathrm{VN}(G) = \lambda(G)''$
$\mathrm{B}(G)^* \cong \mathrm{W}^*(G) = \varpi(G)''$, $\mathrm{B}(G) \cong \mathrm{C}^*(G)^*$

H – another loc. comp. grp.

Ques. Struc. homo's $\varphi : \mathrm{A}(G) \to \mathrm{B}(H)$?
– Will give partial answer.

Affine maps

$C \subset H$ *coset* if

$$r, s, t \in C \quad \Rightarrow \quad rs^{-1}t \in C$$

Prop. C coset $\Leftrightarrow C^{-1}C, CC^{-1}$ groups;
in which case $C = sC^{-1}C = CC^{-1}s$, $\forall\, s \in C$.

$\alpha : C \subset H \to G$, for C a coset, $r, s, t \in C$, is

- *affine*: $\alpha(rs^{-1}t) = \alpha(r)\alpha(s)^{-1}\alpha(t)$
  $\Leftrightarrow \alpha_0 : s_0^{-1}C \to G$, fixed $s_0 \in C$
  $\alpha_0(s_0^{-1}t) = \alpha(s_0)^{-1}\alpha(t)$ homo.
- *anti-affine*: $\alpha(rs^{-1}t) = \alpha(t)\alpha(s)^{-1}\alpha(r)$
  $\Leftrightarrow \alpha_0 : s_0^{-1}C \to G$, fixed $s_0 \in C$
  $\alpha_0(s_0^{-1}t) = \alpha(s_0)^{-1}\alpha(t)$ anti-homo.

Prop. $C \subset H$ open coset
$\alpha : C \subset H \to G$ cts. affine (anti-affine)
$\Rightarrow \varphi_\alpha : \mathsf{B}(G) \to \mathsf{B}(H)$

$$\varphi_\alpha u(s) = u \circ \alpha(s) 1_C(s),\ s \in H$$

is a bdd. homo. Also, α affine $\Rightarrow \varphi_\alpha$ c.b.

Pf. Suppose, C subgroup, so α (anti-)homo.

$1_C(t) = \langle \pi_C(t)\delta_C | \delta_C \rangle$
$\pi_C : G \to \mathcal{U}(\ell^2(G/C))$ – left quasi-reg. rep'n

$u = \langle \pi(\cdot)\xi | \eta \rangle,\ \xi, \eta \in \mathcal{H}_\pi,\ t \in H$

$\varphi_\alpha u(t) = \langle \pi \circ \alpha(t) \xi | \eta \rangle 1_C(t) \Big(= \Big\langle \pi \circ \check{\alpha}(t) \bar{\xi} | \bar{\eta} \Big\rangle 1_C(t) \Big)$

$\pi \circ \alpha\ (\pi \circ \check{\alpha}) : C \to \mathcal{U}(\mathsf{L}^2(G))$ cts. rep'n.
$v \mapsto v|_C : \mathsf{B}(H) \twoheadrightarrow \mathsf{B}(C)$ dualises
$\mathsf{L}^1(C) \hookrightarrow \mathsf{L}^1(H) \rightsquigarrow \mathsf{C}^*(C) \hookrightarrow \mathsf{C}^*(H)$

$\varphi_\alpha^* : \mathsf{W}^*(H) \to \pi(G)''$, $\varphi_\alpha^*(a) = (\pi \circ \alpha)''(M_{1_C} a)$ is c.b. if α homo., since M_{1_C} expectation

If C coset, fix s_0 in C. For $\varphi_\alpha u(t)$ we get

$$\Big\langle \pi \circ \alpha_0(s_0^{-1} t) \xi | \pi \circ \alpha(s_0) \eta \Big\rangle 1_C(t) = s_0 * [\varphi_{\alpha_0} u](t)$$

If α affine dual is c.b. too. □

Mixed piecewise affine maps

$\Omega(H)$ -coset ring, $\Omega_o(H)$ – open coset ring

$\alpha : Y \subset H \to G$ is *(m.)p.a.* if

(i) $Y = \bigcup_{i=1}^{n} Y_i,\ Y_i \in \Omega(H)$

(ii) $\forall i\ \exists$ coset $C_i \supset Y_i$ and affine or anti-affine $\alpha_i : C_i \to G$ s.t. $\alpha_i|_{Y_i} = \alpha|_{Y_i}$.

If each α_i affine, α *p.a.*

Prop. $\alpha : Y \subset H \to G$ cts. m.p.a., $Y_i \in \Omega_o(H)$ $\Rightarrow \varphi_\alpha : \mathrm{B}(G) \to \mathrm{B}(H)$, $\varphi_\alpha u(s) = u\circ\alpha(s)1_Y(s)$ bdd. homo.; c.b. if α p.a.

Pf. Factor φ_α

$$\mathrm{B}(G) \to \ell^1(n)\widehat{\otimes}\mathrm{B}(H),\ u \mapsto \sum_{i=1}^{n} \delta_i \otimes \varphi_{\alpha_i} u$$

$$\to \mathrm{B}(H), \qquad x \mapsto \sum_{i=1}^{n} (\chi_i \otimes m_{Y_i})x$$

$m_{Y_i}u = 1_{Y_i}u$. Note: $\left\|1_{Y_i}\right\|_{\mathrm{B}} = 1 \Leftrightarrow Y_i$ coset. $\square$

The role of graphs

$\alpha : Y \subset H \to G$, $\Gamma_\alpha = \{(s, \alpha(s)) : s \in Y\} \subset H \times G$

Lem. (i) α homo. $\Leftrightarrow$ Γ_α subgrp.
(ii) α affine $\Leftrightarrow$ Γ_α coset.
(iii) α p.a. $\Leftrightarrow$ $\Gamma_\alpha \in \Omega(H \times G)$

Pf. (ii) Γ_α coset $\Rightarrow \forall r, s, t \in Y$

$$(rs^{-1}t, \alpha(r)\alpha(s)^{-1}\alpha(t)) = (r, \alpha(r))(s, \alpha(s))^{-1}(t, \alpha(t)) \in \Gamma_\alpha.$$

Γ_α graph $\Rightarrow \alpha(r)\alpha(s)^{-1}\alpha(t) = \alpha(rs^{-1}t)$.

(i) Γ_α is a coset containing e.

(iii) Fussier. $\square$

$\mathrm{PA}_c(H, G) = \{\alpha : Y \subset H \to G | \text{cts.}, Y_i \in \Omega_o(H)\}$

Ques. Reasonable characterisation Γ_α, α m.p.a.?

Thm. ([Cohen] G, H abel., [Host] G alm. abel.)
[Ilie-S] G amen. $\Rightarrow$

$$\mathrm{Hom}_{cb}(\mathrm{A}(G),\mathrm{B}(H)) \leftrightsquigarrow \mathrm{PA}_c(H,G)$$
$$\varphi \mapsto \lambda_G^{-1}\circ\varphi^*\circ\varpi_H \text{ on } Y = \mathrm{supp}\varphi(\mathrm{A}(G)))$$
$$\varphi_\alpha \leftarrow\!\shortmid \alpha$$

Pf. ($\mapsto$)
$\Phi_{\mathrm{A}(G)} = \lambda_G(G) \Rightarrow \varphi^*\circ\varpi_H(H) \subset \lambda_G(G)$, so $\alpha\ \exists$.

[Effros-Ruan] $\mathrm{A}(G \times G) \cong \mathrm{A}(G)\hat{\otimes}\mathrm{A}(G)$
(op. proj. tens. prod.)
[Losert] $\mathrm{A}(G \times G) \neq \mathrm{A}(G) \otimes^\gamma \mathrm{A}(G)$, G not a.a.

[Ruan] G amen. $\Rightarrow$ $\mathrm{A}(G \times G)$ has b.a.d. (w_i)
Arrange $w_i \in \mathrm{P}(G\times G)$, $\lim_i w_i(s,t) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}$

φ c.b. $\Rightarrow \varphi \otimes \mathrm{id} : \mathrm{A}(G)\hat{\otimes}\mathrm{A}(G) \to \mathrm{B}(H)\hat{\otimes}\mathrm{A}(G)$

$$\tilde{w}_i := \varphi \otimes \mathrm{id}(w_i) \in \mathrm{B}(H)\hat{\otimes}\mathrm{A}(G) \hookrightarrow \mathrm{B}(H \times G)$$
$$\hookrightarrow \mathrm{B}(H_d \times G_d)$$

For $(s,t) \in H \times G$

$$\tilde{w}_i(s,t) = w_i(\alpha(s),t) \xrightarrow{i} \begin{cases} 1 & t = \alpha(s) \\ 0 & t \neq \alpha(s) \end{cases}$$
$$= 1_{\Gamma_\alpha}(s,t)$$

Bdd. nets in $\mathrm{B}(H_d \times G_d)$: w*-conv. = ptwise.
$\Rightarrow 1_{\Gamma_\alpha} \in \mathrm{B}(H_d \times G_d)$
$\Rightarrow$ [Host] $\Gamma_\alpha \in \Omega(H \times G)$
$\Rightarrow$ (Lem. above) α p.a.
More effort $\Rightarrow$ arrange $Y_i \in \Omega_0(H)$ & α cts. $\square$

Notes: (i) φ c.pos. $\Leftrightarrow$ α homo.
(ii) φ c.cont've $\Leftrightarrow$ α affine

Cor. G amen., H connect.
$\Rightarrow$ all φ in $\mathrm{Hom}_{cb}(\mathrm{A}(G), \mathrm{B}(H))$ c.c.

Cor. G amen.

$\mathrm{Hom}_{cb}(\mathrm{A}(G), \mathrm{A}(H)) \leftrightsquigarrow \{\alpha \in \mathrm{PA}_c(H,G) : \text{ proper}\}$

Prop. [Forrest-Runde] $\iota(s) = s^{-1}$
φ_ι c.b. $\Leftrightarrow$ G is virt. abel. $\Leftrightarrow$ ι p.a.

Thm. [Ilie-Stokke] G amen.

$\mathsf{Hom}_{cb}^{w^*}(\mathsf{B}(G), \mathsf{B}(H)) \leftrightsquigarrow \{\alpha \in \mathsf{PA}_c(H, G) : \text{ open}\}$

Ex. (i) translations

(ii) homeo. between open subgrps.

$\mathsf{B}(G) \xrightarrow{\text{rest.}} \mathsf{B}(G_0)$

(iii) $\mathsf{B}(G/N) \hookrightarrow \mathsf{B}(G)$

(iv) $\alpha : H \to G$ p.a. homeo.

$\Rightarrow \varphi_\alpha : \mathsf{B}(G) \to \mathsf{B}(H)$ w*-cts. c. isomor.

Thm. [Pham]

G amen., $\varphi : \mathsf{B}(G) \to \mathsf{B}(H)$ c. isomor.

$\Rightarrow \varphi = \varphi_\alpha$, $\alpha : H \to G$ p.a. homeo., & H amen.

Spine of $\mathsf{B}(H)$

(η_{ap}, H^{ap}) – almost periodic comp'n, τ_H - topol.

$$\mathcal{T}_{ap}(H) = \left\{ \tau \subset \tau_H : \quad \begin{array}{c} \exists \text{ l.c.grp. } G, \text{ cts. homo.} \\ \eta : H \to G \text{ s.t. } \tau = \eta^{-1}(\tau_G) \\ \& \ \tau_{ap} \subset \tau \end{array} \right\}$$

$\tau_1 \vee \tau_2 = \delta^{-1}(\tau_1 \times \tau_2),\ \delta(s) = (\eta_1(s), \eta_2(s))$
$\mathcal{T}_{ap}(H)$ semi-lattice, unit τ_{ap}, ideal τ_G

Thm. Let $\mathsf{A}_\tau(H) = \mathsf{A}(G) \circ \eta_\tau$ if $\tau = \eta^{-1}(\tau_G)$
(i) $\mathsf{A}_{\tau_1}(H) \cap \mathsf{A}_{\tau_2}(H) = \{0\}$ if $\tau_1 \neq \tau_2$ in $\mathcal{T}_{ap}(H)$
(ii) $\mathsf{A}_{\tau_1}(H)\mathsf{A}_{\tau_2}(H) \subset \mathsf{A}_{\tau_1 \vee \tau_2}(H)$
(iii) $\mathsf{A}^*(H) = \ell^1\text{-} \bigoplus_{\tau \in \mathcal{T}_{ap}(H)} \mathsf{A}_\tau(H)$
$\mathcal{T}_{ap}(H)$-graded subalg. of $\mathsf{B}(H)$

Thm. (i) $\mathrm{Idem}(\mathsf{B}(H)) = \{u : u^2 = u\} \subset \mathsf{A}^*(H)$
(ii) $\alpha \in \mathsf{MPA}_c(H, G) \Rightarrow \varphi_\alpha(\mathsf{A}(G)) \subset \mathsf{A}^*(H)$

Note. $(\varepsilon_{\mathsf{A}^*}, \Phi_{\mathsf{A}^*(H)})$ semi-top'l comp'n of H
$(\varepsilon_{\mathsf{A}^*}, \Phi_{\mathsf{A}^*(H)}) \leq (\varepsilon_e, G^e)$ - sub. to Eberlein comp'n

Conj. $\mathsf{A}^*(H)$ largest regular subalg. in $\mathsf{B}(H)$

When G not amenable

Thm. [Leinert, Bozejko-Fendler]
G discrete, $E \subset G$ inf. free set
$\Rightarrow 1_E \in \mathsf{M}_{cb}\mathsf{A}(G)$.

Consequence. $u \mapsto 1_E u : \mathsf{A}(G) \to \mathsf{A}(G)$ c.b.
$m_{1_E} = \varphi_\alpha$, $\alpha : E \hookrightarrow G$
$E \notin \Omega(G)$ since $1_E \notin \mathsf{B}(G)$
$\Rightarrow \alpha$ not m.p.a.

m_{1_E} does extend to $\mathrm{Hom}(\mathsf{B}(G), \mathsf{B}(G))$

A discretisation procedure

Lem. [Pham] Let $\varphi \in \mathrm{Hom}(\mathsf{A}(G), \mathsf{B}(H))$
$Y = \mathrm{supp}\varphi(\mathsf{A}(G))$, $\alpha = \lambda_G^{-1} \circ \varphi^* \circ \varpi_H$ so $\varphi = \varphi_\alpha$.
Then $\varphi_\alpha(\mathsf{A}(G_d)) \subset \mathsf{B}(H_d)$.
φ pos. (i.e. pres'ves pos. def.) $\Rightarrow \varphi_\alpha|_{\mathsf{A}(G_d)}$ pos.

Pf. $\mathsf{A}_c(G_d)$ dense in $\mathsf{A}(G_d)$.
Typ. elem. of $\mathsf{A}_c(G)_{\|\cdot\|_{\mathsf{B}} \leq 1}$:
$u = \left\langle \lambda_{G_d}(\cdot) \sum_{i=1}^n \alpha_i \delta_{s_i} \middle| \sum_{i=1}^n \beta_i \delta_{t_i} \right\rangle$
wh. $\sum_{i=1}^n |\alpha_i|^2 = \sum_{i=1}^n |\beta_i|^2 \leq 1$, each $s_i, t_i \in G$

Let $(\gamma_k)_{k=1}^m \subset \mathbb{C}$, $e_H \in (x_k)_{k=1}^m \subset H$ sat'y

$$\left\| \sum_{k=1}^m \gamma_k \varpi_H(x_k) \right\|_{\mathsf{C}^*(H_d)} \leq 1. \qquad (*)$$

Dual pairing:

$$\left| \sum_{k=1}^m \gamma_k \varphi(u)(x_k) \right| = \left| \sum_{x_k \in Y} \sum_{i,j=1}^n \gamma_k \alpha_i \bar{\beta}_j \delta_{s_i t_j^{-1}}(\alpha(x_k)) \right|$$

Let V nbhd. of e be so

$$VV^{-1} \cap \{s_i^{-1}\alpha(x_k)t_j\} = \{e\}.$$

Let

$$v = \frac{1}{m(V)} \left\langle \lambda_G(\cdot) \sum_{i=1}^{n} \alpha_i 1_{s_i V} \middle| \sum_{i=1}^{n} \beta_i 1_{t_i V} \right\rangle \in \mathsf{A}(G)$$

so $\|v\|_{\mathsf{B}} \leq 1$.

$$\|\varphi\| \geq \|\varphi(v)\|_{\mathsf{B}} \geq \left| \sum_{\substack{k=1 \\ x_k \in Y}}^{m} \gamma_k u \circ \alpha(x_k) \right|$$

$$= \left| \sum_{x_k \in Y} \sum_{i,j=1}^{n} \gamma_k \alpha_i \bar{\beta}_j \delta_{s_i t_j^{-1}}(\alpha(x_k)) \right|$$

Take sup over $(*)$, on right; then $\sup_{u \in \mathsf{A}_c(G_d), \|u\|_{\mathsf{B}} \leq 1}$
$\Rightarrow \|\varphi\| \geq \left\| \varphi\alpha|_{\mathsf{A}(G_d)} \right\|$.

Positivity is checked similarly. $\square$

Thm. [Pham]
$\varphi \in \mathrm{Hom}(\mathrm{A}(G), \mathrm{B}(H))$ pos., $\varphi = \varphi_\alpha$
$\Rightarrow Y$ open subgp. & α homo. or anti-homo.

Sketch. φ homo. $\Rightarrow \varphi(\varpi_H(H)) \subset \lambda_G(G)$
φ pos. $\Rightarrow \varphi^*$ pos. $\Rightarrow \varphi^*(\varpi_H(e_H)) = \lambda_G(e_G)$
Also $\varphi(u)(s^{-1}) = \overline{\varphi(u)(s)} = \overline{u \circ \alpha(s)} = u(\alpha(s)^{-1})$
$\Rightarrow Y^{-1} = Y$ & $\alpha(s^{-1}) = \alpha(s)^{-1}$.

Claim. $s, t \in Y$, $\{\alpha(st), \alpha(ts)\} = \{\alpha(s)\alpha(t), \alpha(t)\alpha(s)\}$.
For $\alpha, \beta \in \mathbb{C}$ let

$$a_{\alpha,\beta} = \alpha\varpi_H(s) + \beta\varpi_H(t) + \bar{\alpha}\varpi_H(s^{-1}) + \bar{\beta}\varpi_H(t^{-1}).$$

Kadison's ineq.: $\varphi^*(a_{\alpha,\beta})^2 \geq \varphi^*(a_{\alpha,\beta}^2)$. (‡)
Trick: $\mathrm{Re}[\alpha^2 a + \beta^2 b + \alpha\bar{\beta}c + \bar{\alpha}\beta d] \geq 0 \; \forall \alpha, \beta$ in $\mathbb{C}$
$\Rightarrow a, b, c, d = 0$
Expand out $\varphi^*(a_{\alpha,\beta})^2 - \varphi^*(a_{\alpha,\beta}^2) \geq 0$ in $\mathrm{VN}(G)$.

Conseq. $\alpha^{-1}(e_G)$ cl. norm. subgrp. in Y

$H_0 := Y/\alpha^{-1}(e_G) \rightsquigarrow \alpha_0 : H_0 \to G$, $G_0 := \alpha(Y)$
$\rho := \varphi_{\alpha_0}|_{\mathsf{A}(G_{0,d})} : \mathsf{A}(G_{0,d}) \to \mathsf{B}(H_{0,d})$,
α bijec. $\Rightarrow \rho(\mathsf{A}(G_{0,d})) \subset \mathsf{A}(H_{0,d})$

For a, b in span$\lambda_{H_{0,d}}(H_{0,d})$ compute that

$$\rho^*(ab) + \rho^*(ba) = \rho^*(a)\rho^*(b) + \rho^*(b)\rho^*(a)$$

– Jordan $*$-homo., extend to VN$(H_{0,d})$
[Kadison] ρ^* isomet'c & onto, hence ρ isomet'c
[Walter] $\Rightarrow \alpha_0$ isom. or anti-isom.
$\Rightarrow \alpha$ homo. or anti-homo. $\square$

(‡) Kadison's inequality:
ψ pos. on C*-alg. $\mathcal{A}$, $aa^* = a^*a$ in $\mathcal{A}$
$\tilde{\psi} = \psi|_{\overline{\mathsf{alg}(a,a^*)}}$ is c.p. as $\overline{\mathsf{alg}(a, a^*)}$ abelian.
Thus $\tilde{\psi}$ is 2-pos. and Kadison ineq.

$$\psi(a)^*\psi(a) = \tilde{\psi}(a)^*\tilde{\psi}(a) \geq \tilde{\psi}(a^*a) = \psi(a^*a)$$

is easy.

Thm. [Pham] $\varphi \in \mathrm{Hom}(\mathsf{A}(G), \mathsf{B}(H))$ cont've
$\Rightarrow \varphi = \varphi_\alpha$, α affine or anti-affine

Pf. $Y = \mathrm{supp}\varphi(\mathsf{A}(G))$, $\alpha = \lambda_G^{-1} \circ \varphi^* \circ \varpi_H$

Fix s_0 in Y, $\alpha_0 : s_0^{-1}Y \to G$, $\alpha_0(s_0^{-1}t) := \alpha(s_0)^{-1}\alpha(t)$
$\alpha_0(e_H) = e_G$ & $\varphi_{\alpha_0} \in \mathrm{Hom}(\mathsf{A}(G), \mathsf{B}(G))$ cont.
If $u \in \mathsf{A}(G)$ pos.

$$\|u\|_{\mathsf{B}} \geq \|\varphi_{\alpha_0}(u)\|_{\mathsf{B}} \geq u \circ \alpha_0(e_H) = u(e_G) = \|u\|_{\mathsf{B}}$$

$\Rightarrow \varphi_{\alpha_0}(u) \Rightarrow \varphi$ pos. $\square$

Cor. $\varphi \in \mathrm{Hom}(\mathsf{A}(G), \mathsf{A}(H))$
$\varphi(\mathsf{A}(G))$ sep. points $\Rightarrow \varphi$ onto

Cor. $\varphi : \mathsf{A}(G) \to \mathsf{A}(H)$ cont. isom. $\Rightarrow \varphi = \varphi_\alpha$,
α affine of anti-affine homeo.

Gen. form of cont've homo.:

$$\mathsf{A}(G) \xrightarrow{\text{trans.}} \mathsf{A}(G) \xrightarrow[\text{closed}]{\text{rest.}} \mathsf{A}(G_0) \xrightarrow{\cong} \mathsf{A}(H_0/K)$$

$$\hookrightarrow \mathsf{A}(H_0) \underset{\text{open}}{\hookrightarrow} \mathsf{A}(H) \xrightarrow{\text{trans.}} \mathsf{A}(H)$$

Questions

(i) General form of $\varphi \in \mathsf{Hom}(\mathsf{A}(G), \mathsf{B}(H))$?

(i') When G amenable?

(ii) General form of $\varphi \in \mathsf{Hom}_{cb}(\mathsf{A}(G), \mathsf{B}(G))$, when G not amenable?

(ii') When G a non-abelian free group?

(ii_0) What are $\mathsf{Idem}(\mathsf{M}_{cb}\mathsf{A}(G))$?

[Forrest-Runde] Contractive $u \in \mathsf{Idem}(\mathsf{M}_{cb}\mathsf{A}(G))$ is $u = 1_C$, where C is a coset.

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