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ARENS REGULARITY OF THE ALGEBRA OF OPERATORS ON A BANACH SPACE

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Abstract

A short proof is given that if E is a super-reflexive Banach space, then $\mathcal{B}(E)$, the Banach algebra of operators on E with composition product, is Arens regular. Some remarks on necessary conditions on E for $\mathcal{B}(E)$ to be Arens regular are made.

1. *Introduction*

Throughout, we denote the dual space of a Banach space E by E'. If $x \in E$ and $\lambda \in E'$ then we write $\langle \lambda, x \rangle = \lambda(x)$. We adopt the convention that the left-hand side of $\langle \cdot, \cdot \rangle$ is a member of the dual of the space which contains the right-hand side member of $\langle \cdot, \cdot \rangle$. We have the canonical isometric map $\kappa = \kappa_E : E \to E''$ defined by $\langle \kappa_E(x), \mu \rangle = \langle \mu, x \rangle$ for each $x \in E$ and $\mu \in E'$. If $T \in \mathcal{B}(E)$ then define $T' \in \mathcal{B}(E')$ by

$$\langle T'(\lambda), x \rangle = \langle \lambda, T(x) \rangle \qquad (x \in E, \lambda \in E'),$$

so that $T \mapsto T'$ is an isometric map. It is a surjection if and only if E is reflexive.

If $\mathcal{A}$ is a Banach algebra, let $\mathcal{A}^{\mathrm{op}}$ be the Banach algebra whose underlying space is $\mathcal{A}$, but with the product reversed. There are two canonical ways to extend the product from $\mathcal{A}$ to $\mathcal{A}''$, called the *Arens products*, which were defined in [**1**] and first extensively studied in [**2**]. We recall the definitions: if $a, b \in \mathcal{A}$, $\lambda \in \mathcal{A}'$ and $\Phi \in \mathcal{A}''$ we define $a.\lambda \in \mathcal{A}'$, $\lambda.a \in \mathcal{A}'$, $\lambda.\Phi \in \mathcal{A}'$ and $\Phi.\lambda \in \mathcal{A}'$ by

$$a.\lambda : b \mapsto \langle \lambda, ba \rangle \quad , \quad \lambda.a : b \mapsto \langle \lambda, ab \rangle$$
$$\lambda.\Phi : b \mapsto \langle \Phi, b.\lambda \rangle \quad , \quad \Phi.\lambda : b \mapsto \langle \Phi, \lambda.b \rangle$$

and then define two products $\Box$ and $\Diamond$ on $\mathcal{A}''$ by

$$\langle \Phi \Box \Psi, \lambda \rangle = \langle \Phi, \Psi.\lambda \rangle \quad , \quad \langle \Phi \Diamond \Psi, \lambda \rangle = \langle \Psi, \lambda.\Phi \rangle \qquad (\Phi, \Psi \in \mathcal{A}'', \lambda \in \mathcal{A}').$$

Then $(\mathcal{A}'', \Box)$ and $(\mathcal{A}'', \Diamond)$ are Banach algebras. We say that $\mathcal{A}$ is *Arens regular* if for all $\Phi, \Psi \in \mathcal{A}''$ we have $\Phi \Box \Psi = \Phi \Diamond \Psi$. For further details we refer the reader to [**11**, Section 1.4] or [**3**, Section 2.6].

If $\mathcal{A}$ is a Banach algebra, then a *Banach left $\mathcal{A}$-module* is a Banach space E together with a bilinear map $\mathcal{A} \times E \to E; (a, e) \mapsto a.e$ such that $\|a.e\| \leq \|a\| \|e\|$ and $a.(b.e) = (ab).e$ for all $a, b \in \mathcal{A}$ and $e \in E$. Similarly we have a *Banach right $\mathcal{A}$-module*. A *Banach $\mathcal{A}$-bimodule* is a Banach left $\mathcal{A}$-module E that is also a Banach right $\mathcal{A}$-module, and for which $a.(e.b) = (a.e).b$ for all $a, b \in \mathcal{A}$ and $e \in E$. If E is a Banach $\mathcal{A}$-bimodule then E' is also, with module operations given by the duality:

$$\langle a.\lambda, e \rangle = \langle \lambda, e.a \rangle \quad \text{and} \quad \langle \lambda.a, e \rangle = \langle \lambda, a.e \rangle \qquad (a \in \mathcal{A}, e \in E, \lambda \in E').$$

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Thus if E is a Banach left $\mathcal{A}$-module, then E' is a Banach right $\mathcal{A}$-module; if E is a Banach right $\mathcal{A}$-module, then E' is a Banach left $\mathcal{A}$-module.

As is well-known, if $\mathcal{A}$ is a unital C*-algebra then there exists a Hilbert space $\mathcal{H}$ and an isometric *-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, called the *universal representation*, such that $\mathcal{A}'' \cong \pi(\mathcal{A})'' = \pi(\mathcal{A})^{cc}$, the double commutant of $\pi(\mathcal{A})$. Further, $\mathcal{A}$ is Arens regular, the Arens product is the same as the composition product from $\mathcal{B}(\mathcal{H})$, and $\mathcal{A}''$ is semi-simple. See, for example, [**2**] or [**3**, Theorem 3.2.36]

In particular, if $E = l^2 = l^2(\mathbb{N})$, then $\mathcal{B}(E)$ is Arens regular. It is known that if $\mathcal{B}(E)$ is Arens regular, then E is reflexive (see 8 below). Thus there is some motivation for the belief that $\mathcal{B}(E)$ should be Arens regular for $E = l^p$, $1 < p < \infty$; indeed, this question was raised in [**12**]. We shall show that if E is a *super-reflexive* Banach space (defined below) then $\mathcal{B}(E)$ is Arens regular. As is known, if $p \in (1, \infty)$ and Ω is an arbitrary measure space, then $L^p(\Omega)$ is super-reflexive (see Corollary 3): certainly it follows that $\mathcal{B}(l^p)$ is Arens regular for $p \in (1, \infty)$.

In [**12**], it was shown that there exist Banach spaces E that are reflexive but for which $\mathcal{B}(E)$ is not Arens regular. We present a new, short proof of this result for a large class of reflexive Banach spaces E. Finally, we give some necessary and sufficient conditions for $\mathcal{B}(E)$ to be Arens regular, and ask if there might exist a reflexive, not super-reflexive Banach space E such that $\mathcal{B}(E)$ is Arens regular.

2. *Ultrapowers*

We shall extensively use the idea of an *ultrapower* of a Banach space: for further details and proofs we refer the reader to [**9**].

We recall the notion of a *filter*, and a maximally refined filter, called an *ultrafilter*. If $\mathcal{U}$ is an ultrafilter, X a topological space, and $(x_\alpha)_{\alpha\in I}$ a family in X, then we write $\lim_{\mathcal{U}} x_\alpha$ or $x = \lim_{\alpha\in\mathcal{U}} x_\alpha$ for the limit of (x_α) along the ultrafilter $\mathcal{U}$. If X is compact and Hausdorff then such a limit always exists and is unique. Such a generalised notion of convergence is why ultrafilters are useful in analysis.

Let I be an indexing set, $(E_\alpha)_{\alpha\in I}$ be a family of Banach spaces, and consider the Banach space $l^\infty((E_\alpha), I)$ of all bounded families $(x_\alpha)_{\alpha\in I}$ with $x_\alpha \in E_\alpha$, under pointwise operations and the supremum norm. Then let $\mathcal{U}$ be an ultrafilter on I and $N_{\mathcal{U}} = \{(x_\alpha) \in l^\infty(E, I) : \lim_{\mathcal{U}} \|x_\alpha\| = 0\}$. It is simple to show that $N_{\mathcal{U}}$ is a closed subspace, so $(E_\alpha)_{\mathcal{U}} := l^\infty((E_\alpha), I)/N_{\mathcal{U}}$ is a Banach space, and that $\|(x_\alpha)\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_\alpha\|$ coincides with the quotient norm. We call $(E_\alpha)_{\mathcal{U}}$ the *ultraproduct* of (E_α) with respect to $\mathcal{U}$. If $E_\alpha = E$ for each α, then $(E_\alpha)_{\mathcal{U}} = (E)_{\mathcal{U}}$ is the *ultrapower* of E with respect to $\mathcal{U}$.

We can regard E as a subspace of $(E)_{\mathcal{U}}$ by the canonical isometric embedding $x \mapsto (x_\alpha)$ where for each α, $x_\alpha = x$.

The following notions were introduced by James in [**10**].

Definition 1. Let E and F be Banach spaces and $T : E \to F$ be a bounded linear map. Then T is a $(1+\epsilon)-$*isomorphism* ($\epsilon > 0$) if T is an isomorphism, $\|T\| \leq 1+\epsilon$ and $\|T^{-1}\| \leq 1+\epsilon$. In this case, we say E and F are $(1+\epsilon)-$*isomorphic*.

If E and F are Banach spaces such that for each finite dimensional subspace M of F and each $\epsilon > 0$, M is $(1+\epsilon)-$isomorphic to some subspace of E, then F is *finitely representable* in E.

The space E is *super-reflexive* if and only if every Banach space that is finitely representable in E is reflexive.

Proposition 1. *Let E and F be Banach spaces. Then F is finitely representable in E if and only if F if isometrically isomorphic to a subspace of $(E)_{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$.*

Further, E is super-reflexive if and only if each ultrapower $(E)_{\mathcal{U}}$ is reflexive.

Proof. The first assertion is [**9**, Theorem 6.3]. The second assertion is then clear, as a subspace of a reflexive space is itself reflexive. □

In [**9**], after Corollary 7.6, it is shown that if $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters and E a Banach space, then $((E)_{\mathcal{U}})_{\mathcal{V}}$ is isometrically isomorphic to $(E)_{\mathcal{W}}$ for some ultrafilter $\mathcal{W}$. Thus if E is super-reflexive, then $((E)_{\mathcal{U}})_{\mathcal{V}}$ is reflexive for every $\mathcal{V}$, and so $(E)_{\mathcal{U}}$ is super-reflexive. Thus we see that E is super-reflexive if and only if each ultrapower of E is super-reflexive.

If we form an ultraproduct $(E_\alpha)_{\mathcal{U}}$ then we have a canonical map $J : (E'_\alpha)_{\mathcal{U}} \to (E_\alpha)'_{\mathcal{U}}$ defined by

$$\langle J(\mu_\alpha), (x_\alpha)\rangle = \lim_{\mathcal{U}} \langle \mu_\alpha, x_\alpha\rangle \qquad ((\mu_\alpha) \in (E'_\alpha)_{\mathcal{U}}, (x_\alpha) \in (E_\alpha)_{\mathcal{U}}).$$

It is an easy exercise to show that J is well-defined in the sense that the definition is independent of the choice of representatives (x_α) and (μ_α). In [**9**, Section 7], it is shown that J is an isometry, and that J is a surjection if and only if $(E_\alpha)_{\mathcal{U}}$ is reflexive (for a countably incomplete ultrafilter $\mathcal{U}$). In particular, if E is super-reflexive then $J : (E')_{\mathcal{U}} \to (E)'_{\mathcal{U}}$ is an isometric isomorphism.

Let E be a Banach space and $\mathcal{U}$ be an ultrafilter. Then, as the unit ball of E'' is weak*-compact, we can well-define a map $\sigma : (E)_{\mathcal{U}} \to E''$ by

$$\sigma((x_\alpha)) = \text{weak}*\!-\!\lim_{\alpha\in\mathcal{U}} \kappa_E(x_\alpha) \qquad ((x_\alpha) \in (E)_{\mathcal{U}}).$$

Clearly σ is well-defined and norm-decreasing, and for each $\mu \in E'$ we have $\langle \sigma((x_\alpha)), \mu\rangle = \lim_{\alpha\in\mathcal{U}} \langle \mu, x_\alpha\rangle$.

Proposition 2. *Let E be a Banach space. Then there exists an ultrafilter $\mathcal{U}$ and a linear isometric embedding $K : E'' \to (E)_{\mathcal{U}}$ such that $\sigma \circ K$ is the identity on E'' and $K \circ \kappa_E$ is the canonical embedding of E into $(E)_{\mathcal{U}}$. Thus $K \circ \sigma$ is a norm 1 projection of $(E)_{\mathcal{U}}$ onto $K(E'')$.*

Proof. This is [**9**, Proposition 6.7]. □

Proposition 3. *Let Ω be a measure space (with σ-additive measure), and choose $p \in (1, \infty)$. Then, for each ultrafilter $\mathcal{U}$, $(L^p(\Omega))_{\mathcal{U}}$ is isometrically isomorphic to $L^p(\Omega')$ for some measure space Ω'. Consequently, $L^p(\Omega)$ is super-reflexive.*

Proof. This is [**9**, Theorem 3.3]. □

3. *The Arens regularity of $\mathcal{B}(E)$*

Definition 2. Let E and F be Banach spaces. Then on $E \otimes F$, the tensor product of E with F, we have the *projective tensor norm*, $\|.\|_\pi$, defined as

$$\left\| \sum_{i=1}^n e_i \otimes f_i \right\|_\pi = \inf \left\{ \sum_{j=1}^m \|x_j\|\|y_j\| : \sum_{j=1}^m x_j \otimes y_j = \sum_{i=1}^n e_i \otimes f_i \right\}.$$

The completion of $E \otimes F$ with respect to $\|.\|_\pi$ is the *projective tensor product* of E with F, denoted $E\widehat{\otimes}F$.

For more details on tensor products of Banach spaces see, for example, [**6**]. We note that for any element $u \in E\widehat{\otimes}F$ and $\epsilon > 0$ we can find sequences (x_n) in E and (y_n) in F with

$$u = \sum_{n=1}^\infty x_n \otimes y_n \quad , \quad \|u\| \leq \sum_{n=1}^\infty \|x_n\|\|y_n\| < \|u\| + \epsilon.$$

It is standard that $(E\widehat{\otimes}F)' = \mathcal{B}(E, F') = \mathcal{B}(F, E')$ by the identification

$$\langle S, \sum_{n=1}^\infty x_n \otimes y_n \rangle = \sum_{n=1}^\infty \langle S(y_n), x_n \rangle \qquad \left(S \in \mathcal{B}(F, E'), \sum_{n=1}^\infty x_n \otimes y_n \in E\widehat{\otimes}F \right).$$

Thus if E is a reflexive Banach space, we have $(E'\widehat{\otimes}E)' = \mathcal{B}(E, E'') = \mathcal{B}(E)$.

Furthermore, if $T : E \times F \to G$ is a bounded bilinear map to a Banach space G then there is a unique bounded linear map $\hat{T} : E\widehat{\otimes}F \to G$ with $\hat{T}(x \otimes y) = T(x, y)$ and $\|T\| = \|\hat{T}\|$.

We shall now show that $\mathcal{B}(E)$ is Arens regular for super-reflexive Banach spaces E. We proceed with a little generality.

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{L}$ a Banach left $\mathcal{A}$-module. Then we can define a bilinear map $\pi : \mathcal{L}' \times \mathcal{L} \to \mathcal{A}'$ by

$$\langle \pi(\mu, x), a \rangle = \langle \mu, a.x \rangle \qquad (x \in \mathcal{L}, \mu \in \mathcal{L}', a \in \mathcal{A}),$$

so that $\|\pi(\mu, x)\| \leq \|\mu\|\|x\|$. Thus we can extend π to a continuous linear map $\pi : \mathcal{L}'\widehat{\otimes}\mathcal{L} \to \mathcal{A}'$. Note that $\mathcal{L}'$ is a (dual) Banach right $\mathcal{A}$-module, and $\mathcal{L}'\widehat{\otimes}\mathcal{L}$ is a Banach $\mathcal{A}$-bimodule with module actions, for $a \in \mathcal{A}$, $x \in \mathcal{L}$ and $\mu \in \mathcal{L}'$, given by

$$\langle \mu.a, x \rangle = \langle \mu, a.x \rangle \quad , \quad a.(\mu \otimes x) = \mu \otimes a.x \quad , \quad (\mu \otimes x).a = \mu.a \otimes x.$$

If $a, b \in \mathcal{A}, x \in \mathcal{L}$ and $\mu \in \mathcal{L}'$, then we have

$$\langle \pi(\mu \otimes x).a, b \rangle = \langle \mu, ab.x \rangle = \langle \mu.a, b.x \rangle = \langle \pi(\mu.a \otimes x), b \rangle$$
$$\langle a.\pi(\mu \otimes x), b \rangle = \langle \mu, ba.x \rangle = \langle \pi(\mu \otimes a.x), b \rangle.$$

Thus we conclude that π is an $\mathcal{A}$-bimodule homomorphism:

$$a.\pi(\tau) = \pi(a.\tau) \quad , \quad \pi(\tau).a = \pi(\tau.a) \qquad (a \in \mathcal{A}, \tau \in \mathcal{L}'\widehat{\otimes}\mathcal{L}).$$

As $(\mathcal{L}'\widehat{\otimes}\mathcal{L})' = \mathcal{B}(\mathcal{L}')$ we have $\pi' : \mathcal{A}'' \to \mathcal{B}(\mathcal{L}')$. Explicitly, π' is given by

$$\langle \pi'(\Phi)(\mu), x \rangle = \langle \Phi, \pi(\mu \otimes x) \rangle \qquad (\Phi \in \mathcal{A}'', \mu \in \mathcal{L}', x \in \mathcal{L}).$$

Then we see that, for $\Phi, \Psi \in \mathcal{A}'', a \in \mathcal{A}, \mu \in \mathcal{L}'$ and $x \in \mathcal{L}$, we have

$$\langle \pi(\mu \otimes x).\Phi, a \rangle = \langle \Phi, \pi(\mu \otimes a.x) \rangle = \langle \pi'(\Phi)(\mu), a.x \rangle = \langle \pi(\pi'(\Phi)(\mu) \otimes x), a \rangle$$

and so

$$\langle \Psi, \pi(\mu \otimes x).\Phi \rangle = \langle \Psi, \pi(\pi'(\Phi)(\mu) \otimes x) \rangle = \langle \pi'(\Psi)(\pi'(\Phi)(\mu)), x \rangle$$

hence

$$\pi'(\Phi\Diamond\Psi) = \pi'(\Psi) \circ \pi'(\Phi) \qquad (\Phi, \Psi \in \mathcal{A}'')$$

so that $\pi' : (\mathcal{A}'', \Diamond) \to (\mathcal{B}(\mathcal{L}'), \circ)$ is an anti-homomorphism.

In the case that $\mathcal{L}$ is reflexive, we have

$$\langle \Phi.\pi(\mu \otimes x), a\rangle = \langle \Phi, \pi(\mu.a \otimes x)\rangle = \langle \pi'(\Phi)(\mu.a), x\rangle = \langle \pi(\mu \otimes \pi'(\Phi)'(x)), a\rangle$$

and so

$$\langle \Phi, \Psi.\pi(\mu \otimes x)\rangle = \langle \Phi, \pi(\mu \otimes \pi'(\Psi)'(x))\rangle = \langle \pi'(\Psi)'(\pi'(\Phi)(\mu)), x\rangle$$

hence

$$\pi'(\Phi\Box\Psi) = \pi'(\Psi) \circ \pi'(\Phi) \qquad (\Phi, \Psi \in \mathcal{A}'')$$

so that $\pi' : (\mathcal{A}'', \Box) \to (\mathcal{B}(\mathcal{L}'), \circ)$ is an anti-homomorphism.

Finally, we conclude that if $\mathcal{L}$ is reflexive and π' is injective, then $\mathcal{A}$ is Arens regular. Note that if $\mathcal{L}$ is reflexive, then the map $\mathcal{B}(\mathcal{L}) \to \mathcal{B}(\mathcal{L}')$, $T \mapsto T'$, is an isometric anti-homomorphism. Thus we can get homomorphisms $(\mathcal{A}'', \Box) \to \mathcal{B}(\mathcal{L})$ and $(\mathcal{A}'', \Diamond) \to \mathcal{B}(\mathcal{L})$.

For $p \in (1, \infty)$ and E a Banach space, define $l^p(E)$ to be the Banach space of all p-summable sequences in E:

$$l^p(E) = \left\{ (x_n)_{n=1}^{\infty} \subset E : \|(x_n)\| = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} < \infty \right\}.$$

Then $l^p(E)$ becomes a left $\mathcal{B}(E)$-module with pointwise module action, that is $T.(x_n) = (T(x_n))$ for $T \in \mathcal{B}(E)$ and $(x_n) \in l^p(E)$. So if $\mathcal{U}$ is an ultrafilter, $(l^p(E))_{\mathcal{U}}$ is also a left $\mathcal{B}(E)$-module with pointwise action, that is $T.(x_i) = (T.x_i)$ for $(x_i) \in (l^p(E))_{\mathcal{U}}$.

Proposition 4. *If E is super-reflexive then $l^p(E)$ is super-reflexive for each $p \in (1, \infty)$.*

Proof. It is a classical result of Enflo and James (see, for example, [**8**]) that E is super-reflexive if and only if E can be given an equivalent, uniformly-convex norm. By a result in [**5**], E is uniformly-convex if and only if $l^p(E)$ is uniformly-convex for any $p \in (1, \infty)$. Thus we are done. □

Thus if E is super-reflexive and $1 < p < \infty$ then for each ultrafilter $\mathcal{U}$, $(l^p(E))_{\mathcal{U}}$ is reflexive. Furthermore, $l^p(E) = l^q(E')$ where $p^{-1} + q^{-1} = 1$, and thus $(l^p(E))'_{\mathcal{U}} = (l^q(E'))_{\mathcal{U}}$.

Proposition 5. *If E is super-reflexive and $1 < p < \infty$, then we can find an ultrafilter $\mathcal{U}$ so that if $\mathcal{L} = (l^p(E))_{\mathcal{U}}$ then π, defined as above, is a linear metric surjection.*

Proof. By Proposition 2, applied to $(E\widehat{\otimes}E')'' = \mathcal{B}(E)'$, we can find an ultrafilter $\mathcal{U}$ and an isometry $K : \mathcal{B}(E)' \to (E\widehat{\otimes}E')_{\mathcal{U}}$. For $\lambda \in \mathcal{B}(E)'$ let $(\tau_i) = K(\lambda)$ where we may assume that $\|\tau_i\| = \|\lambda\|$ for each i. For each i, as $\tau_i \in E\widehat{\otimes}E'$ we can let

$$\tau_i = \sum_{j=1}^\infty y_{i,j} \otimes \phi_{i,j} \quad , \quad \sum_{j=1}^\infty \|y_{i,j}\|\|\phi_{i,j}\| \leq \|\lambda\| + \epsilon_i$$

where $\lim_{i\in\mathcal{U}} \epsilon_i = 0$ (if we examine the proof of Proposition 2 in [**9**, Proposition 6.7], it is clear that we can find such a (ϵ_i)). For each i and j let $x_{i,j} = \|y_{i,j}\|^{-1+1/p}\|\phi_{i,j}\|^{1/p}y_{i,j}$ and $\mu_{i,j} = \|y_{i,j}\|^{1/q}\|\phi_{i,j}\|^{-1+1/q}\phi_{i,j}$. Then for each i we have

$$\left(\sum_{j=1}^\infty \|x_{i,j}\|^p\right)^{1/p} = \left(\sum_{j=1}^\infty \|y_{i,j}\|\|\phi_{i,j}\|\right)^{1/p} \leq (\|\lambda\| + \epsilon_i)^{1/p}$$

and similarly

$$\left(\sum_{j=1}^\infty \|\mu_{i,j}\|^q\right)^{1/q} = \left(\sum_{j=1}^\infty \|y_{i,j}\|\|\phi_{i,j}\|\right)^{1/q} \leq (\|\lambda\| + \epsilon_i)^{1/q}.$$

Thus we can let $x_i = (x_{i,j})_{j=1}^\infty \in l^p(E)$ and $\mu_i = (\mu_{i,j})_{j=1}^\infty \in l^q(E')$. Then

$$\|x_i\| \leq (\|\lambda\| + \epsilon_i)^{1/p} \qquad \|\mu_i\| \leq (\|\lambda\| + \epsilon_i)^{1/q}$$

and so we can set $x = (x_i) \in (l^p(E))_\mathcal{U}$ and $\mu = (\mu_i) \in (l^q(E'))_\mathcal{U}$.

Then if $T \in \mathcal{B}(E)$,

$$\langle \pi(x\otimes\mu), T\rangle = \langle \mu, T.x\rangle = \lim_{i\in\mathcal{U}} \sum_{j=1}^\infty \langle \mu_{i,j}, T(x_{i,j})\rangle = \lim_{i\in\mathcal{U}} \langle T, \tau_i\rangle = \langle \lambda, T\rangle.$$

Thus $\pi(x\otimes\mu) = \lambda$ and finally note that

$$\|x\otimes\mu\| = \lim_{i\in\mathcal{U}} \|x_i\|\|\mu_i\| \leq \lim_{i\in\mathcal{U}} (\|\lambda\| + \epsilon_i)^{1/p}(\|\lambda\| + \epsilon_i)^{1/q} = \|\lambda\|.$$

As π is norm-decreasing, we have $\|x\|\|\mu\| = \|\lambda\|$. □

THEOREM 1. *Let E be a super-reflexive Banach space and set $\mathcal{A} = \mathcal{B}(E)$. Then $\mathcal{A}$ is Arens regular, and $\mathcal{A}''$ is isometrically a subalgebra of $\mathcal{B}(F)$ for some super-reflexive Banach space F.*

Proof. Pick $p \in (1,\infty)$ and set $F = \mathcal{L} = (l^p(E))_\mathcal{U}$ for a suitable $\mathcal{U}$, so by Proposition 5, $\pi : F\widehat{\otimes}F' \to \mathcal{B}(E)'$ is a surjection. The last remark in the proof of Proposition 5 shows that $\mathcal{B}(E)'$ can be identified with $F\widehat{\otimes}F'$ quotiented by the kernel of π. Thus $\pi' : \mathcal{B}(E)'' \to \mathcal{B}(F')$ is both an anti-homomorphism and an isomorphism onto its range. As $\|\pi\| \leq 1$, this isomorphism is an isometry. Hence, by the remark above, composing with the isometric anti-homomorphism $\mathcal{B}(F') \to \mathcal{B}(F)$, we have an isometric homomorphism $\mathcal{B}(E)'' \to \mathcal{B}(F)$ for either Arens product. In particular, $\mathcal{B}(E)$ is Arens regular. □

COROLLARY 1. *Let E be a super-reflexive Banach space and $\mathcal{C}$ a subalgebra of $\mathcal{B}(E)$. Then $\mathcal{C}''$ can be identified with a subalgebra of $\mathcal{B}(F)$ for some super-reflexive F.*

Proof. By Theorem 1, we can a super-reflexive F so that $\mathcal{B}(E)''$ is isometrically identified with a subalgebra of $\mathcal{B}(F)$. As $\mathcal{C}''$ is the weak*-closure of $\mathcal{C}$ in $\mathcal{B}(E)''$, we can thus idenfity $\mathcal{C}''$ with a subalgebra of (the image of $\mathcal{B}(E)''$ in) $\mathcal{B}(F)$. □

If E is super-reflexive, then $\mathcal{B}(E)''$ is a subalgebra of $\mathcal{B}(F)$ for a super-reflexive F, hence $\mathcal{B}(E)''$ is Arens regular, and $\mathcal{B}(E)''''$ is a subalgebra of $\mathcal{B}(G)$ for a super-reflexive G, by Corollary 1. Hence we see that every even dual of $\mathcal{B}(E)$ is Arens regular.

Thus we have, almost completely, extended the result that $\mathcal{B}(l^2)$ is Arens regular (as it is a C*-algebra). Indeed, let $\mathcal{A}$ be a C*-algebra, so that $\mathcal{A}$ is a closed subalgebra of $\mathcal{B}(H)$ for some Hilbert space H. Then $\mathcal{A}''$ is a subalgebra of $\mathcal{B}(\widehat{H})$ for some larger Hilbert space $\widehat{H}$, and H can be isometrically embedded in $\widehat{H}$.

However, we leave open the question of whether the second dual of $\mathcal{B}(E)$ is semi-simple or not. By considering the C*-algebra case, there is again some belief that $\mathcal{B}(l^p)''$ should be semi-simple for $1 < p < \infty$.

4. *Banach spaces E for which $\mathcal{B}(E)$ is not Arens regular*

Can we find a Banach space E which is reflexive but for which $\mathcal{B}(E)$ is not Arens regular. Young [**12**, p. 108] showed that we can; in this section we will present a shorter proof of this fact for a wide class of Banach spaces.

PROPOSITION 6. *Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:*
(i) *$\mathcal{A}$ is Arens regular;*
(ii) *for each $\lambda \in \mathcal{A}'$, the map $a \mapsto \lambda.a, \mathcal{A} \to \mathcal{A}'$ is weakly compact;*
(iii) *for each pair of bounded sequences $(a_n), (b_m)$ in $\mathcal{A}$ and each $\lambda \in \mathcal{A}'$,*
$$\lim_n \lim_m \langle \lambda, a_n b_m \rangle = \lim_m \lim_n \langle \lambda, a_n b_m \rangle$$
when both iterated limits exist.

Proof. See [**11**, Theorem 1.4.11]. □

PROPOSITION 7. *Let E be a Banach space. Then the following are equivalent:*
(i) *E is reflexive;*
(ii) *for each bounded sequence (x_n) in E and each bounded sequence (λ_m) in E', if $\lim_n \lim_m \langle \lambda_m, x_n \rangle$ and $\lim_n \lim_m \langle \lambda_m, x_n \rangle$ both exist, then they are equal.*

Further, the following are equivalent:
(i) *E is not reflexive;*
(ii) *for each $\theta \in (0,1)$, there exist sequences (x_n) in E and (λ_m) in E' such that for each n, $\|x_n\| = \|\lambda_n\| = 1$, and for all n and m we have*
$$\langle \lambda_m, x_n \rangle = \begin{cases} \theta & m \leq n, \\ 0 & m > n. \end{cases}$$

Proof. The second equivalence is [**7**, Theorem I.6.1], which also gives (2) implies (1) in the first equivalence. In the first equivalence (1) implies (2) follows from the weak compactness of the unit ball in E. □

PROPOSITION 8. *Let E be a Banach space such that $\mathcal{B}(E)$ is Arens regular. Then E is reflexive.*

Proof. This follows from [**12**, Theorems 2 and 3] or [**3**, Theorem 2.6.23]. □

THEOREM 2. *Let F be a non-reflexive Banach space and $(M_n, \|.\|_n)$ a sequence of Banach spaces such that for some $\epsilon > 0$ and each finite dimensional subspace M of F, M is $(1+\epsilon)$-isomorphic to some subspace of some M_n. Let $E = \oplus_{n=1}^{\infty} M_n$ as a linear space and suppose that E is a normed space with a norm $\|.\|$ which satisfies:*
(i) *there exists C such that if (x_n) and (y_n) are sequences in E with $\|y_n\|_n \leq \|x_n\|_n$ for all n, then $\|(y_n)\| \leq C\|(x_n)\|$;*
(ii) *there exists K_1 such that if $m \in \mathbb{N}$ and $(x_n) \in E$ with $x_n = 0$ for all $n \neq m$, then $\|(x_n)\| \leq K_1 \|x_m\|_m$;*
(iii) *there exists K_2 such that for all $(x_n) \in E$ and $m \in \mathbb{N}$, $\|x_m\|_m \leq K_2\|(x_n)\|$.*
Let $\widehat{E}$ be the norm-completion of E, then $\mathcal{B}(\widehat{E})$ is not Arens regular.

Note that (2) and (3) say that the canonical projections $E \to M_n$, and the canonical embeddings $M_n \to E$, are continuous and uniformly bounded. This (essentially) ensures that F is crudely finitely representable in E (which implies that $\hat{E}$ is not super-reflexive).

Proof. First note that condition (3) on the norm implies that if $\lambda \in M_N'$, then we can define $\hat{\lambda} \in E'$ by

$$\langle \hat{\lambda}, (x_n) \rangle = \langle \lambda, x_N \rangle \qquad ((x_n) \in E),$$

so that $\|\hat{\lambda}\| \leq K_2\|\lambda\|$ and we can view M_N' as a subspace of E' (note also that $E' = \hat{E}'$).

Choose $(x_n) \subseteq F, (\lambda_m) \subseteq F'$ as in Proposition 7, for some $0 < \theta < 1$. For each i let $N_i = \operatorname{span}\{x_1, \dots, x_i\}$, so that there exists $n(i) \in \mathbb{N}$ with N_i being $(1+\epsilon)$-isomorphic to $M_{n(i)}$. Then N_i' is $(1+\epsilon)$-isomorphic to $M_{n(i)}'$, and we can regard, for each m, λ_m as being in N_i' by restriction.

Thus for some increasing sequence $(n(i))_{i=1}^{\infty}$ we can find, for each i and each j such that $1 \leq j \leq n(i)$, $x_j^{(i)} \in M_{n(i)}$ and $\lambda_j^{(i)} \in M_{n(i)}'$ so that, if $1 \leq k \leq n(i)$, then

$$\langle \lambda_j^{(i)}, x_k^{(i)} \rangle = \begin{cases} \theta & j \leq k, \\ 0 & j > k. \end{cases}$$

and we have $(1+\epsilon)^{-1} \leq \|x_j^{(i)}\|_{n(i)} \leq (1+\epsilon)$ and $(1+\epsilon)^{-1} \leq \|\lambda_j^{(i)}\|_{n(i)} \leq (1+\epsilon)$.

For each $N \in \mathbb{N}$, define $T_N : E \to E$ by setting, for $(x_n) \in E$, $T_N(x_n) = (y_n)$ where

$$y_n = \begin{cases} \langle \lambda_N^{(i)}, x_{n(i)} \rangle x_N^{(i)} & n(i) = n \geq N, \\ 0 & \text{otherwise}. \end{cases}$$

Thus for all n, $\|y_n\|_n \leq (1+\epsilon)^2 \|x_n\|_n$. By condition (1) on the norm, T_N is continuous, and so T_N extends to a member of $\mathcal{B}(\hat{E})$. Also, the family (T_N) is bounded.

Then for $N, M \in \mathbb{N}$ and $(x_n) \in E$, let $T_N(x_n) = (y_n)$ and $T_M(y_n) = (z_n)$ so that

$$z_n = \begin{cases} \theta \langle \lambda_N^{(i)}, x_{n(i)} \rangle x_M^{(i)} & n(i) = n \geq N, M \leq N, \\ 0 & \text{otherwise}. \end{cases}$$

Hence $T_M T_N = 0$ for $M > N$.

Then if $M \leq N$, we have $T_M T_N(x_j^{(j)}) = \theta^2 x_M^{(j)}$ if $j \geq M$ or 0 otherwise. Thus $\langle \lambda_1^{(j)}, T_M T_N(x_j^{(j)}) \rangle = \theta$ if $j \geq M$ or 0 otherwise. Via condition (2) on the norm,

$(x_j^{(j)})$ is a bounded sequence in $\hat{E}$; by the remark at the start of the proof, $(\lambda_1^{(j)})$ is a bounded sequence in $\hat{E}'$. Thus we can define $\lambda \in \mathcal{B}(E)'$ by

$$\langle \lambda, T \rangle = \lim_{\mathcal{U}} \langle \lambda_1^{(j)}, T(x_j^{(j)}) \rangle \qquad (T \in \mathcal{B}(E))$$

for some non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

Then $\langle \lambda, T_M T_N \rangle = \theta$ if $M \leq N$ or 0 if $M > N$. Thus $\mathcal{B}(\hat{E})$ is not Arens regular in light of Proposition 6. □

Corollary 2. *If $p \in (1, \infty)$ and $E = l^p(\oplus_{n=1}^{\infty} l_n^1)$, so*

$$E = \left\{ (x_n) : \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} < \infty \ \text{and} \, \forall n, x_n \in l_n^1 \right\},$$

then E is reflexive, but $\mathcal{B}(E)$ is not Arens regular.

Proof. It is easy to see that if $M_n = l_n^1$, then for each finite dimensional subspace M of l^1 and each $\epsilon > 0$, M is $(1+\epsilon)$-isomorphic to some subspace of some M_n. Clearly the l^p norm satisfies conditions (1), (2) and (3), so we are done. □

5. *Towards a converse*

The following is sketched in [**11**, Section 1.7.8], and was first proved in [**4**]:

Theorem 3. *Let E and F be Banach spaces and $T : E \to F$ be a weakly compact linear map. Then there exists a reflexive Banach space Z and linear maps $S : E \to Z$, $R : Z \to F$ such that $T = R \circ S$. Further, we can choose Z, S and R so that S has dense range and the same norm and kernel as T, and R is injective and norm-decreasing.*

Proposition 9. *Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:*
- (i) *$\mathcal{A}$ is Arens regular;*
- (ii) *for each $\lambda \in \mathcal{A}'$, there exists a reflexive Banach space Z and continuous linear maps $\phi : \mathcal{A} \to Z$, $\psi : \mathcal{A} \to Z'$ such that we have $\langle \lambda, ab \rangle = \langle \psi(a), \phi(b) \rangle$ for each $a, b \in \mathcal{A}$.*

Proof. To show that (2) implies (1), let $(a_n), (b_m)$ be bounded sequences in $\mathcal{A}$ and pick λ, X, ϕ and ψ as in the hypotheses. Assume that $\lim_n \lim_m \langle \lambda, a_n b_m \rangle$ and $\lim_m \lim_n \langle \lambda, a_n b_m \rangle$ both exist, so by Proposition 6 we need to show that they are equal. However,

$$\lim_n \lim_m \langle \lambda, a_n b_m \rangle = \lim_n \lim_m \langle \psi(a_n), \phi(b_m) \rangle$$
$$\lim_m \lim_n \langle \lambda, a_n b_m \rangle = \lim_m \lim_n \langle \psi(a_n), \phi(b_m) \rangle$$

so as $(\phi(a_n))$ and $(\psi(b_m))$ are bounded sequences, we are done by Proposition 7.

Conversely, by Theorem 6, for $\lambda \in \mathcal{A}'$ the map $a \mapsto \lambda.a$ is weakly compact. Thus by Proposition 3, there is a reflexive Banach space Z and maps $\phi : \mathcal{A} \to Z$ and $R : Z \to \mathcal{A}'$ such that $R(\phi(a)) = a.\lambda$ for each $a \in \mathcal{A}$. Let $\psi = R' \circ \kappa_{\mathcal{A}}$ so $\psi : \mathcal{A} \to Z'$ and for $a, b \in \mathcal{A}$ we have

$$\langle \psi(a), \phi(b) \rangle = \langle (R' \circ \kappa_{\mathcal{A}})(a), \phi(b) \rangle = \langle (R'' \circ \kappa_Z \circ \phi)(b), \kappa_{\mathcal{A}}(a) \rangle$$

$$= \langle (\kappa_{\mathcal{A}'} \circ R \circ \phi)(b), \kappa_{\mathcal{A}}(b) \rangle = \langle (R \circ \phi)(b), a \rangle = \langle b.\lambda, a \rangle = \langle \lambda, ab \rangle$$

□

This provides an alternative way to show that $\mathcal{B}(E)$ is Arens regular if E is super-reflexive. Indeed, as above, we use the space $Z = (l^2(E))_{\mathcal{U}}$ for a suitable ultrafilter $\mathcal{U}$, and $\psi(T) = (T(x_{n,\alpha}))$ for a suitably chosen $(x_{n,\alpha})$ in Z, depending on λ.

Conversely, suppose that $\mathcal{B}(E)$ is Arens regular. Then we have, for each $\lambda \in \mathcal{B}(E)'$, some unknown reflexive space Z as in the theorem. How to relate this to, say, an ultrapower of E, is a somewhat open question.

6. *Conclusion*

We leave open some interesting algebraic questions about $\mathcal{B}(E)''$ — principally whether $\mathcal{B}(E)''$ is semi-simple, even in the case $E = l^p$, $p \in (1, \infty), p \neq 2$. We also make little progress in the direction of seeing if $\mathcal{B}(E)$ Arens regular implies that E is super-reflexive.

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