On higher-dimensional amenability of Banach algebras

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Leeds, June 2010

Amenability

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(Here $(a \cdot f)(x) = f(x \cdot a)$ and $(f \cdot a)(x) = f(a \cdot x)$ for $a \in \mathcal{A}$, $f \in X^*$, $x \in X$.)

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$\iff$ there exists a **virtual diagonal** $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that for all $a \in \mathcal{A}$,

$$aM = Ma, \;\; \pi^{**}(M)a = a.$$

Here π is the product map on $\mathcal{A}$.

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N. Gronbaek, B.E. Johnson and G.A. Willis (1994) proved that, for a Banach $\mathcal{L}_\infty$-space X, $\mathcal{K}(X)$ is amenable, hence $\mathcal{B}(X) = \mathcal{K}(X) \oplus \mathbf{C}$ is amenable too.

Open Problem. Describe infinite-dimensional Banach spaces E such that the Banach algebra $\mathcal{B}(E)$ is not amenable/ is amenable.

Note: it is known that $\mathcal{B}(l_p)$ is not amenable for $1 \leq p \leq \infty$ (S. Wassermann, C.J. Read, G. Pisier, N. Ozawa, V. Runde).

The Hochschild (co)homology groups of $\mathcal{A}$

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. The **continuous homology** $\mathcal{H}_n(\mathcal{A}, X)$ **of** A **with coefficients in** X is defined to be the nth homology

$$\mathcal{H}_n(\mathcal{C}_\sim(\mathcal{A}, X)) = \mathrm{Ker}\ b_{n-1}/\operatorname{Im} b_n$$

of the standard homological chain complex $(\mathcal{C}_\sim(\mathcal{A}, X))$:

$$0 \longleftarrow X \xleftarrow{b_0} X\hat{\otimes}\mathcal{A} \longleftarrow \ldots \longleftarrow X\hat{\otimes}\mathcal{A}^{\hat{\otimes}n} \xleftarrow{b_n} X\hat{\otimes}\mathcal{A}^{\hat{\otimes}(n+1)} \longleftarrow \ldots,$$

where the differentials b_* are given by

$$b_n(x \otimes a_1 \otimes \cdots \otimes a_{n+1}) = (x \cdot a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) +$$

$$\sum_{i=1}^{n}(-1)^i(x \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1}(a_{n+1} \cdot x \otimes a_1 \otimes \cdots \otimes a_n).$$

The Hochschild cohomology groups of $\mathcal{A}$ with coefficients in the dual $\mathcal{A}$-bimodule X^*

$$\mathcal{H}^n(\mathcal{A}, X^*) \cong H^n((\mathcal{C}_\sim(\mathcal{A}, X))^*)$$

the cohomology groups of the dual complex $(\mathcal{C}_\sim(\mathcal{A}, X))^*$.

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Virtual diagonals and higher-dimensional amenability of Banach algebras were investigated by E.G. Effros and A. Kishimoto (1987) for unital algebras and by A.L.T. Paterson and R.R. Smith (1996, 1997) in the non-unital case.

The weak bidimension $db_w\mathcal{A}$ of a Banach algebra $\mathcal{A}$

Yu.V. Selivanov (1995): The **weak bidimension** of $\mathcal{A}$ is

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$\mathcal{A}$ is n-amenable but not $(n-1)$-amenable $\iff$ $db_w\mathcal{A} = n - 1$.

$\mathcal{A}$ is amenable $\iff$ $\mathcal{A}$ is 1-amenable $\iff$ $db_w\mathcal{A} = 0$.

Examples of n-amenable Banach algebras

The upper triangular 2×2-complex matrices T_2 and $\mathcal{K}(l_2 \hat{\otimes} l_2)$ are 2-amenable but not (1-)amenable, that is,

$db_w T_2 = 1$ and $db_w \mathcal{K}(l_2 \hat{\otimes} l_2) = 1$.

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l_1 and $\mathcal{N}(H)$ are 3-amenable but not 2-amenable, that is,

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l_2 and $\mathcal{HS}(H)$ are not n-amenable for all n, that is,

$db_w l_2 = \infty$ and $db_w \mathcal{HS}(H) = \infty$.

(A.L.T. Paterson, Yu.V. Selivanov, A. Ya. Helemskii)

Examples of n-amenable Banach algebras

In 1997 A.L.T. Paterson and R.R. Smith proved that the Banach algebra

$B_n = S^{n-1}(A_4)$ is $(n+1)$-amenable but not n-amenable.

Here A_4 is the Banach subalgebra of $M_4(\mathbb{C})$ of elements of the form

$$\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ * & * & * & 0 \\ * & * & 0 & * \end{bmatrix}$$

The two-point suspension $S(A_4)$ of the algebra A_4 is the subalgebra of $B(\mathbb{C}^2 \oplus \mathbb{C}^4)$ whose elements are of the form

$$\begin{bmatrix} d & 0 \\ u & a \end{bmatrix}$$

$d \in \mathcal{D}_2$ the diagonal 2×2 complex matrices, $a \in A_4$, $u \in \mathcal{B}(\mathbb{C}^2, \mathbb{C}^4)$.

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The following is known:

H. Leptin: $A(G)$ has b.a.i $\iff$ G is amenable.

B.E. Forrest and V. Runde (2005): $A(G)$ is amenable $\iff$ G admits an abelian subgroup of finite index.

V. Runde (2009): If $A(G)$ is biflat then either (a) G admits an abelian subgroup of finite index, or (b) G is non-amenable and does not contain a discrete copy of the free group of two generators.

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Open Problem. Find $db_w\mathcal{B}(H)$.

Amenability and flatness

A.Ya. Helemskii (1984): $\mathcal{A}$ is amenable

$\Longleftrightarrow$ $\mathcal{A}_+$ is biflat

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A module $Y \in \mathcal{A}\text{-mod-}\mathcal{A}$ is called *flat* if for any admissible complex $\mathcal{X}$ of Banach $\mathcal{A}$-bimodules

$$0 \longleftarrow X_0 \xleftarrow{\phi_0} X_1 \xleftarrow{\phi_1} X_2 \xleftarrow{\phi_2} X_3 \longleftarrow \cdots$$

the complex $\mathcal{X}\widehat{\otimes}_{\mathcal{A}-\mathcal{A}}Y$:

$$0 \longleftarrow X_0\widehat{\otimes}_{\mathcal{A}-\mathcal{A}}Y \xleftarrow{\phi_0\widehat{\otimes}_{\mathcal{A}-\mathcal{A}}id_Y} X_1\widehat{\otimes}_{\mathcal{A}-\mathcal{A}}Y \xleftarrow{\phi_1\widehat{\otimes}_{\mathcal{A}-\mathcal{A}}id_Y} X_2\widehat{\otimes}_{\mathcal{A}-\mathcal{A}}Y \longleftarrow \cdots$$

is exact.

Here $\widehat{\otimes}_{\mathcal{A}-\mathcal{A}}$ is the projective tensor product of Banach $\mathcal{A}$-bimodules.

n-amenability and flat resolutions of $\mathcal{A}_+$

Theorem 1. [Yu.V. Selivanov (1995)] *Let $\mathcal{A}$ be a Banach algebra. For each integer $n \geq 0$ the following properties of $\mathcal{A}$ are equivalent:*

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(ii) $\mathcal{A}$ *is* $(n+1)$*-amenable, that is,* $\mathcal{H}^{n+1}(\mathcal{A}, X^*) = \{0\}$ for all $X \in \mathcal{A}\text{-mod-}\mathcal{A}$;

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(iii) *the* $\mathcal{A}$*-bimodule* $\mathcal{A}_+$ *has a flat admissible resolution of length* $\mathbf{n}$ *in the category of* $\mathcal{A}$-mod-$\mathcal{A}$*;*

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(iii) *the* $\mathcal{A}$*-bimodule* $\mathcal{A}_+$ *has a flat admissible resolution of length* $\mathbf{n}$ *in the category of* $\mathcal{A}$-mod-$\mathcal{A}$;

(iv) *if* $0 \longleftarrow \mathcal{A}_+ \stackrel{\varepsilon}{\longleftarrow} P_0 \stackrel{\phi_0}{\longleftarrow} P_1 \stackrel{\phi_1}{\longleftarrow} \cdots P_{n-1} \stackrel{\phi_{n-1}}{\longleftarrow} Y \longleftarrow 0 \qquad (0 \leftarrow \mathcal{A} \leftarrow \mathcal{P})$

is an admissible resolution of $\mathcal{A}_+$ *in which all the modules* P_i *are flat in* $\mathcal{A}$-mod-$\mathcal{A}$*, then* Y *is also flat in* $\mathcal{A}$-mod-$\mathcal{A}$.

It is well known that $db_w\mathcal{A} = db_w\mathcal{A}_+$.

Ideals with b.a.i and essential modules

Proposition 1. [Z.A. Lykova and M.C. White (1998)] *Let $\mathcal{A}$ be a Banach algebra and let I be a closed two-sided ideal of $\mathcal{A}$. Suppose that I has a b.a.i. Then,*

(i) *for any Banach I-bimodule Z,*

$$\mathcal{H}_n(I, Z) = \mathcal{H}_n(\mathcal{A}, \overline{IZI}) \text{ and } \mathcal{H}^n(I, Z^*) = \mathcal{H}^n(\mathcal{A}, (\overline{IZI})^*) \text{ for all } n \geq 1.$$

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(ii) *for any Banach* A/I*-bimodule* Y*,*

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Remark 1. *Proposition 1 shows that in the case of Banach algebras $\mathcal{A}$ with b.a.i. we can restrict ourselves to the category of $essential\ Banach\ modules$ in questions on db_w and $\mathcal{H}^n(\mathcal{A}, X^*)$.*

$db_w\mathcal{A} \geq \max\{db_w I, db_w \mathcal{A}/I\}$ for a closed two-sided ideal I with a b.a.i.

Proposition 2. [Z.A. Lykova and M.C. White (1998)] *Let $\mathcal{A}$ be a Banach algebra and let I be a closed two-sided ideal of $\mathcal{A}$. Suppose that I has a b.a.i.. Then*

(i) *the n-amenability of A implies the n-amenability of the two Banach algebras A/I and I;*

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(i) *the* n*-amenability of* A *implies the* n*-amenability of the two Banach algebras* A/I *and* I*;*

(ii) $db_w I \leq db_w A$ *and* $db_w A/I \leq db_w A$*.*

$db_w \mathcal{A} \widehat{\otimes} \mathcal{B}$

B.E. Johnson (1972):
If Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are amenable then their tensor product $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is amenable too.

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Suppose $db_w \mathcal{A} = m$ and $db_w \mathcal{B} = q$. Question: *What can we say about the higher-dimensional amenability of* $\mathcal{A} \widehat{\otimes} \mathcal{B}$*?*

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In 1996 Yu. Selivanov remarked without proof that, for $\mathcal{A}$ and $\mathcal{B}$ with bounded approximate identities,

$$db_w \mathcal{A} \widehat{\otimes} \mathcal{B} = db_w \mathcal{A} + db_w \mathcal{B}.$$

In 2002 he gave a proof of the formula in the particular case of algebras with identities and his proof depends heavily on the existence of identities.

In this talk

– we show that the formula

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is correct for algebras $\mathcal{A}$ and $\mathcal{B}$ b.a.i..

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– We show further that the formula does **not** hold for algebras with no b.a.i, nor for algebras with only 1-sided b.a.i.

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– We show further that the formula does **not** hold for algebras with no b.a.i, nor for algebras with only 1-sided b.a.i.

– The well-known trick adjoining of an identity to the algebra does not work for the tensor product of algebras.

The homological properties of the tensor product algebras $\mathcal{A}\widehat{\otimes}\mathcal{B}$ and $\mathcal{A}_+\widehat{\otimes}\mathcal{B}_+$ are different.

Pseudo-resolutions in categories of Banach modules $\mathcal{K}$

Definition 1. *For $X \in \mathcal{K}$, a complex*

$$0 \longleftarrow X \xleftarrow{\varepsilon} Q_0 \xleftarrow{\phi_0} Q_1 \xleftarrow{\phi_1} Q_2 \longleftarrow \dots$$

is called a $pseudo\text{-}resolution\ of\ X\ in\ \mathcal{K}$ if it is weakly admissible,

and a $flat\ pseudo\text{-}resolution\ of\ X\ in\ \mathcal{K}$ if, in addition, all the modules in $\mathcal{Q}$ are flat in $\mathcal{K}$.

Flat pseudo-resolution of $\mathcal{A}$ with b.a.i. in $\mathcal{A}$-mod-$\mathcal{A}$

We put $\beta_n(\mathcal{A}) = \mathcal{A}^{\widehat{\otimes}^{n+2}}$, $n \geq 0$, and let $d_n : \beta_{n+1}(\mathcal{A}) \to \beta_n(\mathcal{A})$ be given by

$$d_n(a_0 \otimes \ldots \otimes a_{n+2}) =$$

$$\sum_{k=0}^{n+1} (-1)^k (a_0 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_{n+2}).$$

One can prove that the complex

$$0 \leftarrow \mathcal{A} \xleftarrow{\pi} \beta_0(\mathcal{A}) \xleftarrow{d_0} \beta_1(\mathcal{A}) \xleftarrow{d_1} \cdots \leftarrow \beta_n(\mathcal{A}) \xleftarrow{d_n} \beta_{n+1}(\mathcal{A}) \leftarrow \ldots,$$

where $\pi : \beta_0(\mathcal{A}) \to \mathcal{A} : a \otimes b \mapsto ab$, is a flat pseudo-resolution of the $\mathcal{A}$-bimodule $\mathcal{A}$. We denote it by $0 \leftarrow \mathcal{A} \xleftarrow{\pi} \beta(\mathcal{A})$.

Banach algebras $\mathcal{A}$ with b.a.i. and with $db_w\mathcal{A} \le n$

Theorem 2. [B.E. Johnson; Z.A. Lykova] *Let $\mathcal{A}$ be a Banach algebra with b.a.i.. For each integer $n \ge 0$ the following properties of $\mathcal{A}$ are equivalent:*

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(ii) $\mathcal{A}$ *is* $(n+1)$*-amenable, that is,* $\mathcal{H}^{n+1}(\mathcal{A}, X^*) = \{0\}$ for all $X \in \mathcal{A}$-mod-$\mathcal{A}$;

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(iii) $\mathcal{H}^m(\mathcal{A}, X^*) = \{0\}$ for all $m \geq n+1$ and for all $X \in \mathcal{A}$-essmod-$\mathcal{A}$;

(iv) $\mathcal{H}_{n+1}(\mathcal{A}, X) = \{0\}$ *and* $\mathcal{H}_n(\mathcal{A}, X)$ *is a Hausdorff space for all* $X \in \mathcal{A}$-essmod-$\mathcal{A}$;

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(v) *if* $0 \longleftarrow \mathcal{A} \xleftarrow{\varepsilon} P_0 \xleftarrow{\phi_0} P_1 \xleftarrow{\phi_1} \cdots P_{n-1} \xleftarrow{\phi_{n-1}} Y \longleftarrow 0 \qquad (0 \leftarrow \mathcal{A} \leftarrow \mathcal{P})$

is a pseudo-resolution of $\mathcal{A}$ *in* $\mathcal{A}\text{-essmod-}\mathcal{A}$ *such that all the modules* P_i *are flat in* $\mathcal{A}\text{-essmod-}\mathcal{A}$*, then* Y *is also flat in* $\mathcal{A}\text{-essmod-}\mathcal{A}$*.*

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(vi) *the* $\mathcal{A}$*-bimodule* $\mathcal{A}$ *has a flat pseudo-resolution of length* n *in the category of* $\mathcal{A}$-essmod-$\mathcal{A}$*.*

$$db_w(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}) \geq \max\{db_w\mathcal{A}, db_w\mathcal{B}\}$$

Proposition 3. [Yu.V Selivanov; Z.A. Lykova] *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be Banach algebras with b.a.i.. Then*

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Questions. Is it true that for all Banach algebras $\mathcal{A}$ and $\mathcal{B}$

$$db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) \leq db_w\mathcal{A} + db_w\mathcal{B}?$$

$$db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) \geq \max\{db_w\mathcal{A}, db_w\mathcal{B}\}?$$

The tensor product algebra $\mathcal{A} \widehat{\otimes} \mathcal{B}$ of biflat Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is biflat.

Proposition 4. [Z.A. Lykova] *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, let X be an essential Banach $\mathcal{A}$-bimodule and let Y be an essential Banach $\mathcal{B}$-bimodule. Suppose X is flat in $\mathcal{A}$-mod-$\mathcal{A}$ and Y is flat in $\mathcal{B}\text{-mod-}\mathcal{B}$. Then $X \widehat{\otimes} Y$ is flat in $\mathcal{A} \widehat{\otimes} \mathcal{B}\text{-mod-}\mathcal{A} \widehat{\otimes} \mathcal{B}$.*

Theorem 3. [Z.A. Lykova] *The tensor product algebra $\mathcal{A} \widehat{\otimes} \mathcal{B}$ of biflat Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is biflat.*

Proof. A biflat Banach algebra is essential (Helemskii). Hence, by Proposition 4, $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is flat in $\mathcal{A} \widehat{\otimes} \mathcal{B}\text{-mod-}\mathcal{A} \widehat{\otimes} \mathcal{B}$.

The tensor product $\mathcal{X} \widehat{\otimes} \mathcal{Y}$ of bounded complexes

Definition 2. *Let $\mathcal{X}$, $\mathcal{Y}$ be chain complexes in $\mathcal{B}an$:*

$$0 \xleftarrow{\phi_{-1}} X_0 \xleftarrow{\phi_0} X_1 \xleftarrow{\phi_1} X_2 \xleftarrow{\phi_2} X_3 \longleftarrow \cdots$$

and

$$0 \xleftarrow{\psi_{-1}} Y_0 \xleftarrow{\psi_0} Y_1 \xleftarrow{\psi_1} Y_2 \xleftarrow{\psi_2} Y_3 \longleftarrow \cdots .$$

The tensor product $\mathcal{X} \widehat{\otimes} \mathcal{Y}$ of bounded complexes $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{B}an$ is the chain complex

$$0 \xleftarrow{\delta_{-1}} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_0 \xleftarrow{\delta_0} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_1 \xleftarrow{\delta_1} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_2 \longleftarrow \cdots , \tag{1}$$

where

$$(\mathcal{X}\widehat{\otimes}\mathcal{Y})_n = \bigoplus_{m+q=n} X_m \widehat{\otimes} Y_q$$

and

$$\delta_{n-1}(x \otimes y) = \phi_{m-1}(x) \otimes y + (-1)^m x \otimes \psi_{q-1}(y),$$

$x \in X_m, y \in Y_q$ *and* $m + q = n$.

Proposition 5. [Z.A. Lykova] *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be Banach algebras.*

Let $0 \leftarrow X \stackrel{\varepsilon_1}{\longleftarrow} \mathcal{X}$ *be a pseudo-resolution of* X *in* $\mathcal{A}_1\text{-essmod-}\mathcal{A}_1$ *such that all modules in* $\mathcal{X}$ *are flat in* $\mathcal{A}_1\text{-mod-}\mathcal{A}_1$ *and*

$0 \leftarrow Y \stackrel{\varepsilon_2}{\longleftarrow} \mathcal{Y}$ *be a pseudo-resolution of* Y *in* $\mathcal{A}_2\text{-essmod-}\mathcal{A}_2$ *such that all modules in* $\mathcal{Y}$ *are flat in* $\mathcal{A}_2\text{-mod-}\mathcal{A}_2$*.*

Then $0 \leftarrow X \widehat{\otimes} Y \stackrel{\varepsilon_1 \otimes \varepsilon_2}{\longleftarrow} \mathcal{X} \widehat{\otimes} \mathcal{Y}$ *is a pseudo-resolution of* $X \widehat{\otimes} Y$ *in* $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2\text{-essmod-}\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2$ *such that all modules in* $\mathcal{X} \widehat{\otimes} \mathcal{Y}$ *are flat in* $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2\text{-mod-}\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2$*.*

$db_w\mathcal{A} < n$ for $\mathcal{A}$ with b.a.i.

Recall that a continuous linear operator $T : X \to Y$ between Banach spaces X and Y is *topologically injective* if it is injective and its image is closed, that is, $T : X \to \mathrm{Im}\, T$ is a topological isomorphism.

Proposition 6. [Yu. V. Selivanov; Z.A. Lykova] *Let $\mathcal{A}$ be a Banach algebra with b.a.i. and let $(0 \leftarrow \mathcal{A} \leftarrow \mathcal{P})$:*

$$0 \longleftarrow \mathcal{A} \overset{\varepsilon}{\longleftarrow} P_0 \overset{\phi_0}{\longleftarrow} \cdots P_{n-1} \overset{\phi_{n-1}}{\longleftarrow} P_n \longleftarrow 0 \tag{2}$$

be a flat pseudo-resolution of $\mathcal{A}$ in $\mathcal{A}$-essmod-$\mathcal{A}$. Then

$$db_w\mathcal{A} < n \quad \Longleftrightarrow$$

for every X in $\mathcal{A}$-essmod-$\mathcal{A}$, the operator

$$\phi_{n-1} \otimes_{\mathcal{A}-\mathcal{A}} 1_X : P_n \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} X \to P_{n-1} \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} X$$

is topologically injective.

Lemma 1. [Yu. V. Selivanov] *Let E_0, E, F_0 and F be Banach spaces, and let $S : E_0 \to E$ and $T : F_0 \to F$ be continuous linear operators. Suppose S and T are not topologically injective. Then the continuous linear operator*

$$\Delta : E_0 \widehat{\otimes} F_0 \to (E_0 \widehat{\otimes} F) \oplus (E \widehat{\otimes} F_0)$$

defined by

$$\Delta(x \otimes y) = (x \otimes T(y), S(x) \otimes y) \ \ (x \in E_0, y \in F_0).$$

is not topologically injective.

$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B})$ for Banach algebras with b.a.i.

Theorem 4. [Z.A. Lykova] *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with bounded approximate identities. Then*

$$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B}) = db_w\mathcal{A} + db_w\mathcal{B}.$$

Proof. Suppose $db_w\mathcal{A} = m$ and $db_w\mathcal{B} = q$ where $0 < m, q < \infty$.

By Theorem 2, $db_w\mathcal{A} = m$ implies there is a flat pseudo-resolution $0 \leftarrow \mathcal{A} \xleftarrow{\varepsilon_1} (\mathcal{P}, \phi)$ of length m in the category $\mathcal{A}$-essmod-$\mathcal{A}$. By Proposition 6, there exists $X \in \mathcal{A}$-essmod-$\mathcal{A}$ such that the operator

$$\phi_{m-1} \otimes_{\mathcal{A}-\mathcal{A}} 1_X : P_m\widehat{\otimes}_{\mathcal{A}-\mathcal{A}}X \to P_{m-1}\widehat{\otimes}_{\mathcal{A}-\mathcal{A}}X$$

is not topologically injective.

Similarly, $db_w\mathcal{B} = q$ implies that there is a flat pseudo-resolution $0 \leftarrow \mathcal{B} \xleftarrow{\varepsilon_2} (\mathcal{Q}, \psi)$ of length q in the category $\mathcal{B}$-essmod-$\mathcal{B}$ and there exist $Y \in \mathcal{B}$-essmod-$\mathcal{B}$

such that the operator

$$\psi_{q-1} \otimes_{\mathcal{B}-\mathcal{B}} 1_Y : Q_q \widehat{\otimes}_{\mathcal{B}-\mathcal{B}} Y \to Q_{q-1} \widehat{\otimes}_{\mathcal{B}-\mathcal{B}} Y$$

is not topologically injective.

By Proposition 5, $0 \leftarrow \mathcal{A}\widehat{\otimes}\mathcal{B} \stackrel{\varepsilon_1 \otimes \varepsilon_2}{\longleftarrow} (\mathcal{P}\widehat{\otimes}\mathcal{Q}, \delta)$ is a flat pseudo-resolution of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ in $\mathcal{A}\widehat{\otimes}\mathcal{B}\text{-essmod-}\mathcal{A}\widehat{\otimes}\mathcal{B}$ of length $m+q$.

Take $Z = X\widehat{\otimes}Y$ in $\mathcal{A}\widehat{\otimes}\mathcal{B}\text{-essmod-}\mathcal{A}\widehat{\otimes}\mathcal{B}$.

By Lemma 1, the operator

$$\delta_{m+q-1} \otimes_{\mathcal{A}\widehat{\otimes}\mathcal{B}-\mathcal{A}\widehat{\otimes}\mathcal{B}} 1_Z : (\mathcal{P}\widehat{\otimes}\mathcal{Q})_{m+q} \widehat{\otimes}_{\mathcal{A}\widehat{\otimes}\mathcal{B}-\mathcal{A}\widehat{\otimes}\mathcal{B}} Z$$

$$\to (\mathcal{P}\widehat{\otimes}\mathcal{Q})_{m+q-1} \widehat{\otimes}_{\mathcal{A}\widehat{\otimes}\mathcal{B}-\mathcal{A}\widehat{\otimes}\mathcal{B}} Z$$

is not topologically injective. Therefore, by Proposition 6, $db_w(\mathcal{A}\widehat{\otimes}\mathcal{B}) = m+q$.

Corollary 1. [Z.A. Lykova] *Let* $\mathcal{A}$ *be an amenable Banach algebra and* $\mathcal{B}$ *be Banach algebras with b.a.i. Then*

(i)

$$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B}) = db_w\mathcal{B}.$$

(ii) $\mathcal{A}\widehat{\otimes}\mathcal{B}$ *is* n*-amenable* $\iff$ $\mathcal{B}$ *is* n*-amenable.*

$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B})$ for biflat $\mathcal{A}$ and $\mathcal{B}$

Theorem 5. [Z.A. Lykova] *Let $\mathcal{A}$ and $\mathcal{B}$ be biflat Banach algebras. Then*
(i)

$$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B}) = 0 \ \text{ and}$$

$$db_w(\mathcal{A}_+\widehat{\otimes}\mathcal{B}_+) = db_w\mathcal{A} + db_w\mathcal{B} = 0$$

if $\mathcal{A}$ and $\mathcal{B}$ have two-sided b.a.i.;
(ii)

$$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B}) \leq 1 \ \text{ and}$$

$$db_w(\mathcal{A}_+\widehat{\otimes}\mathcal{B}_+) = db_w\mathcal{A} + db_w\mathcal{B} = 2$$

if $\mathcal{A}$ and $\mathcal{B}$ have left [right], but not two-sided b.a.i.;
(iii)

$$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B}) \leq 2 \ \text{ and}$$

$$db_w(\mathcal{A}_+\widehat{\otimes}\mathcal{B}_+) = db_w\mathcal{A} + db_w\mathcal{B} = 4$$

if $\mathcal{A}$ and $\mathcal{B}$ have neither left nor right b.a.i.

The algebra $\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)$ of compact operators

Example 1. *The algebra* $\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)$ *of compact operators on* $\ell_2 \widehat{\otimes} \ell_2$ *is a biflat Banach algebra with a left, but not two-sided bounded approximate identity (Gronbaek, Johnson, Willis; Selivanov). By Theorem 5, for* $n \geq 1$*,*

$$db_w[\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)]^{\widehat{\otimes} n} \leq 1$$

and

$$db_w[\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)_+]^{\widehat{\otimes} n} = n.$$

The tensor algebra $E\widehat{\otimes}F$ generated by the duality $(E, F, \langle\cdot,\cdot\rangle)$

Example 2. *Let (E, F) be a pair of infinite-dimensional Banach spaces endowed with a nondegenerate jointly continuous bilinear form*

$$\langle\cdot,\cdot\rangle : E \times F \to \mathbf{C}$$

that is not identically zero. The space $\mathcal{A} = E\widehat{\otimes}F$ is a $\widehat{\otimes}$-algebra with respect to the multiplication defined by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = \langle x_2, y_1\rangle x_1 \otimes y_2,\ x_i \in E,\ y_i \in F.$$

Then $\mathcal{A} = E\widehat{\otimes}F$ is called the tensor algebra generated by the duality $(E, F, \langle\cdot,\cdot\rangle)$.

It is known that $\mathcal{A}$ is biprojective, and has neither a left nor a right b.a.i. (Yu.V. Selivanov; A. Grothendieck).

In particular, if E is a Banach space with the approximation property, then the algebra $\mathcal{A} = E\widehat{\otimes}E^*$ is isomorphic to the algebra $\mathcal{N}(E)$ of nuclear operators on E.

By Theorem 5, for $n \geq 1$,

$$db_w[E\widehat{\otimes}F]^{\widehat{\otimes}n} \leq 2 \;\text{ and }\; db_w[(E\widehat{\otimes}F)_+]^{\widehat{\otimes}n} = 2n.$$

Example 3. *Let* $\mathcal{B}$ *be the algebra of* 2×2*-complex matrices of the form*

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

with matrix multiplication and norm. It is known that $\mathcal{B}$ *is* 2*-amenable, biprojective, has a left, but not right identity (A.L.T. Paterson). By Theorem 5, for* $n \geq 1$*,*

$$db_w[\mathcal{B}]^{\widehat{\otimes} n} = 1, \text{ and } db_w[\mathcal{B}_+]^{\widehat{\otimes} n} = n;$$

$$db_w[\mathcal{B} \widehat{\otimes} \mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)]^{\widehat{\otimes} n} = 1,$$

and

$$db_w[\mathcal{B}_+ \widehat{\otimes} \mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)_+]^{\widehat{\otimes} n} = 2n.$$

External products of Hochschild cohomology of Banach algebras with b.a.i.

Theorem 6. [Z.A. Lykova] *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with b.a.i., let X be an essential Banach $\mathcal{A}$-bimodule and let Y be an essential Banach $\mathcal{B}$-bimodule. Then for $n \geq 0$, up to topological isomorphism,*

$$\mathcal{H}^n(\mathcal{A}\widehat{\otimes}\mathcal{B}, (X\widehat{\otimes}Y)^*) = H^n((\mathcal{C}_\sim(\mathcal{A}, X)\widehat{\otimes}\mathcal{C}_\sim(\mathcal{B}, Y))^*).$$

Theorem 7. [Z.A. Lykova] *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with b.a.i., let X be an essential Banach $\mathcal{A}$-bimodule and let Y be an essential Banach $\mathcal{B}$-bimodule. Suppose $\mathcal{A}$ is amenable. Then, for $n \geq 0$, up to topological isomorphism,*

$$\mathcal{H}^n(\mathcal{A}\,\widehat{\otimes}\,\mathcal{B}, (X\,\widehat{\otimes}\,Y)^*) = \mathcal{H}^n(\mathcal{B}, (X/[X, \mathcal{A}]\,\widehat{\otimes}\,Y)^*),$$

where $b \cdot (\bar{x} \otimes y) = (\bar{x} \otimes b \cdot y)$ and $(\bar{x} \otimes y) \cdot b = (\bar{x} \otimes y \cdot b)$ for $\bar{x} \in X/[X, \mathcal{A}]$, $y \in Y$ and $b \in \mathcal{B}$.

The simplicial cohomology of $L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}$

Let $\mathcal{A}$ be the Banach algebra $L^1(\mathbf{R}_+)$ of complex-valued, Lebesgue measurable functions f on $\mathbf{R}_+$ with finite L^1-norm and convolution multiplication.

Theorem 8. [F. Gourdeau, Z.A.L. and M.C. White; Z.A.Lykova] *Let $\mathcal{C}$ be an amenable Banach algebra. Then*

$$\mathcal{H}_n(L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}, L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}) \cong \{0\} \text{ if } \; n > k;$$

$$\mathcal{H}^n\left(L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}, \left(L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}\right)^*\right) \cong \{0\} \text{ if } \; n > k;$$

up to topological isomorphism,

$$\mathcal{H}_n(L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}, L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}) \cong \bigoplus^{\binom{k}{n}} L^1(\mathbf{R}_+^k) \widehat{\otimes} (\mathcal{C}/[\mathcal{C},\mathcal{C}]) \text{ if } n \leq k;$$

and

$$\mathcal{H}^n(L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}, (L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C})^*) \cong \bigoplus^{\binom{k}{n}} \left[L^1(\mathbf{R}_+^k) \widehat{\otimes} (\mathcal{C}/[\mathcal{C},\mathcal{C}])\right]^*$$

if $n \leq k$.

Applications to the cyclic cohomology

Theorem 9. [Z.A. Lykova] *Let $\mathcal{A}$ be a Banach algebra. Suppose that $db_w\mathcal{A} = m$ and $m = 2L$ is an even integer. Then,*

(i) *for all $\ell \geq L$, $HC_{2\ell+2}(\mathcal{A}) = HC_m(\mathcal{A})$ and $HC_{2\ell+3}(\mathcal{A}) = HC_{m+1}(\mathcal{A})$;*

(ii) *$HP_0(\mathcal{A}) = HC_m(\mathcal{A})$ and $HP_1(\mathcal{A}) = HC_{m+1}(\mathcal{A})$;*

(iii) *for all $\ell \geq L$, $HC^{2\ell+2}(\mathcal{A}) = HC^m(\mathcal{A})$ and $HC^{2\ell+3}(\mathcal{A}) = HC^{m+1}(\mathcal{A})$;*

(iv) *$HP^0(\mathcal{A}) = HC^m(\mathcal{A})$ and $HP^1(\mathcal{A}) = HC^{m+1}(\mathcal{A})$.*

(v) *$HE^0(\mathcal{A}) = HP^0(\mathcal{A}) = HC^m(\mathcal{A})$ and $HE^1(\mathcal{A}) = HP^1(\mathcal{A}) = HC^{m+1}(\mathcal{A})$.*

There are similar formulae for odd m.

Theorem 10. [A. Ya. Helemskii; M. Khalkhali; Z.A. Lykova] *Let* $\mathcal{A}$ *be a biflat Banach algebra. Then*

(i) *for all* $\ell \geq 0$, $HC_{2\ell}(\mathcal{A}) = \mathcal{A}/[\mathcal{A}, \mathcal{A}]$ *and* $HC_{2\ell+1}(\mathcal{A}) = \{0\}$*;*

(ii) $HP_0(\mathcal{A}) = \mathcal{A}/[\mathcal{A}, \mathcal{A}]$ *and* $HP_1(\mathcal{A}) = \{0\}$*;*

(iii) *for all* $\ell \geq 0$, $HC^{2\ell}(\mathcal{A}) = \mathcal{A}^{tr}$ *and* $HC^{2\ell+1}(\mathcal{A}) = \{0\}$*;*

(iv) $HE^0(\mathcal{A}) = HP^0(\mathcal{A}) = \mathcal{A}^{tr}$ *and* $HE^1(\mathcal{A}) = HP^1(\mathcal{A}) = \{0\}$.

Thank you