Dichotomy theorems for random matrices and ideals of operators on $\left(\bigoplus_{n=1}^{\infty} \ell_1^n\right)_{c_0}$

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The problem

Set $X = \left(\bigoplus_{n=1}^{\infty} \ell_1^n \right)_{c_0}$.

The aim is to classify to closed ideals of $\mathcal{B}(X)$.

(Part of a study of the relationship between the geometry of a Banach space E on the one hand and the Banach algebra structure of $\mathcal{B}(E)$ on the other hand.)

An easy theorem

We have the following closed ideals in $\mathcal{B}(X)$:

$$\{0\} \subsetneq \mathcal{K}(X) \subsetneq \overline{\mathcal{G}}_{c_0}(X) \subsetneq \mathcal{B}(X) \ .$$

Moreover, if T is a non-compact operator on X, then the closed ideal generated by T contains $\overline{\mathcal{G}}_{c_0}(X)$.

It follows that any closed ideal of $\mathcal{B}(X)$ not in the above list must lie strictly between $\overline{\mathcal{G}}_{c_0}(X)$ and $\mathcal{B}(X)$.

Reduction to the finite-dimensional case

An operator

$$T\colon \Big(\bigoplus_{n=1}^{\infty} \ell_1^n\Big)_{c_0} \to \Big(\bigoplus_{n=1}^{\infty} \ell_1^n\Big)_{c_0}$$

can be thought of as an infinite matrix $(T_{m,n})$ of operators $T_{m,n}\colon \ell_1^n \to \ell_1^m$.

Lemma: $\forall\, \varepsilon > 0$ there is a compact operator K with $\|K\| < \varepsilon$ such that $T + K$ has finite rows and columns.

We write $T^{(m)}$ for the m^{th} row of T :

$$T^{(m)}\colon \Big(\bigoplus_{n\in R_m} \ell_1^n\Big)_{\ell_\infty} \to \ell_1^m$$

for some finite set $R_m \subset \mathbb{N}$.

Reduction to the finite-dimensional case (contd.)

We consider sequences of operators

$$T^{(m)}\colon \ell_\infty^m(\ell_1^m) \to L_1$$

with $\sup\|T^{(m)}\| < \infty$.

Denote by $e_{i,j} = e_{i,j}^{(m)}$ the unit vector basis of $\ell_\infty^m(\ell_1^m)$.

The norm of $\sum_{i,j} a_{i,j} e_{i,j}$ is given by $\max_i \sum_j |a_{i,j}|$.

We let $T_{i,j}^{(m)} = T^{(m)}(e_{i,j})$ and identify $T^{(m)}$ with the $m \times m$ matrix $\big(T_{i,j}^{(m)}\big)$.

The finite-dimensional problem

Let $T^{(m)}\colon \ell_\infty^m(\ell_1^m) \to L_1$ be a uniformly bounded sequence of operators. Is the following true:

(i) either the identity operators $\mathrm{Id}_{\ell_1^k}$ uniformly factor through the $T^{(m)}$,

(ii) or the $T^{(m)}$ uniformly approximately factor through ℓ_∞^k?

Dichotomy theorem I

Let $X_1, X_2, \ldots$ be arbitrary Banach spaces.

Let $T_m \colon X_m \to L_1$ be a uniformly bounded sequence of operators. Then the following dichotomy holds:

(i) either the identity operators $\mathrm{Id}_{\ell_1^k}$ uniformly factor through the T_m

(ii) or the T_m have uniform approximate lattice bounds.

Lattice bounds and factorization

Let $T_m\colon X_m \to L_1$ be a uniformly bounded sequence of operators.

(i) If the T_m have uniform lattice bounds then they uniformly factor through ℓ_∞^n's.

(ii) Assume that for each $m \in \mathbb{N}$ we have $X_m = \ell_1^{N_m}$ for some $N_m \in \mathbb{N}$. If the T_m have uniform approximate lattice bounds, then they uniformly approximately factor through ℓ_∞^n's.

Proof of Dichotomy I

Two ingredients:

Theorem (Dor): Let μ and ν be measures and $T\colon L_1(\nu) \to L_1(\mu)$ an isomorphic embedding with $\|T\| \cdot \|T^{-1}\| = \lambda < \sqrt{2}$. Then there is a projection P of $L_1(\mu)$ onto the range of T with

$$\|P\| \leq \left(2\lambda^{-2} - 1\right)^{-1} .$$

Theorem: Let $T_m\colon X_m \to L_1$ be operators with $\|T_m\| \leq 1$. Let us assume that $\exists\, \delta > 0\ \forall\, n \in \mathbb{N}\ \exists\, m \in \mathbb{N}$ such that there exist $f_1, \ldots, f_n \in T_m(B_{X_m})$ and pairwise disjoint sets $E_1, \ldots, E_n$ with $\|f_i\restriction_{E_i}\| \geq \delta$ for all i. Then the identity operators $\mathrm{Id}_{\ell_1^k}$ uniformly factor through the T_m.

Surjective operator ideals

An operator ideal $\mathcal{J}$ is *surjective* if, given any operator $T\colon E \to F$ and a quotient map $Q\colon D \to E$, we have $TQ \in \mathcal{J}(D, F)$ implies $T \in \mathcal{J}(E, F)$.

The *surjective hull* of $\mathcal{J}$ is

$$\mathcal{J}^{(\mathrm{sur})}(E, F) = \big\{ T \in \mathcal{B}(E, F) : \exists\, Q\colon D \twoheadrightarrow E \text{ with } TQ \in \mathcal{J}(D, F) \big\}$$

Fact: $\mathcal{J}^{(\mathrm{sur})}$ is a surjective operator ideal and it is the smallest such object containing $\mathcal{J}$.

A consequence of Dichotomy I

We have the following closed ideals in $\mathcal{B}(X)$:

$$\{0\} \subsetneq \mathcal{K}(X) \subsetneq \overline{\mathcal{G}}_{\mathrm{c}_0}(X) \subseteq \overline{\mathcal{G}}_{\mathrm{c}_0}^{(\mathrm{sur})}(X) \subsetneq \mathcal{B}(X) \ .$$

Moreover, $\overline{\mathcal{G}}_{\mathrm{c}_0}^{(\mathrm{sur})}(X)$ is the unique maximal ideal in $\mathcal{B}(X)$.

Dichotomy theorem II

For each $m \in \mathbb{N}$ let $T^{(m)} \colon \ell_\infty^m(\ell_1^m) \to L_1$ be an operator such that the entries of the corresponding random matrix $\big(T_{i,j}^{(m)}\big)$ are independent, symmetric random variables with

$$\mathbb{E}\Big|\sum_{i=1}^{m} T_{i,j_i}^{(m)}\Big| \leq 1$$

for all functions $j \colon \{1,\dots,m\} \to \{1,\dots,m\}$. Then

(i) either the identity operators $\mathrm{Id}_{\ell_1^k}$ uniformly factor through the $T^{(m)}$

(ii) or the $T^{(m)}$ uniformly approximately factor through ℓ_∞^k's.

Factorization and 2-summing norm

The 2-summing norm of an operator $U\colon E \to F$ is defined as

$$\pi_2(U) = \sup\Big\{ \big(\textstyle\sum_{s=1}^k \|Uz^{(s)}\|^2\big)^{1/2} :\ k \in \mathbb{N},\ z^{(1)}, \dots, z^{(k)} \in E\ , \\ \textstyle\sum_{s=1}^k |\langle z^{(s)}, z^* \rangle|^2 \leq 1 \quad \forall\, z^* \in B_{E^*} \Big\}\ .$$

Let $T^{(m)}\colon \ell_\infty^m(\ell_1^m) \to L_1$ be a uniformly bounded sequence of operators. Then the following are equivalent.

(i) The $T^{(m)}$ uniformly factor through ℓ_∞^k's

(ii) $\sup_m \pi_2\big(T^{(m)}\big) < \infty$.

Estimating the 2-summing norm

Let $U\colon \ell_\infty^m(\ell_1^m) \to L_1$ be an operator such that the $U_{i,j}$ form a symmetric sequence of random variables. Then

$$\pi_2(U) \leq \left(\sum_{i=1}^{m} \max_{1\leq j\leq m} \|U_{i,j}\|_{L_2}^2 \right)^{1/2} .$$

The square-function inequality

If $f_1, \ldots, f_n \in L_1$ form a symmetric sequence of random variables, then

$$\frac{1}{K}\Big\|\Big(\sum_{i=1}^{n}|f_i|^2\Big)^{1/2}\Big\|_{L_1} \le \Big\|\sum_{i=1}^{n} f_i\Big\|_{L_1} \le \Big\|\Big(\sum_{i=1}^{n}|f_i|^2\Big)^{1/2}\Big\|_{L_1} .$$

Given symmetric sequences $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ of random variables in L_1, if $|f_i| \le |g_i|$ for all i, then

$$\Big\|\sum_{i=1}^{n} f_i\Big\|_{L_1} \le K \cdot \Big\|\sum_{i=1}^{n} g_i\Big\|_{L_1}$$

Proof of Dichotomy II

We consider two cases

(i) $\exists\, \varepsilon > 0 \;\forall\, C > 0 \;\forall\, n \in \mathbb{N} \;\exists\, m \in \mathbb{N}$ such that there exist pairwise disjoint functions $j^{(s)} \in \mathcal{F}_m$ $(s = 1, \ldots, n)$ such that

$$\Big\| \sum_{i=1}^{m} T^{(m)}_{i,j_i^{(s)}} \cdot \mathbb{1}_{\left\{\left|T^{(m)}_{i,j_i^{(s)}}\right| > C\right\}} \Big\|_{L_1} \geq \varepsilon \qquad s = 1, \ldots, n\ .$$

(ii) $\forall\, \varepsilon > 0 \;\exists\, C > 0 \;\exists\, n \in \mathbb{N} \;\forall\, m \in \mathbb{N}$ there exist pairwise disjoint functions $j^{(s)} \in \mathcal{F}_m$ $(s = 1, \ldots, n)$ such that

$$\Big\| \sum_{i=1}^{m} T^{(m)}_{i,j_i} \cdot \mathbb{1}_{\{|T^{(m)}_{i,j_i}| > C\}} \Big\|_{L_1} < \varepsilon$$

for all $j \in \mathcal{F}_m$ that is disjoint from all the $j^{(s)}$.

A consequence of the Hoffman-Jørgensen inequality

Given $0 < p, q < \infty$, there is a constant $K_{p,q}$ such that if $X_1, \dots, X_N$ are independent, symmetric random variables in L_p then

$$\Big\| \sum_{i=1}^{N} X_i \Big\|_{L_p} \overset{K_{p.q}}{\sim} \Big\| \max_{1 \leq i \leq N} |X_i| \Big\|_{L_p} + \Big\| \sum_{i=1}^{N} X_i \cdot \mathbb{1}_{\{|X_i| \leq \delta_0\}} \Big\|_{L_q}$$

where $\delta_0 = \inf \Big\{ t > 0 : \sum_{i=1}^{N} \mathbb{P}\big(|X_i| > t\big) \leq \frac{1}{8 \cdot 3^p} \Big\}$.