Reading course: Banach spaces and algebras (MATH5002)

Here are some further exercises, loosely collected under the same headings I have used elsewhere. Some questions might be harder than I expect!

1 Revision of normed spaces; dual spaces; Hahn-Banach

2 Weak and weak*-topologies; second duals; geometric forms of Hahn-Banach; Krein-Milman

- Let E be a normed space, and let (x_n) be a sequence in E which converges weakly to x. Show that we can find a sequence (y_n) in E, which converges to x in norm, and with y_n in the convex hull of $\{x_1, x_2, \cdots, x_n\}$, for each n.

- Let E be an infinite dimensional normed space. Show that the weak closure of the unit sphere $S = \{x \in E : \|x\| = 1\}$ is precisely the closed unit ball of E.

- Give $[0, 1]$ Lebesgue measure (though this question works for any "reasonable" measure space). Let $1 < p < \infty$. Show that the extreme points of the closed unit ball of $L^p([0, 1])$ is the unit sphere, $\{f \in L^p([0, 1]) : \|f\|_p = 1\}$.

- What are the extreme points of the closed unit ball of $L^1([0, 1])$?

- What are the extreme points of the closed unit ball of ℓ^1?

3 Baire category, Open Mapping, Closed Graph, Uniform boundedness theorems

- Let E be a vector space, and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on E such that E is Banach for either norm. Let τ_1 and τ_2 be the corresponding topologies on E, and suppose that $\tau_1 \subseteq \tau_2$ (that is, if a subset of E is open for $\|\cdot\|_1$ then it's open for $\|\cdot\|_2$). Show that $\tau_1 = \tau_2$.

- Let $1 \leq p \leq \infty$. Let $(a_{i,j})$ be an infinite matrix, and suppose that $(Ax)_i = \sum_j a_{i,j} x_j$ is an element of ℓ^p, whenever $x = (x_j) \in \ell^p$. Show that A defines a bounded linear map on ℓ^p.

- Let E be a normed space, let (f_n) be a sequence in E^* which converges weak* to $f \in E^*$. Show that there is $K > 0$ such that $\|f_n\| \leq K$ for all n.

4 Basics of Banach algebras; constructions; group of units

- Let A be an algebra. Suppose that there are $a \in A$ and a sequence (b_n) in A, each b_n is non-zero, such that $ab_n = nb_n$ for all n. Show that there is no algebra norm on A.

  Use this result to show that $C(\mathbb{R})$, the algebra of all continuous functions on $\mathbb{R}$, cannot be given an algebra norm.

- Let A be a commutative Banach algebra such that for each $a \in A$, there is $n \in \mathbb{N}$ with $a^n = 0$. Prove that there is $N \in \mathbb{N}$ with $a^N = 0$ for all $a \in A$. *Hint:* Baire Category.

  Can you prove the same for a non-commutative Banach algebra?

5 Spectrum; Characters; Gelfand Theory

- Let A be a Banach algebra, and let $a, b \in A$. Show that $\mathrm{Sp}(ab) \setminus \{0\} = \mathrm{Sp}(ba) \setminus \{0\}$ (this is probably in the book– check that you understand the proof!)

  Can it happen that $\mathrm{Sp}(ab) \neq \mathrm{Sp}(ba)$?

  Give a proof (by contradiction!) that $ab - ba$ cannot be a multiple of 1 (assuming that A is unital).

- Find examples of 2×2 complex matrices A, B such that $\rho(AB) > \rho(A)\rho(B)$ and $\rho(A + B) > \rho(A) + \rho(B)$. *Hint:* Remember that $\mathrm{Sp}(A)$ is just the collection of eigenvalues of A.

- Let A be a Banach algebra, and suppose that for $C > 0$, we have that $\|a\| \leq C\rho(a)$ for all $a \in A$. Show that A is commutative.

  Hint: Let $a, b \in A$, and define $f(z) = e^{-za}be^{za}$, for $z \in \mathbb{C}$. Prove that f is analytic and constant. Deduce the result from this.

6 Commutative Banach algebras; holomorphic functional calculus

- Let A be a Banach algebra, let $a \in A$, and suppose that 0 and ∞ belongs to the same unbounded component of $\mathbb{C} \setminus \mathrm{Sp}(a)$. Show that:

  1. $a = e^b$ for some $b \in A$;
  2. for any $n \in \mathbb{N}$ there is $c \in A$ with $c^n = a$.
  3. for $\epsilon > 0$, we can find a complex polynomial P such that $\|a^{-1} - P(a)\| < \epsilon$.

  Show that if M is an $n \times n$ invertible matrix, then $M = e^L$ for some matrix L.

7 C*-algebras; continuous functional calculus

1. Let A be a C*-algebra, and let $a \in A$. Supposing that a is normal, show that $\mathrm{Sp}(a^*a) = \{|\lambda|^2 : \lambda \in \mathrm{Sp}(a)\}$. Is this always true if a is not normal?

2. Let X be a compact Hausdorff space, let $A = C(X)$ with the usual norm. Let $\| \cdot \|_0$ be some other algebra norm on A (we do not assume that $(A, \| \cdot \|_0)$ is Banach). Show that:

   (a) Let B be the completion of $(A, \| \cdot \|_0)$, so that B is a Banach algebra. Let E be the collection of all characters φ on B, restricted to the algebra A. Show that E forms a non-empty, closed subset of the character space of A (which we identify with X).

   (b) Using Urysohn's Lemma, show that if $E \neq X$, then there are non-zero $a, b \in A$ with $ab = 0$ but with $\varphi(a) = 1$ for all $\varphi \in E$. Show that this leads to a contradiction; so $E = X$.

   (c) Deduce that for each $f \in A$, we have $\|f\| = \rho_B(f)$.

(d) Deduce that $\|f\| \leq \|f\|_0$ for each $f \in A$.

3. Let X, Y be compact Hausdorff spaces, and let $T : C(X) \to C(Y)$ be a unital homomorphism. Show that there is a continuous map $f : Y \to X$ such that $T(a) = a \circ f$ for all $a \in C(X)$.

   If you know what the words mean: Show that the category of compact Hausdorff spaces with continuous maps is anti-equivalent to the category of unital commutative C*-algebras with unital homomorphisms.

4. In the book, Corollary 2.19 is stated for C*-algebras A and B. Prove that the result still holds if A is merely a Banach $*$-algebra.

5. Consider the Hilbert space $H = \ell^2 = \ell^2(\mathbb{N})$, with the standard orthonormal basis (e_n) (so $e_1 = (1, 0, 0, \cdots), e_2 = (0, 1, 0, \cdots)$ and so forth). Let (a_n) be a sequence of complex numbers. Show that there is a bounded linear operator T on H with $T(e_n) = a_n e_n$ for all n, if and only if (a_n) is a bounded sequence. Show that T is a normal operator. In terms of the sequence (a_n), determine when T is: (i) unitary, (ii) self-adjoint.

6. We continue with the same notation. For T defined by a sequence (a_n), determine the spectrum of T.

7. We continue with the same notation. Let A be the C*-algebra (in $\mathcal{B}(H)$) generated by T. Show that:

   (a) As $T^*T = TT^*$, we can talk about a "polynomial in T and T^*". Show that the collection of all such polynomials, $\mathbb{C}[T, T^*]$ is dense in A. *Hint:* By definition, A is the smallest C*-algebra containing T. Show that any C*-algebra containing T contains $\mathbb{C}[T, T^*]$, and then show that the closure of $\mathbb{C}[T, T^*]$ is a C*-algebra.

   (b) It follows that A is commutative. Using the results of Section 6.4 in the book, show that if $\varphi \in \Phi_A$, then φ is uniquely determined by the value $\varphi(T)$.

   (c) By Commutative Gel'fand–Naimark (Theorem 6.24) A is isomorphic to $C(\Phi_A)$. Show that the compact Hausdorff spaces Φ_A and $\mathrm{Sp}(T)$ are homeomorphic. *Hint:* Show firstly that the map $\Phi_A \to \mathrm{Sp}(T); \varphi \mapsto \varphi(T)$ is well-defined and injective. Now prove that it is surjective (and then appeal to the result that a continuous bijection between compact, Hausdorff spaces is a homeomorphism).

8. We continue with the same notation. Let f be a continuous function on the spectrum of T, so by the Continuous Functional Calculus, we can make sense of $f(T)$. Now consider the map $\Phi : C(\mathrm{Sp}(T)) \to \mathcal{B}(H)$ which maps f to S, where

$$S(e_n) = f(a_n)e_n \qquad \text{for all } n.$$

   Using the previous two questions, show that this is well-defined (that is, $f(a_n)$ makes sense, and that S is bounded). Show that Φ is a unital $*$-homomorphism with $\Phi(Z) = T$. Conclude that Φ agrees with the Continuous Functional Calculus. *Remark:* So in this case, we have a very concrete picture of what the Continuous Functional Calculus actually is!

8 Representation theory; modules; radicals; uniqueness of norm

- I find the discussion in Section 5.3 hard to follow. Check *carefully* that you understand why the definition of the Radical given for commutative algebras on page 193 agrees with the general definition give on page 232.

- This one is in the book, but let's try to give a nicer proof. Firstly, check that you understand that a unital commutative Banach algebra A is semisimple if and only if the Gelfand transform $\mathcal{G} : A \to C(\Phi_A)$ is injective.

  Theorem: Let A and B be unital commutative Banach algebras, with B semisimple. Then any unital homomorphism $T : A \to B$ is continuous.

  Here is a strategy for proving this:

  - Let φ be a character on B. Show that $\phi = \varphi \circ T$ is a character on A, and hence conclude that ϕ is bounded.
  - Let (a_n) be a sequence in A converging to 0, and suppose that $b = \lim_n T(a_n)$ exists in B. Show that $\mathcal{G}(b) = 0$, and hence that $b = 0$.
  - Use the closed graph theorem to conclude that T is continuous.
  - Now write all that up neatly!

- Check that you understand why this result implies that a unital commutative semisimple Banach algebra has a unique (complete algebra) norm.

9 Applications and examples to group algebras

10 More additional questions on later parts of the course

- Let u be a unitary element in a unital C*-algebra A. Suppose that $\mathrm{Sp}(u)$ is not the whole of the unit circle. Show that there is $a \in A$ with $a^* = a$ and $u = \exp(ia)$. *Hint:* Functional calculus.

- Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$, and let $u \in C(\mathbb{T})$ be the element $u(z) = z$. Show that there is no $a \in C(\mathbb{T})$ with $u = \exp(ia)$.

- Let $T, S \in \mathcal{B}(H)$ satisfy $T^*T \leq S^*S$. (Recall that for $A, B \in \mathcal{B}(H)$ we define $A \leq B$ to mean that $(Ax|x) \leq (Bx|x)$ for all $x \in H$). Show that there exists $U \in \mathcal{B}(H)$ with $T = US$ and $\|U\| \leq 1$. *Hint:* Show that $U : S(H) \to H; S(x) \mapsto T(x)$ is well-defined, linear, and bounded. Extend U to all of H by orthogonal decomposition.