ARENS REGULARITY OF THE ALGEBRA OF OPERATORS ON A BANACH SPACE

MATTHEW DAWS

Abstract

A short proof is given that if \( E \) is a super-reflexive Banach space, then \( \mathcal{B}(E) \), the Banach algebra of operators on \( E \) with composition product, is Arens regular. Some remarks on necessary conditions on \( E \) for \( \mathcal{B}(E) \) to be Arens regular are made.

1. Introduction

Throughout, we denote the dual space of a Banach space \( E \) by \( E' \). If \( x \in E \) and \( \lambda \in \mathbb{E}' \) then we write \( \langle \lambda, x \rangle = \lambda(x) \). We adopt the convention that the left-hand side of \( \langle \cdot, \cdot \rangle \) is a member of the dual of the space which contains the right-hand side member of \( \langle \cdot, \cdot \rangle \). We have the canonical isometric map \( \kappa = \kappa_E : E \to E'' \) defined by \( \langle \kappa_E(x), \mu \rangle = \langle \mu, x \rangle \) for each \( x \in E \) and \( \mu \in E' \). If \( T \in \mathcal{B}(E) \) then define \( T' \in \mathcal{B}(E') \) by

\[
\langle T'(\lambda), x \rangle = \langle \lambda, T(x) \rangle \quad (x \in E, \lambda \in E'),
\]

so that \( T \mapsto T' \) is an isometric map. It is a surjection if and only if \( E \) is reflexive.

If \( \mathcal{A} \) is a Banach algebra, let \( \mathcal{A}^{\text{op}} \) be the Banach algebra whose underlying space is \( \mathcal{A} \), but with the product reversed. There are two canonical ways to extend the product from \( \mathcal{A} \) to \( \mathcal{A}'' \), called the Arens products, which were defined in [1] and first extensively studied in [2]. We recall the definitions: if \( a, b \in \mathcal{A} \), \( \lambda \in \mathcal{A}' \) and \( \Phi \in \mathcal{A}'' \) we define \( a.\lambda \in \mathcal{A}' \), \( \lambda.a \in \mathcal{A}' \), \( \lambda.\Phi \in \mathcal{A}' \) and \( \Phi.\lambda \in \mathcal{A}' \) by

\[
\begin{align*}
&a.\lambda : b \mapsto \langle \lambda, ba \rangle, &\lambda.a : b \mapsto \langle \lambda, ab \rangle \\
&\lambda.\Phi : b \mapsto \langle \Phi, b.\lambda \rangle, &\Phi.\lambda : b \mapsto \langle \Phi, \lambda.b \rangle
\end{align*}
\]

and then define two products \( \Box \) and \( \Diamond \) on \( \mathcal{A}'' \) by

\[
\langle \Phi \Box \Psi, \lambda \rangle = \langle \Phi, \Psi.\lambda \rangle, \quad \langle \Phi \Diamond \Psi, \lambda \rangle = \langle \Psi, \lambda.\Phi \rangle \quad (\Phi, \Psi \in \mathcal{A}'', \lambda \in \mathcal{A}').
\]

Then \( \mathcal{A}'', \Box \) and \( \mathcal{A}'', \Diamond \) are Banach algebras. We say that \( \mathcal{A} \) is Arens regular if for all \( \Phi, \Psi \in \mathcal{A}' \) we have \( \Phi \Box \Psi = \Phi \Diamond \Psi \). For further details we refer the reader to [11, Section 1.4] or [3, Section 2.6].

If \( \mathcal{A} \) is a Banach algebra, then a Banach left \( \mathcal{A} \)-module is a Banach space \( E \) together with a bilinear map \( \mathcal{A} \times E \to E; (a, e) \mapsto a.e \) such that \( \|a.e\| \leq \|a\|\|e\| \) and \( a.(b.e) = (ab).e \) for all \( a, b \in \mathcal{A} \) and \( e \in E \). Similarly we have a Banach right \( \mathcal{A} \)-module. A Banach \( \mathcal{A} \)-bimodule is a Banach left \( \mathcal{A} \)-module \( E \) that is also a Banach right \( \mathcal{A} \)-module, and for which \( a.(e.b) = (a.e).b \) for all \( a, b \in \mathcal{A} \) and \( e \in E \). If \( E \) is a Banach \( \mathcal{A} \)-bimodule then \( E'' \) is also, with module operations given by the duality:

\[
\langle a.\lambda, e \rangle = \langle \lambda, e.a \rangle \quad \text{and} \quad \langle \lambda.a, e \rangle = \langle \lambda, a.e \rangle \quad (a \in \mathcal{A}, e \in E, \lambda \in \mathcal{A}'').
\]
Thus if $E$ is a Banach right $\mathcal{A}$-module, then $E'$ is a Banach right $\mathcal{A}$-module; if $E$ is a Banach right $\mathcal{A}$-module, then $E'$ is a Banach left $\mathcal{A}$-module.

As is well-known, if $\mathcal{A}$ is a unital $C^*$-algebra then there exists a Hilbert space $\mathcal{H}$ and an isometric $^*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, called the universal representation, such that $\mathcal{A}' \cong \pi(\mathcal{A})'' = \pi(\mathcal{A})''^c$, the double commutant of $\pi(\mathcal{A})$. Further, $\mathcal{A}$ is Arens regular, the Arens product is the same as the composition product from $\mathcal{B}(\mathcal{H})$, and $\mathcal{A}''$ is semi-simple. See, for example, [2] or [3, Theorem 3.2.36]

In particular, if $E = l^2 = l^2(\mathbb{N})$, then $\mathcal{B}(E)$ is Arens regular. It is known that if $\mathcal{B}(E)$ is Arens regular, then $E$ is reflexive (see 8 below). Thus there is some motivation for the belief that $\mathcal{B}(E)$ should be Arens regular for $E = l^p$, $1 < p < \infty$; indeed, this question was raised in [12]. We shall show that if $E$ is a super-reflexive Banach space (defined below) then $\mathcal{B}(E)$ is Arens regular. As is known, if $p \in (1, \infty)$ and $\Omega$ is an arbitrary measure space, then $L^p(\Omega)$ is super-reflexive (see Corollary 3): certainly it follows that $\mathcal{B}(l^p)$ is Arens regular for $p \in (1, \infty)$.

In [12], it was shown that there exist Banach spaces $E$ that are reflexive but for which $\mathcal{B}(E)$ is not Arens regular. We present a new, short proof of this result for a large class of reflexive Banach spaces $E$. Finally, we give some necessary and sufficient conditions for $\mathcal{B}(E)$ to be Arens regular, and ask if there might exist a reflexive, not super-reflexive Banach space $E$ such that $\mathcal{B}(E)$ is Arens regular.

2. Ultrapowers

We shall extensively use the idea of an ultrapower of a Banach space: for further details and proofs we refer the reader to [9].

We recall the notion of a filter, and a maximally refined filter, called an ultrafilter. If $\mathcal{U}$ is an ultrafilter, $X$ a topological space, and $(x_\alpha)_{\alpha \in I}$ a family in $X$, then we write $\lim_{\mathcal{U}} x_\alpha$ or $x = \lim_{\mathcal{U}} x_\alpha$ for the limit of $(x_\alpha)$ along the ultrafilter $\mathcal{U}$. If $X$ is compact and Hausdorff then such a limit always exists and is unique. Such a generalised notion of convergence is why ultrafilters are useful in analysis.

Let $I$ be an indexing set, $(E_\alpha)_{\alpha \in I}$ be a family of Banach spaces, and consider the Banach space $l^\infty((E_\alpha), I)$ of all bounded families $(x_\alpha)_{\alpha \in I}$ with $x_\alpha \in E_\alpha$, under pointwise operations and the supremum norm. Then let $\mathcal{U}$ be an ultrafilter on $I$ and $N_\mathcal{U} = \{ (x_\alpha) \in l^\infty(E, I) : \lim_{\mathcal{U}} \|x_\alpha\| = 0 \}$. It is simple to show that $N_\mathcal{U}$ is a closed subspace, so $(E_\alpha)_\mathcal{U} := l^\infty((E_\alpha), I)/N_\mathcal{U}$ is a Banach space, and that $\|x_\alpha\|_\mathcal{U} = \lim_{\mathcal{U}} \|x_\alpha\|$ coincides with the quotient norm. We call $(E_\alpha)_\mathcal{U}$ the ultraproduct of $(E_\alpha)$ with respect to $\mathcal{U}$. If $E_\alpha = E$ for each $\alpha$, then $(E)_\mathcal{U} = (E)_\mathcal{U}$ is the ultraproduct of $E$ with respect to $\mathcal{U}$.

We can regard $E$ as a subspace of $(E)_\mathcal{U}$ by the canonical isometric embedding $x \mapsto (x_\alpha)$ where for each $\alpha$, $x_\alpha = x$.

The following notions were introduced by James in [10].

**Definition 1.** Let $E$ and $F$ be Banach spaces and $T : E \to F$ be a bounded linear map. Then $T$ is a $(1 + \epsilon)$–isomorphism ($\epsilon > 0$) if $T$ is an isomorphism, $\|T\| \leq 1 + \epsilon$ and $\|T^{-1}\| \leq 1 + \epsilon$. In this case, we say $E$ and $F$ are $(1 + \epsilon)$–isomorphic.

If $E$ and $F$ are Banach spaces such that for each finite dimensional subspace $M$ of $F$ and each $\epsilon > 0$, $M$ is $(1 + \epsilon)$–isomorphic to some subspace of $E$, then $F$ is finitely representable in $E$.

The space $E$ is super-reflexive if and only if every Banach space that is finitely representable in $E$ is reflexive.
Proposition 1. Let $E$ and $F$ be Banach spaces. Then $F$ is finitely representable in $E$ if and only if $F$ if isometrically isomorphic to a subspace of $(E)_U$ for some ultrafilter $U$.

Further, $E$ is super-reflexive if and only if each ultrapower $(E)_U$ is reflexive.

Proof. The first assertion is [9, Theorem 6.3]. The second assertion is then clear, as a subspace of a reflexive space is itself reflexive.

In [9], after Corollary 7.6, it is shown that if $U$ and $V$ are ultrafilters and $E$ a Banach space, then $((E)_U)_V$ is isometrically isomorphic to $(E)_W$ for some ultrafilter $W$. Thus if $E$ is super-reflexive, then $((E)_U)_V$ is reflexive for every $V$, and so $(E)_U$ is super-reflexive. Thus we see that $E$ is super-reflexive if and only if each ultrapower of $E$ is super-reflexive.

If we form an ultraproduct $(E_\alpha)_U$ then we have a canonical map $J : (E'_\alpha)_U \to (E_\alpha)_U$ defined by

$$J(\mu_\alpha)(x_\alpha) = \lim \langle \mu_\alpha, x_\alpha \rangle \quad (\mu_\alpha) \in (E'_\alpha)_U, (x_\alpha) \in (E_\alpha)_U).$$

It is an easy exercise to show that $J$ is well-defined in the sense that the definition is independent of the choice of representatives $(x_\alpha)$ and $(\mu_\alpha)$. In [9, Section 7], it is shown that $J$ is an isometry, and that $J$ is a surjection if and only if $(E_\alpha)_U$ is reflexive (for a countably incomplete ultrafilter $U$). In particular, if $E$ is super-reflexive then $J : (E'_\alpha)_U \to (E'_\alpha)_U$ is an isometric isomorphism.

Let $E$ be a Banach space and $U$ be an ultrafilter. Then, as the unit ball of $E''$ is weak$^*$-compact, we can well-define a map $\sigma : (E)_U \to E''$ by

$$\sigma((x_\alpha)) = \lim_{\alpha \in U} \kappa_E(x_\alpha) \quad ((x_\alpha) \in (E)_U).$$

Clearly $\sigma$ is well-defined and norm-decreasing, and for each $\mu \in E'$ we have $\langle \sigma((x_\alpha)), \mu \rangle = \lim_{\alpha \in U} \langle \mu, x_\alpha \rangle$.

Proposition 2. Let $E$ be a Banach space. Then there exists an ultrafilter $U$ and a linear isometric embedding $K : E'' \to (E)_U$ such that $\sigma \circ K$ is the identity on $E''$ and $K \circ \kappa_E$ is the canonical embedding of $E$ into $(E)_U$. Thus $K \circ \sigma$ is a norm $1$ projection of $(E)_U$ onto $K(E'')$.

Proof. This is [9, Proposition 6.7].

Proposition 3. Let $\Omega$ be a measure space (with $\sigma$-additive measure), and choose $p \in (1, \infty)$. Then, for each ultrafilter $U$, $(L^p(\Omega))_U$ is isometrically isomorphic to $L^p(\Omega')$ for some measure space $\Omega'$. Consequently, $L^p(\Omega)$ is super-reflexive.

Proof. This is [9, Theorem 3.3].

3. The Arens regularity of $\mathcal{B}(E)$

Definition 2. Let $E$ and $F$ be Banach spaces. Then on $E \otimes F$, the tensor
product of \(E\) with \(F\), we have the projective tensor norm, \(\|\cdot\|_\pi\), defined as

\[
\left\| \sum_{i=1}^{n} e_i \otimes f_i \right\|_{\pi} = \inf \left\{ \sum_{j=1}^{m} \|x_j\| y_j \right\} : \sum_{j=1}^{m} x_j \otimes y_j = \sum_{i=1}^{n} e_i \otimes f_i \}.
\]

The completion of \(E \otimes F\) with respect to \(\|\cdot\|_\pi\) is the projective tensor product of \(E\) with \(F\), denoted \(E\tilde{\otimes} F\).

For more details on tensor products of Banach spaces see, for example, [6]. We note that for any element \(u \in E\tilde{\otimes} F\) and \(\epsilon > 0\) we can find sequences \((x_n)\) in \(E\) and \((y_n)\) in \(F\) with

\[
u = \sum_{n=1}^{\infty} x_n \otimes y_n, \quad \|\nu\| \leq \sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \|\nu\| + \epsilon.
\]

It is standard that \((E\tilde{\otimes} F)' = \mathcal{B}(E, F') = \mathcal{B}(F, E')\) by the identification

\[
(S, \sum_{n=1}^{\infty} x_n \otimes y_n) = (S, \sum_{n=1}^{\infty} (S(y_n), x_n) \quad (S \in \mathcal{B}(F, E')) , \sum_{n=1}^{\infty} x_n \otimes y_n \in E\tilde{\otimes} F).
\]

Thus if \(E\) is a reflexive Banach space, we have \((E\tilde{\otimes} F)' = \mathcal{B}(E, E'') = \mathcal{B}(E)\).

Furthermore, if \(T : E \times F \to G\) is a bounded bilinear map to a Banach space \(G\) then there is a unique bounded linear map \(\hat{T} : E\tilde{\otimes} F \to G\) with \(\hat{T}(x \otimes y) = T(x, y)\) and \(\|\hat{T}\| = \|T\|\).

We shall now show that \(\mathcal{B}(E)\) is Arens regular for super-reflexive Banach spaces \(E\). We proceed with a little generality.

Let \(A\) be a Banach algebra and \(L\) a Banach left \(A\)-module. Then we can define a bilinear map \(\pi : L' \times L \to A'\) by

\[
\langle \pi(\mu, x), a \rangle = \langle \mu, a \cdot x \rangle \quad (x \in L, \mu \in L', a \in A),
\]

so that \(\|\pi(\mu, x)\| \leq \|\mu\| \|x\|\). Thus we can extend \(\pi\) to a continuous linear map \(\pi : L'\tilde{\otimes} L \to A'\). Note that \(L'\) is a (dual) Banach right \(A\)-module, and \(L'\tilde{\otimes} L\) is a Banach \(A\)-bimodule with module actions, for \(a \in A, x \in L\) and \(\mu \in L'\), given by

\[
\langle \mu \cdot a, x \rangle = \langle \mu, a \cdot x \rangle, \quad \langle a \cdot (\mu \otimes x), b \rangle = \langle \mu, a \cdot x \rangle, \quad \langle \mu - a, x \rangle = \langle \mu \otimes x, \rangle.
\]

If \(a, b \in A, x \in L\) and \(\mu \in L'\), then we have

\[
\langle \pi(\mu \otimes x), a, b \rangle = \langle \mu, ab \cdot x \rangle = \langle \mu, a, b \cdot x \rangle = \langle \pi(\mu \otimes x), b \rangle
\]

\[
\langle a \cdot \pi(\mu \otimes x), b \rangle = \langle \mu, ab \cdot x \rangle = \langle \pi(\mu \otimes x), b \rangle.
\]

Thus we conclude that \(\pi\) is an \(A\)-bimodule homomorphism:

\[
\pi(a \cdot \tau) = \pi(\tau) \cdot a = \pi(\tau \cdot a) \quad (a \in A, \tau \in L'\tilde{\otimes} L).
\]

As \((L'\tilde{\otimes} L)' = \mathcal{B}(L')\) we have \(\pi' : A'' \to \mathcal{B}(L')\). Explicitly, \(\pi'\) is given by

\[
\langle \pi'(\Phi)(\mu), x \rangle = \langle \Phi, \pi(\mu \otimes x) \rangle \quad (\Phi \in A'', \mu \in L', x \in L).
\]

Then we see that, for \(\Phi, \Psi \in A'', a \in A, \mu \in L'\) and \(x \in L\), we have

\[
\langle \pi(\mu \otimes x), \Phi, a \rangle = \langle \Phi, \pi(\mu \otimes x) \rangle = \langle \pi'(\Phi)(\mu), a, x \rangle = \langle \pi(\pi'\Phi)(\mu) \otimes x, a \rangle
\]

and so

\[
\langle \Phi, \pi(\pi'\Phi)(\mu) \otimes x \rangle = \langle \pi(\pi'\Phi)(\mu), x \rangle
\]
hence
\[ \pi'(\Phi \circ \psi) = \pi'(\psi) \circ \pi'(\Phi) \quad (\Phi, \psi \in A'') \]
so that \( \pi' : (A'', \circ) \rightarrow (B(L^\prime), \circ) \) is an anti-homomorphism.

In the case that \( L \) is reflexive, we have
\[ \langle \Phi, \pi(\mu \otimes x), a \rangle = \langle \Phi, \pi(\mu.a \otimes x) \rangle = \langle \pi'(\Phi)(\mu.a), x \rangle = \langle \pi(\mu \otimes \pi'(\Phi)'(x)), a \rangle \]
and so
\[ \langle \Phi, \psi, \pi(\mu \otimes x) \rangle = \langle \Phi, \pi(\mu \otimes \pi'(\psi)'(x)) \rangle = \langle \pi'(\psi)'(\pi'(\Phi)(\mu)), x \rangle \]
hence
\[ \pi'(\Phi \circ \psi) = \pi'(\psi) \circ \pi'(\Phi) \quad (\Phi, \psi \in A'') \]
so that \( \pi' : (A'', \circ) \rightarrow (B(L^\prime), \circ) \) is an anti-homomorphism.

Finally, we conclude that if \( L \) is reflexive and \( \pi' \) is injective, then \( A \) is Arens regular. Note that if \( L \) is reflexive, then the map \( B(L) \rightarrow B(L^\prime), T \mapsto T' \), is an isometric anti-homomorphism. Thus we can get homomorphisms \( (A'', \circ) \rightarrow B(L) \) and \( (A'', \circ) \rightarrow B(L) \).

For \( p \in (1, \infty) \) and \( E \) a Banach space, define \( l^p(E) \) to be the Banach space of all \( p \)-summable sequences in \( E \):
\[ l^p(E) = \left\{ (x_n)_{n=1}^\infty \subset E : \|x_n\| = \left( \sum_{n=1}^\infty \|x_n\|^p \right)^{1/p} < \infty \right\}. \]
Then \( l^p(E) \) becomes a left \( B(E) \)-module with pointwise module action, that is \( T.(x_n) = (T(x_n)) \) for \( T \in B(E) \) and \( (x_n) \in l^p(E) \). So if \( U \) is an ultrafilter, \( (l^p(E))_U \) is also a left \( B(E) \)-module with pointwise action, that is \( T.(x_i) = (T.x_i) \) for \( (x_i) \in (l^p(E))_U \).

**Proposition 4.** If \( E \) is super-reflexive then \( l^p(E) \) is super-reflexive for each \( p \in (1, \infty) \).

**Proof.** It is a classical result of Enflo and James (see, for example, [8]) that \( E \) is super-reflexive if and only if \( E \) can be given an equivalent, uniformly-convex norm. By a result in [5], \( E \) is uniformly-convex if and only if \( l^p(E) \) is uniformly-convex for any \( p \in (1, \infty) \). Thus we are done.

Thus if \( E \) is super-reflexive and \( 1 < p < \infty \) then for each ultrafilter \( U \), \( (l^p(E))_U \) is reflexive. Furthermore, \( l^p(E) = l^q(E') \) where \( p^{-1} + q^{-1} = 1 \), and thus \( (l^p(E))_U = (l^q(E'))_U \).

**Proposition 5.** If \( E \) is super-reflexive and \( 1 < p < \infty \), then we can find an ultrafilter \( U \) so that if \( L = (l^p(E))_U \) then \( \pi \), defined as above, is a linear metric surjection.

**Proof.** By Proposition 2, applied to \( (E \hat{\otimes} E')'' = B(E)' \), we can find an ultrafilter \( U \) and an isometry \( K : B(E)' \rightarrow (E \hat{\otimes} E')_U \). For \( \lambda \in B(E)' \) let \( (\tau_{\lambda}) = K(\lambda) \) where
we may assume that \( \|\tau_i\| = \|\lambda\| \) for each \( i \). For each \( i \), as \( \tau_i \in E \otimes E' \) we can let

\[
\tau_i = \sum_{j=1}^{\infty} y_{i,j} \otimes \phi_{i,j}, \quad \sum_{j=1}^{\infty} \|y_{i,j}\| \|\phi_{i,j}\| \leq \|\lambda\| + \epsilon_i
\]

where \( \lim_{i \in \mathcal{U}} \epsilon_i = 0 \) (if we examine the proof of Proposition 2 in [9, Proposition 6.7], it is clear that we can find such an \( \epsilon_i \)). For each \( i \) and \( j \) let \( x_{i,j} = y_{i,j}^{-1+1/p} \|\phi_{i,j}\|^{1/p} y_{i,j} \) and \( y_{i,j} = \|y_{i,j}\|^{1/q} \|\phi_{i,j}\|^{-1+1/q} \phi_{i,j} \). Then for each \( i \) we have

\[
\left( \sum_{j=1}^{\infty} \|x_{i,j}\|^p \right)^{1/p} = \left( \sum_{j=1}^{\infty} \|y_{i,j}\| \|\phi_{i,j}\| \right)^{1/p} \leq (\|\lambda\| + \epsilon_i)^{1/p}
\]

and similarly

\[
\left( \sum_{j=1}^{\infty} \|y_{i,j}\|^q \right)^{1/q} = \left( \sum_{j=1}^{\infty} \|y_{i,j}\| \|\phi_{i,j}\| \right)^{1/q} \leq (\|\lambda\| + \epsilon_i)^{1/q}.
\]

Thus we can let \( x_i = (x_{i,j})_{j=1}^{\infty} \in l^p(E) \) and \( \mu_i = (\mu_{i,j})_{j=1}^{\infty} \in l^q(E') \). Then

\[
\|x_i\| \leq (\|\lambda\| + \epsilon_i)^{1/p}, \quad \|\mu_i\| \leq (\|\lambda\| + \epsilon_i)^{1/q}
\]

and so we can set \( x = (x_i) \in (l^p(E))_{\mathcal{U}} \) and \( \mu = (\mu_i) \in (l^q(E'))_{\mathcal{U}} \).

Then if \( T \in \mathcal{B}(E) \),

\[
\langle \pi(x \otimes \mu), T \rangle = \langle \mu, T.x \rangle = \lim_{i \in \mathcal{U}} \sum_{j=1}^{\infty} \langle \mu_{i,j}, T(x_{i,j}) \rangle = \lim_{i \in \mathcal{U}} \langle T, \tau_i \rangle = \langle \lambda, T \rangle.
\]

Thus \( \pi(x \otimes \mu) = \lambda \) and finally note that

\[
\|x \otimes \mu\| = \lim_{i \in \mathcal{U}} \|x_i\| \|\mu_i\| \leq \lim_{i \in \mathcal{U}} (\|\lambda\| + \epsilon_i)^{1/p} (\|\lambda\| + \epsilon_i)^{1/q} = \|\lambda\|.
\]

As \( \pi \) is norm-decreasing, we have \( \|x\| \|\mu\| = \|\lambda\| \).

\[
\text{Theorem 1.} \quad \text{Let} \ E \ \text{be a super-reflexive Banach space and set} \ A = \mathcal{B}(E). \ \text{Then} \ A \ \text{is Arens regular, and} \ A'' \ \text{is isometrically a subalgebra of} \ \mathcal{B}(F) \ \text{for some super-reflexive Banach space} \ F.
\]

**Proof.** Pick \( p \in (1, \infty) \) and set \( F = L = (l^p(E))_{\mathcal{U}} \) for a suitable \( \mathcal{U} \), so by Proposition 5, \( \pi : F \otimes F' \to B(E)' \) is a surjection. The last remark in the proof of Proposition 5 shows that \( B(E)' \) can be identified with \( F \otimes F' \) quotiented by the kernel of \( \pi \). Thus \( \pi' : B(E)' \to B(F)' \) is both an anti-homomorphism and an isomorphism onto its range. As \( \|\pi'\| \leq 1 \), this isomorphism is an isometry. Hence, by the remark above, composing with the isometric anti-homomorphism \( B(F') \to B(F) \), we have an isometric homomorphism \( B(E)' \to B(F) \) for each Arens product. In particular, \( B(E) \) is Arens regular.

\[
\text{Corollary 1.} \quad \text{Let} \ E \ \text{be a super-reflexive Banach space and} \ C \ \text{a subalgebra of} \ \mathcal{B}(E). \ \text{Then} \ C'' \ \text{can be identified with a subalgebra of} \ \mathcal{B}(F) \ \text{for some super-reflexive} \ F.
\]

**Proof.** By Theorem 1, we can a super-reflexive \( F \) so that \( B(E)' \) is isometrically
arens regularity of the algebra of operators

7

identified with a subalgebra of \( \mathcal{B}(F) \). As \( \mathcal{C}'' \) is the weak\(^*\)-closure of \( \mathcal{C} \) in \( \mathcal{B}(E)' \), we can thus identify \( \mathcal{C}'' \) with a subalgebra of \((\text{the image of } \mathcal{B}(E)'' \text{ in } \mathcal{B}(F))\).

If \( E \) is super-reflexive, then \( \mathcal{B}(E)'' \) is a subalgebra of \( \mathcal{B}(F) \) for a super-reflexive \( F \), hence \( \mathcal{B}(E)'' \) is Arens regular, and \( \mathcal{B}(E)''' \) is a subalgebra of \( \mathcal{B}(G) \) for a super-reflexive \( G \), by Corollary 1. Hence we see that every even dual of \( \mathcal{B}(E) \) is Arens regular.

Thus we have, almost completely, extended the result that \( \mathcal{B}(l^2) \) is Arens regular (as it is a \( C^* \)-algebra). Indeed, let \( \mathcal{A} \) be a \( C^* \)-algebra, so that \( \mathcal{A} \) is a closed subalgebra of \( \mathcal{B}(H) \) for some Hilbert space \( H \). Then \( \mathcal{A}'' \) is a subalgebra of \( \mathcal{B}(\hat{H}) \) for some larger Hilbert space \( \hat{H} \), and \( H \) can be isometrically embedded in \( \hat{H} \).

However, we leave open the question of whether the second dual of \( \mathcal{B}(E) \) is semi-simple or not. By considering the \( C^* \)-algebra case, there is again some belief that \( \mathcal{B}(l^p)'' \) should be semi-simple for \( 1 < p < \infty \).

4. Banach spaces \( E \) for which \( \mathcal{B}(E) \) is not Arens regular

Can we find a Banach space \( E \) which is reflexive but for which \( \mathcal{B}(E) \) is not Arens regular. Young [12, p. 108] showed that we can; in this section we will present a shorter proof of this fact for a wide class of Banach spaces.

**Proposition 6.** Let \( \mathcal{A} \) be a Banach algebra. Then the following are equivalent:

(i) \( \mathcal{A} \) is Arens regular;

(ii) for each \( \lambda \in \mathcal{A}' \), the map \( a \mapsto \lambda.a, \mathcal{A} \to \mathcal{A}' \) is weakly compact;

(iii) for each pair of bounded sequences \( (a_n), (b_m) \) in \( \mathcal{A} \) and each \( \lambda \in \mathcal{A}' \),

\[
\lim_n \lim_m \langle \lambda, a_n b_m \rangle = \lim_m \lim_n \langle \lambda, a_n b_m \rangle
\]

when both iterated limits exist.

**Proof.** See [11, Theorem 1.4.11].

**Proposition 7.** Let \( E \) be a Banach space. Then the following are equivalent:

(i) \( E \) is reflexive;

(ii) for each bounded sequence \( (x_n) \) in \( E \) and each bounded sequence \( (\lambda_m) \) in \( E' \), if \( \lim_n \lim_m (\lambda_m, x_n) \) and \( \lim_m \lim_n (\lambda_m, x_n) \) both exist, then they are equal.

Further, the following are equivalent:

(i) \( E \) is not reflexive;

(ii) for each \( \theta \in (0, 1) \), there exist sequences \( (x_n) \) in \( E \) and \( (\lambda_m) \) in \( E' \) such that for each \( n \), \( \|x_n\| = \|\lambda_n\| = 1 \), and for all \( n \) and \( m \) we have

\[
\langle \lambda_m, x_n \rangle = \begin{cases} \theta & m \leq n, \\ 0 & m > n. \end{cases}
\]

**Proof.** The second equivalence is [7, Theorem I.6.1], which also gives (2) implies (1) in the first equivalence. In the first equivalence (1) implies (2) follows from the weak compactness of the unit ball in \( E \).

**Proposition 8.** Let \( E \) be a Banach space such that \( \mathcal{B}(E) \) is Arens regular. Then \( E \) is reflexive.
Theorem 2. Let $F$ be a non-reflexive Banach space and $(M_n, \| \cdot \|_n)$ a sequence of Banach spaces such that for some $\epsilon > 0$ and each finite dimensional subspace $M$ of $F$, $M$ is $(1 + \epsilon)$-isomorphic to some subspace of some $M_n$. Let $E = \oplus_{n=1}^\infty M_n$ as a linear space and suppose that $E$ is a normed space with a norm $\| \cdot \|$ which satisfies:

(i) there exists $C$ such that if $(x_n)$ and $(y_n)$ are sequences in $E$ with $\|y_n\|_n \leq ||x_n||_n$ for all $n$, then $\|(y_n)\| \leq C \|(x_n)\|$;

(ii) there exists $K_1$ such that if $m \in \mathbb{N}$ and $(x_n) \in E$ with $x_n = 0$ for all $n \neq m$, then $\|(x_n)\| \leq K_1 \|x_m\|_m$;

(iii) there exists $K_2$ such that for all $(x_n) \in E$ and $m \in \mathbb{N}$, $\|x_m\| \leq K_2 \|(x_n)\|$.

Let $\hat{E}$ be the norm-completion of $E$, then $B(\hat{E})$ is not Arens regular.

Note that (2) and (3) say that the canonical projections $E \to M_n$, and the canonical embeddings $M_n \to E$, are continuous and uniformly bounded. This (essentially) ensures that $F$ is crudely finitely representable in $E$ (which implies that $\hat{E}$ is not super-reflexive).

Proof. First note that condition (3) on the norm implies that if $\lambda \in M'_N$, then we can define $\hat{\lambda} \in E'$ by

$$\langle \hat{\lambda}, (x_n) \rangle = \langle \lambda, x_N \rangle \quad ((x_n) \in E),$$

so that $\|\hat{\lambda}\| \leq K_2 \|\lambda\|$ and we can view $M'_N$ as a subspace of $E'$ (note also that $E' = \hat{E}'$).

Choose $(x_n) \subseteq F, (\lambda_m) \subseteq F'$ as in Proposition 7, for some $0 < \theta < 1$. For each $i$ let $N_i = \text{span}\{x_1, \ldots, x_i\}$, so that there exists $n(i) \in \mathbb{N}$ with $N_i$ being $(1 + \epsilon)$-isomorphic to $M_{n(i)}$. Then $N'_i$ is $(1 + \epsilon)$-isomorphic to $M'_{n(i)}$, and we can regard, for each $m, \lambda_m$ as being in $N'_i$ by restriction.

Thus for some increasing sequence $(n(i))_{i=1}^\infty$ we can find, for each $i$ and each $j$ such that $1 \leq j \leq n(i)$, $x^{(i)}_j \in M_{n(i)}$ and $\lambda^{(i)}_j \in M'_{n(i)}$ so that, if $1 \leq k \leq n(i)$, then

$$\langle \lambda^{(i)}_j, x^{(i)}_k \rangle = \begin{cases} \theta & j \leq k, \\ 0 & j > k. \end{cases}$$

and we have $(1 + \epsilon)^{-1} \leq \|x^{(i)}_j\|_{n(i)} \leq (1 + \epsilon)$ and $(1 + \epsilon)^{-1} \leq \|\lambda^{(i)}_j\|_{n(i)} \leq (1 + \epsilon)$.

For each $N \in \mathbb{N}$, define $T_N : E \to E$ by setting, for $(x_n) \in E$, $T_N(x_n) = (y_n)$ where

$$y_n = \begin{cases} \frac{1}{\theta} \langle \lambda^{(i)}_j, x_{n(i)} \rangle x^{(i)}_n & n(i) = n \geq N, \\ 0 & \text{otherwise}. \end{cases}$$

Thus for all $n, \|y_n\| \leq (1 + \epsilon)^2 \|x_n\|$. By condition (1) on the norm, $T_N$ is continuous, and so $T_N$ extends to a member of $B(\hat{E})$. Also, the family $(T_N)$ is bounded.

Then for $N, M \in \mathbb{N}$ and $(x_n) \in E$, let $T_N(x_n) = (y_n)$ and $T_M(y_n) = (z_n)$ so that

$$z_n = \begin{cases} \theta \langle \lambda^{(i)}_j, x_{n(i)} \rangle x^{(i)}_M & n(i) = n \geq N, M \leq N, \\ 0 & \text{otherwise}. \end{cases}$$

Hence $T_M T_N = 0$ for $M > N$.

Then if $M \leq N$, we have $T_M T_N(x^{(i)}_j) = \theta^2 x^{(i)}_M$ if $j \geq M$ or 0 otherwise. Thus

$$\langle \lambda^{(i)}_j, T_M T_N(x^{(i)}_j) \rangle = \theta \text{ if } j \geq M \text{ or } 0 \text{ otherwise.}$$

Via condition (2) on the norm,
(x^{(j)}_j) is a bounded sequence in \( \hat{E} \); by the remark at the start of the proof, \( (\lambda^{(j)}_1) \) is a bounded sequence in \( \hat{E}' \). Thus we can define \( \lambda \in \mathcal{B}(E)' \) by

\[
\langle \lambda, T \rangle = \lim_{\mathcal{U}} \langle \lambda^{(j)}_1, T(x^{(j)}_j) \rangle \quad (T \in \mathcal{B}(E))
\]

for some non-principal ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \).

Then \( \langle \lambda, T_M T_N \rangle = 0 \) if \( M \leq N \) or 0 if \( M > N \). Thus \( \mathcal{B}(\hat{E}) \) is not Arens regular in light of Proposition 6.

**Corollary 2.** If \( p \in (1, \infty) \) and \( E = l^p(\oplus_{n=1}^\infty l^1_n) \), so

\[
E = \left\{ (x_n) : \left( \sum_{n=1}^\infty \|x_n\|^p \right)^{1/p} < \infty \text{ and } \forall n, x_n \in l^1_n \right\},
\]

then \( E \) is reflexive, but \( \mathcal{B}(E) \) is not Arens regular.

**Proof.** It is easy to see that if \( M_n = l^1_n \), then for each finite dimensional subspace \( M \) of \( l^1 \) and each \( \epsilon > 0 \), \( M \) is \( (1 + \epsilon) \)-isomorphic to some subspace of some \( M_n \). Clearly the \( p \) norm satisfies conditions (1), (2) and (3), so we are done.

5. Towards a converse

The following is sketched in [11, Section 1.7.8], and was first proved in [4]:

**Theorem 3.** Let \( E \) and \( F \) be Banach spaces and \( T : E \to F \) be a weakly compact linear map. Then there exists a reflexive Banach space \( Z \) and linear maps \( S : E \to Z \), \( R : Z \to F \) such that \( T = R \circ S \). Further, we can choose \( Z \), \( S \) and \( R \) so that \( S \) has dense range and the same norm and kernel as \( T \), and \( R \) is injective and norm-decreasing.

**Proposition 9.** Let \( \mathcal{A} \) be a Banach algebra. Then the following are equivalent:

(i) \( \mathcal{A} \) is Arens regular;

(ii) for each \( \lambda \in \mathcal{A}' \), there exists a reflexive Banach space \( Z \) and continuous linear maps \( \phi : \mathcal{A} \to Z \), \( \psi : \mathcal{A} \to Z' \) such that we have \( \langle \lambda, ab \rangle = \langle \psi(a), \phi(b) \rangle \) for each \( a, b \in \mathcal{A} \).

**Proof.** To show that (2) implies (1), let \( (a_n), (b_m) \) be bounded sequences in \( \mathcal{A} \) and pick \( \lambda, X, \phi \) and \( \psi \) as in the hypotheses. Assume that \( \lim_n \lim_m \langle \lambda, a_n b_m \rangle \) and \( \lim_m \lim_n \langle \lambda, a_n b_m \rangle \) both exist, so by Proposition 6 we need to show that they are equal. However,

\[
\lim_n \lim_m \langle \lambda, a_n b_m \rangle = \lim_m \lim_n \langle \psi(a_n), \phi(b_m) \rangle \quad \text{and} \quad \lim_m \lim_n \langle \lambda, a_n b_m \rangle = \lim_m \lim_n \langle \psi(a_n), \phi(b_m) \rangle
\]

so as \( (\psi(a_n)) \) and \( (\phi(b_m)) \) are bounded sequences, we are done by Proposition 7.

Conversely, by Theorem 6, for \( \lambda \in \mathcal{A}' \) the map \( a \mapsto \lambda a \) is weakly compact. Thus by Proposition 3, there is a reflexive Banach space \( Z \) and maps \( \phi : \mathcal{A} \to Z \) and \( R : Z \to \mathcal{A}' \) such that \( R(\phi(a)) = a \lambda \) for each \( a \in \mathcal{A} \). Let \( \psi = R' \circ \kappa_A \) so \( \psi : \mathcal{A} \to Z' \) and for \( a, b \in \mathcal{A} \) we have

\[
\langle \psi(a), \phi(b) \rangle = \langle (R' \circ \kappa_A)(a), \phi(b) \rangle = \langle (R'' \circ \kappa_Z \circ \phi)(b), \kappa_A(a) \rangle
\]
\[ \langle (\kappa A' \circ R \circ \phi)(b), \kappa A(b) \rangle = \langle (R \circ \phi)(b), a \rangle = \langle b, \lambda, a \rangle = \langle \lambda, ab \rangle. \]

This provides an alternative way to show that \( B(E) \) is Arens regular if \( E \) is super-reflexive. Indeed, as above, we use the space \( Z = (l^2(E))_U \) for a suitable ultrafilter \( U \), and \( \psi(T) = (T(x_{n,\alpha})) \) for a suitably chosen \( (x_{n,\alpha}) \) in \( Z \), depending on \( \lambda \).

Conversely, suppose that \( B(E) \) is Arens regular. Then we have, for each \( \lambda \in B(E)' \), some unknown reflexive space \( Z \) as in the theorem. How to relate this to, say, an ultrapower of \( E \), is a somewhat open question.

6. Conclusion

We leave open some interesting algebraic questions about \( B(E)'' \) — principally whether \( B(E)' \) is semi-simple, even in the case \( E = l^p, p \in (1, \infty), p \neq 2 \). We also make little progress in the direction of seeing if \( B(E) \) Arens regular implies that \( E \) is super-reflexive.

Acknowledgements. This work was undertaken while the author was studying for a PhD. at Leeds University, with financial support from the EPSRC. The author wishes to thank his supervisors Charles Read and H. Garth Dales for much advice and encouragement.

References


Matthew Daws
School of Mathematics
University of Leeds
LEEDS
LS2 9JT
matt.daws@cantab.net