Closed ideals in the Banach algebra of operators on classical non-separable spaces

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Abstract

The classical result of Gohberg, Markus and Feldman states that, when E is one of the classical Banach sequence spaces $E = l^p$ for $1 \leq p < \infty$ or $E = c_0$, the only closed, two-sided, non-trivial ideal in $\mathcal{B}(E)$, the Banach algebra of operators on a Banach space E, is $\mathcal{K}(E)$, the ideal of compact operators. Gramsch and Luft completely classified the closed, two-sided ideals in $\mathcal{B}(H)$ for an arbitrary Hilbert space H through the idea of κ-compact operators, for infinite cardinals κ. This paper presents an extension of this result to the non-separable versions of l^p and c_0.

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1. *Introduction*

The structure of closed, two-sided ideals (from now on, all our ideals will be two-sided) of the Banach algebra of operators on a Banach space E, written $\mathcal{B}(E)$, seems to be a little-understood area. The finite-rank operators, $\mathcal{F}(E)$, form the smallest non-zero ideal in $\mathcal{B}(E)$, and thus their closure, $\mathcal{A}(E)$ (the approximable operators) forms the smallest non-zero, closed ideal in $\mathcal{B}(E)$. For classical sequence spaces (or, more generally, spaces with the approximation property), this closed ideal is equal to the closed ideal of compact operators, $\mathcal{K}(E)$. As first shown in [**2**], in the special case where $E = l^p$ for $1 \leq p < \infty$, or $E = c_0$, this is the only (non-trivial) closed ideal in $\mathcal{B}(E)$. It seems to be unknown if this is true for any other Banach spaces. See [**7**] for a survey of known results.

In [**9**] and [**3**], Gramsch and Luft independently extended this result to non-separable Hilbert spaces (via the introduction of κ-compact operators, for cardinals κ — see below for the precise statement). This paper presents a direct generalisation of this result to non-separable versions of l^p and c_0. We shall see that, unlike the separable case, there seems to be a difference between the l^1 case and the other cases.

2. *Non-separable Banach spaces*

We shall sketch the theory of unconditional bases in non-separable Banach spaces (called *extended unconditional bases* in [**11**, Chapter 17]). The proofs of these results follow in a simple way from the standard theory of unconditional bases, as laid out in, for example, [**8**].

When X is a topological vector space and $(x_\alpha)_{\alpha \in I}$ is a family in X, we say that (x_α)

sums unconditionally to $x \in X$, written $x = \sum_{\alpha\in I} x_\alpha$, if, for each open neighbourhood U of x, there is a finite $A \subseteq I$ such that, if $B \subseteq I$ is finite and $A \subseteq B$, then $\sum_{\alpha\in B} x_\alpha \in U$. This definition agrees with the usual one for sequences.

For a Banach space E, a family of vectors $(e_\alpha)_{\alpha\in I}$ is an *unconditional basis* for E if, for each $x \in E$, there is a unique family of scalars (a_α) such that

$$x = \sum_{\alpha\in I} a_\alpha e_\alpha,$$

with summation interpreted as above. Again, if I is countable, then E is separable, and this definition agrees with the usual one of an unconditional basis.

As in the separable case, we can define bounded linear functionals $e_\alpha^* \in E'$ such that

$$\langle e_\alpha^*, e_\beta\rangle = \begin{cases} 1 & \alpha = \beta, \\ 0 & \alpha \neq \beta. \end{cases}$$

Note that, for $x \in E$ and $\mu \in E'$, we write $\langle \mu, x\rangle = \mu(x)$. If $\|e_\alpha\| = 1$ for each α, then the unconditional basis (x_α) is *normalised*. In this case, the family (e_α^*) is bounded.

For each $A \subseteq I$, we can define a map $P_A : E \to E$,

$$P_A(x) = \sum_{\alpha\in A} \langle e_\alpha^*, x\rangle e_\alpha.$$

A closed-graph argument shows that P_A is bounded, so that P_A is a projection onto the subspace $P_A(E)$. For $x \in E$, we define the *support* of x to be

$$\mathrm{supp}\,(x) = \{\alpha \in I : \langle e_\alpha^*, x\rangle \neq 0\}.$$

Thus $P_A(E)$ is the subspace of vectors in E with support contained in A. From our meaning of summation, we can see that the support of x is always a countable subset of I.

A uniform boundedness argument shows that the family $(P_A)_{A\subseteq I}$ is bounded, and by a standard re-norming, we may suppose that $\|P_A\| = 1$ for each $A \subseteq I$ (and so, in particular, that $\|e_\alpha^*\| = 1$ for each $\alpha \in I$). Henceforth we shall suppose that an unconditional basis is normalised and that $\|P_A\| = 1$ for each $A \subseteq I$.

The family $(e_\alpha^*)_{\alpha\in I}$ forms an unconditional basis for the closure of its span in E'. When this closure is the whole of E', we say that (e_α) is *shrinking*. For an operator $T \in \mathcal{B}(E)$, define its *adjoint* $T' \in \mathcal{B}(E')$ by

$$\langle T'(\mu), x\rangle = \langle \mu, T(x)\rangle \qquad (x \in E, \mu \in E'),$$

so that $\|T'\| = \|T\|$. Then one can show that (e_α) is shrinking if and only if

$$\inf\{\|P_A'(\mu)\| : A \subseteq I, |I \setminus A| < \infty\} = 0 \tag{2·1}$$

for each $\mu \in E'$, where $|A|$ is the cardinality of A.

For an infinite set I, write $I^{<\infty} = \{A \subseteq I : |A| < \infty\}$. Then we define

$$c_0(I) = \{(x_i)_{i\in I} : \forall\, \epsilon > 0, \{i \in I : |x_i| \geq \epsilon\} \in I^{<\infty}\}$$

so that $c_0(I)$ is a Banach space with the supremum norm. Similarly, for $1 \leq p < \infty$, we define

$$l^p(I) = \left\{ (x_i)_{i\in I} : \|(x_i)\|_p := \Big(\sum_{i\in I} |x_i|^p\Big)^{1/p} < \infty \right\}.$$

Then the family of vectors $(e_i)_{i\in I}$, defined such that $e_i = (\delta_{ij})_{j\in I}$, is an unconditional basis for $c_0(I)$ and $l^p(I)$. Here δ_{ij} denotes the Kronecker delta. In all cases except $p = 1$, this basis is also shrinking; when $p = 1$, (e_i^*) spans $c_0(I) \subseteq l^\infty(I) = l^1(I)'$.

The *density character* of a Banach space E is the least cardinality of a dense subset of E. Thus E is separable if and only if E has density character $\aleph_0$.

3. *Generalisation of compact operators*

These definitions are given in [**9**]. We follow the presentation of cardinal numbers as given in, for example, [**4**] or [**6**]. In particular, the cardinal numbers are ordinal numbers α such that, if β is an ordinal equipotent with α, then $\alpha \leq \beta$. If κ is a cardinal number, then κ^+ is the *successor* of κ, that is, the least cardinal strictly greater than κ. If κ is not the successor of any cardinal, then κ is a *limit cardinal*; for example $\aleph_0$ is a limit cardinal.

For Banach spaces E and F, we write $\mathcal{B}(E, F)$ for the space of bounded linear operators between E and F. For $t > 0$ we write

$$E_{[t]} = \{x \in E : \|x\| \leq t\},$$

so that $E_{[1]}$ is the closed unit ball of E.

For an infinite cardinal κ and $T \in \mathcal{B}(E, F)$, we say that T is *κ-compact* if, for each $\epsilon > 0$, we can find a subset X of $E_{[1]}$ with $|X| < \kappa$, and such that

$$\inf\{\|T(x - y)\| : y \in X\} \leq \epsilon$$

for each $x \in E_{[1]}$. We write $\mathcal{K}_\kappa(E, F)$ for the set of κ-compact operators. As this definition does not have a useful meaning for finite cardinals, we shall henceforth assume that any cardinals are infinite. In [**9**] it is shown that $\mathcal{K}_\kappa(E, F)$ is a closed operator ideal in the sense of Pietsch (see [**10**]); that is, we have the following.

PROPOSITION 3·1. *Let E and F be Banach spaces. Then $\mathcal{K}_\kappa(E, F)$ is a closed subspace of $\mathcal{B}(E, F)$. Let D and G be Banach spaces, and $T \in \mathcal{K}_\kappa(E, F)$, $S \in \mathcal{B}(D, E)$ and $R \in \mathcal{B}(F, G)$. Then $RTS \in \mathcal{K}_\kappa(D, G)$.*

We define $\mathcal{K}_\kappa(E)$ to be $\mathcal{K}_\kappa(E, E)$. Then $\mathcal{K}_\kappa(E)$ is a closed ideal in $\mathcal{B}(E)$. The $\aleph_0$-compact operators are just the usual compact operators, so that $\mathcal{K}_{\aleph_0}(E, F) = \mathcal{K}(E, F)$. For higher cardinals, there is an easier description of κ-compact operators, subject to a technicality. Recall that for a cardinal κ, the *cofinality* of κ, $\mathrm{cf}(\kappa)$, is the least ordinal $\sigma \leq \kappa$ such that there is an order-preserving map $f : \sigma \to \kappa$ which is not bounded above. See, for example, [**6**, Chapter 9, Section 2]. Then $\mathrm{cf}(\kappa)$ is a cardinal; if $\mathrm{cf}(\kappa) = \kappa$ we say that κ is *regular*, otherwise κ is *singular*. In particular, if κ is singular, then κ is a limit cardinal.

LEMMA 3·2. *Let κ be a cardinal with $\mathrm{cf}(\kappa) > \aleph_0$ (so that $\kappa > \aleph_0$). Then, if (A_n) is a sequence of sets, each of cardinality less than κ, then $|\bigcup_n A_n| < \kappa$.*

Proof. For each $n \in \mathbb{N}$, let $B_n = \bigcup_{m\leq n} A_m$ so that $(|B_n|)$ is an increasing sequence of cardinals with, for each n, $|B_n| \leq \sum_{m=1}^n |A_m| < \kappa$. As $\mathrm{cf}(\kappa) > \aleph_0$, $(|B_n|)$ is bounded above by some $\sigma < \kappa$. Thus $|\bigcup_n A_n| \leq \sup_n |B_n| \leq \sigma < \kappa$ as required.

LEMMA 3·3. *Let κ be a cardinal with $\mathrm{cf}(\kappa) > \aleph_0$, and let E and F be Banach spaces. Then $T \in \mathcal{B}(E, F)$ is κ-compact if and only if there is a set $A \subseteq E$ with $|A| < \kappa$ and such that $T(A) := \{T(x) : x \in A\}$ is dense in $T(E)$.*

Proof. For $T \in \mathcal{K}_\kappa(E,F)$ and $n \in \mathbb{N}$, let $A_n \subset E_{[1]}$ be a set with $|A_n| < \kappa$ and

$$\inf\{\|T(x-y)\| : y \in A_n\} \leq n^{-1}$$

for each $x \in E_{[1]}$. Then let $B = \bigcup_n A_n$, so, by Lemma 3·2, $|B| < \kappa$, and $T(B)$ is dense in $T(E_{[1]})$. Then we can let $A = \bigcup_{n\in\mathbb{N}} nB$, so that $|A| = |B|$ and $T(A)$ is dense in $T(E)$.

The converse statement is clear.

We write $\mathcal{X}(E,F)$ for the closed operator ideal of $\mathcal{B}(E,F)$ formed by those operators with separable image. Thus $\mathcal{X}(E,F) = \mathcal{K}_{\aleph_1}(E,F)$. The lemma does not hold more generally, for consider $l^1(\aleph_\omega)$, noting that $\mathrm{cf}(\aleph_\omega) = \aleph_0$. As $\aleph_\omega$ is an ordinal, we have $\aleph_\omega = \{\alpha \text{ is an ordinal } : \alpha < \aleph_\omega\}$ and thus, if $\alpha \in \aleph_\omega$, either α is finite, or $\aleph_{n-1} \leq \alpha < \aleph_n$ for some $n \geq 1$. Define $T \in \mathcal{B}(l^1(\aleph_\omega))$ by, for $\alpha \in \aleph_\omega$, $T(e_\alpha) = e_\alpha$ if α is finite, or $T(e_\alpha) = n^{-1}e_\alpha$ if $\aleph_{n-1} \leq \alpha < \aleph_n$. Then T is clearly $\aleph_\omega$-compact, but if A is a dense subset of $T(E_{[1]})$, then $|A| = \aleph_\omega$.

Lemma 3·4. *Let E be a Banach space with density character κ. Then $\mathcal{B}(E) = \mathcal{K}_{\kappa^+}(E)$, and, if $\mathrm{cf}(\kappa) > \aleph_0$, then $\mathcal{K}_\kappa(E) \subsetneq \mathcal{B}(E)$.*

Proof. As E contains a dense subset of cardinality κ, clearly every operator on T is κ^+-compact. If, further, $\mathrm{cf}(\kappa) > \aleph_0$, then by Lemma 3·3, if id_E is κ-compact, then for some $A \subseteq E$ with $|A| < \kappa$, we have that A is dense in E, a contradiction. Thus $\mathcal{K}_\kappa(E)$ is a proper ideal in $\mathcal{B}(E)$.

Recall that for any Banach space E, $T \in \mathcal{B}(E)$ is compact if and only if $T' \in \mathcal{B}(E')$ is compact. This is not true for higher cardinalities, as the identity on l^1 has separable range, but its adjoint is the identity on l^∞, which does not have separable range. The relation between T being κ-compact and T' being κ-compact is only considered for the Hilbert space case in [**9**].

Proposition 3·5. *Let E and F be Banach spaces, let κ be an infinite cardinal, and let $T \in \mathcal{B}(E,F)$. If $T' \in \mathcal{K}_\kappa(F',E')$, then $T \in \mathcal{K}_\kappa(E,F)$.*

Proof. We may suppose that $\kappa > \aleph_0$. Fix $\epsilon > 0$. As $T' \in \mathcal{K}_\kappa(F',E')$, there exists $Y \subset F'_{[1]}$ with $|Y| < \kappa$ such that, for each $\mu \in F'_{[1]}$,

$$\inf\{\|T'(\mu-\lambda)\| : \lambda \in Y\} < \epsilon.$$

For each $\lambda \in Y$, pick $x_\lambda \in E_{[1]}$ with $|\langle T'(\lambda), x_\lambda\rangle| > (1-\epsilon)\|T'(\lambda)\|$. Let $\mathbb{Q}[\imath]$ be the subfield of $\mathbb{C}$ comprising those complex numbers with rational real and imaginary parts. Then let

$$X = \left\{ \sum_{i=1}^n a_i x_{\lambda_i} : n \in \mathbb{N}, (a_i)_{i=1}^n \subseteq \mathbb{Q}[\imath], (\lambda_i)_{i=1}^n \subseteq Y \right\}$$

so that X is dense in $\mathrm{lin}(x_\lambda)_{\lambda\in Y}$. We can write X as

$$X = \bigcup_{n=1}^\infty \Big\{ \sum_{i=1}^n a_i x_{\lambda_i} : (a_i)_{i=1}^n \subseteq \mathbb{Q}[\imath], (\lambda_i)_{i=1}^n \subseteq Y \Big\},$$

so that $|X| \leq \aleph_0 \times \aleph_0 \times |Y| < \kappa$. Let $\delta > 0$ and $y \in E_{[1]}$ be such that $\|T(x-y)\| \geq \delta$ for every $x \in X$. Then $\|T(x-y)\| \geq \delta$ for every $x \in \mathrm{lin}(x_\lambda)_{\lambda\in Y}$. Thus there exists $\mu \in F'$ with $\langle \mu, T(x)\rangle = 0$ for each $x \in X$, with $\langle \mu, T(y)\rangle = \delta$, and with $\|\mu\| \leq 1$. We can then find $\lambda \in Y$ with $\|T'(\mu - \lambda)\| < \epsilon$. Then

$$(1-\epsilon)\|T'(\lambda)\| \leq |\langle \lambda, T(x_\lambda)\rangle| = |\langle \lambda - \mu, T(x_\lambda)\rangle| = |\langle T'(\lambda - \mu), x_\lambda\rangle| < \epsilon,$$

so that $\|T'(\lambda)\| < \epsilon/(1-\epsilon)$. Hence

$$\delta = |\langle \mu, T(y)\rangle| \leq \|T'(\mu)\| \leq \|T'(\mu - \lambda)\| + \|T'(\lambda)\| < \epsilon + \epsilon/(1-\epsilon) < 3\epsilon,$$

if $\epsilon < 1/2$. Consequently, for each $y \in E_{[1]}$, we must have that $\|T(x-y)\| \leq 3\epsilon$ for some $x \in X$. Thus $T(X \cap E_{[1]})$ is 3ϵ-dense in $T(E_{[1]})$, so as $\epsilon > 0$ was arbitrary, we are done.

We now restrict ourselves to spaces with an unconditional basis.

LEMMA 3·6. *Let E have an unconditional basis $(e_i)_{i\in I}$, and let κ be an infinite cardinal. If $A \subseteq I$ with $|A| = \kappa$, then $P_A(E)$ has density character κ, and $P_A \in \mathcal{K}_{\kappa^+}(E) \setminus \mathcal{K}_\kappa(E)$.*

Proof. By taking linear combinations over $\mathbb{Q}[\imath]$, it is clear that $P_A \in \mathcal{K}_{\kappa^+}(E)$, and thus that $P_A(E)$ has density character $\leq \kappa$.

If $\kappa = \aleph_0$ then $P_A \in \mathcal{K}_\kappa(E)$ means that P_A is compact, and thus that $P_A(E)$ is finite-dimensional, which in turn means that A is finite, a contradiction. Thus, if $P_A \in \mathcal{K}_\kappa(E)$, then $\kappa > \aleph_0$, and we can find a set $Y \subseteq P_A(E_{[1]})$ such that $|Y| < \kappa$ and, for each $x \in E_{[1]}$,

$$\inf\{\|P_A(x) - y\| : y \in Y\} \leq 1/2.$$

Then let $B = \bigcup_{y\in Y} \operatorname{supp}(y) \subseteq I$, so that $|B| \leq \aleph_0 \times |Y| < \kappa$. As $|B| < \kappa = |A|$, we can find $\alpha \in A \setminus B$. Then $e_\alpha \in P_A(E)$, and, for each $y \in Y$, $P_B(y) = y$, so that $P_{A\setminus B}(y) = 0$. Thus, for $y \in Y$, we have $1 = \|e_\alpha\| = \|P_{A\setminus B}(e_\alpha)\| = \|P_{A\setminus B}(e_\alpha - y)\| \leq \|e_\alpha - y\|$, a contradiction which shows that $P_A \notin \mathcal{K}_\kappa(E)$, and hence that $P_A(E)$ does have density character κ.

PROPOSITION 3·7. *Let E be a Banach space with an unconditional basis $(e_i)_{i\in I}$. For cardinals $\kappa, \sigma \leq |I|$, we have that $\mathcal{K}_\kappa(E) \neq \mathcal{K}_\sigma(E)$ if $\kappa \neq \sigma$. Furthermore, $\mathcal{K}_{|I|}(E) \neq \mathcal{B}(E)$.*

Proof. We may suppose that $\kappa < \sigma$, so that $\mathcal{K}_\kappa(E) \subseteq \mathcal{K}_\sigma(E)$. By Lemma 3·6, we can find $T \in \mathcal{K}_{\kappa^+}(E) \setminus \mathcal{K}_\kappa(E)$ (indeed, we can have $T = P_A$ for a suitable set $A \subseteq I$), as $\kappa \leq |I|$. Then, as $\kappa^+ \leq \sigma$, $T \in \mathcal{K}_\sigma(E)$ but $T \notin \mathcal{K}_\kappa(E)$.

By Lemma 3·6, applied with $A = I$, we see that id_E is $|I|^+$-compact, but not $|I|$-compact, so that $\mathcal{K}_{|I|}(E) \neq \mathcal{B}(E)$.

Note that this is an improvement on Lemma 3·4, in the case where our Banach space has an unconditional basis.

Thus, when E has an unconditional basis $(e_i)_{i\in I}$, we have a chain of closed ideals in $\mathcal{B}(E)$,

$$\{0\} \subsetneq \mathcal{K}(E) \subsetneq \mathcal{K}_{\aleph_1}(E) \subsetneq \cdots \subsetneq \mathcal{K}_{|I|}(E) \subsetneq \mathcal{K}_{|I|^+}(E) = \mathcal{B}(E).$$

Let E be a Banach space such that every closed ideal J in $\mathcal{B}(E)$ has the form $J = \mathcal{K}_\kappa(E)$ for some cardinal κ. Then we say that $\mathcal{B}(E)$ has *compact ideal structure.* If, further, when κ and σ are infinite cardinals less than or equal to the density character of E, we have $\mathcal{K}_\kappa(E) = \mathcal{K}_\sigma(E)$ only when $\kappa = \sigma$, and that $\mathcal{B}(E) = \mathcal{K}_{\tau^+}(E)$ where τ is the density character of E, then $\mathcal{B}(E)$ has *perfect compact ideal structure.*

Luft and Gramsch proved the following. Recall that the density character of an infinite-dimensional Hilbert space is the same as its Hilbert space dimension, that is, the cardinality of a complete orthonormal system.

THEOREM 3·8. *Let H be an infinite-dimensional Hilbert space with density character κ. Then $\mathcal{B}(H)$ has perfect compact ideal structure. Thus, the closed ideals in $\mathcal{B}(H)$ form an ordered chain*

$$\{0\} \subsetneq \mathcal{K}(H) \subsetneq \mathcal{K}_{\aleph_1}(H) \subsetneq \cdots \subsetneq \mathcal{K}_\kappa(H) \subsetneq \mathcal{K}_{\kappa^+}(H) = \mathcal{B}(H).$$

We shall show that this result holds for $l^p(I)$, $1 \leq p < \infty$ and $c_0(I)$, the $l^2(I)$ case being precisely the theorem above.

4. *Closed ideals in $\mathcal{B}(l^p)$ and $\mathcal{B}(c_0)$*

Our starting point is the classical result that the only closed ideals in $\mathcal{B}(l^p)$ and $\mathcal{B}(c_0)$ are the ideals of compact operators. This was first proved in [**2**], and first shown in a unified manner in [**5**]. We sketch an approach which is essentially laid out in [**10**, Section 5.1], and can also be derived from results in [**8**].

Let $E = l^p$ for $1 \leq p < \infty$, or $E = c_0$. Recall that a *block-basis* in E is a sequence of vectors $(u_n)_{n=1}^\infty$ with finite, pairwise disjoint support, and such that $\max \operatorname{supp}(u_n) < \min \operatorname{supp}(u_{n+1})$ for each $n \in \mathbb{N}$. Recall (see [**8**, Chapter 2.a]) that $\overline{\operatorname{lin}}(u_n)$, the closed span of a normalised block-basis $(u_n)_{n=1}^\infty$, is 1-complemented in E, and that (u_n) is a basic sequence isometrically equivalent to (e_n), the canonical unit basis of E.

LEMMA 4·1. *Let $E = l^p$ for $1 \leq p < \infty$, or $E = c_0$, and let $T \in \mathcal{B}(E) \setminus \mathcal{K}(E)$. Then there exists $\delta > 0$ such that for each $\epsilon > 0$, we can find normalised block-bases $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ so that $\sum_{n=1}^\infty \|T(u_n) - \delta v_n\| < \epsilon$.*

Proof. This is just a combinatorial argument, starting with a sequence (x_n) in E for which $(T(x_n))$ has no convergent subsequences, and extracting a subsequence (u_n) which has the required properties. For further details, see [**10**, Chapter 5].

THEOREM 4·2. *Let $E = l^p$ for $1 \leq p < \infty$, or $E = c_0$. If J is a non-trivial closed ideal in $\mathcal{B}(E)$, then $J = \mathcal{K}(E)$.*

Proof. This is well known, but as we shall essentially generalise this proof later, it is worth presenting the easier case. It is enough to show that, if $T \in \mathcal{B}(E) \setminus \mathcal{K}(E)$, then the ideal generated by T is $\mathcal{B}(E)$. For such a T, find $\delta > 0$, (u_n) and (v_n) using the above lemma, where we shall choose $\epsilon > 0$ later. Let $S : E \to \overline{\operatorname{lin}}(u_n)$ be the isomorphism defined by $S(e_n) = u_n$.

Let $F = \overline{\operatorname{lin}}(v_n)$, and define $R : F \to \overline{\operatorname{lin}}(T(u_n))$ by $R(v_n) = \delta^{-1}T(u_n)$. Then, if $x = \sum_n a_n v_n \in F$, we have

$$\|R(x)\| = \Big\| \sum_n a_n \delta^{-1} T(u_n) \Big\| \leq \delta^{-1}\|T\| \Big\| \sum_n a_n u_n \Big\| = \delta^{-1}\|T\|\|x\|,$$

so that R is bounded. Let P be a projection onto F of norm 1, recalling that F is spanned by a block-basis. If $x = \sum_n a_n v_n \in F$, then

$$\|P(x) - PR(x)\| \le \|x - R(x)\| = \Big\| \sum_n a_n(v_n - \delta^{-1}T(u_n)) \Big\|$$
$$\le \|(a_n)\|_\infty \sum_n \|v_n - \delta^{-1}T(u_n)\| < \delta^{-1}\epsilon\|x\|,$$

noting that, for each of our spaces E, $|a_n| \le \|x\|$ for each n. Now, $\mathrm{id}_F - PR \in \mathcal{B}(F)$, so we see that $\|\mathrm{id}_F - PR\| < \delta^{-1}\epsilon < 1$ if ϵ is sufficiently small. Hence $PR \in \mathcal{B}(F)$ is invertible, so let $U = (PR)^{-1}P \in \mathcal{B}(E)$.

Finally, let $V : F \to E$ be the isomorphism defined by $V(v_n) = e_n$. Then, for $n \in \mathbb{N}$, $R(v_n) = \delta^{-1}T(u_n)$, so that

$$v_n = (PR)^{-1}PR(v_n) = \delta^{-1}(PR)^{-1}PT(u_n) = \delta^{-1}UT(u_n) = \delta^{-1}UTS(e_n)$$

and thus $e_n = V(v_n) = \delta^{-1}VUTS(e_n)$. Hence we see that $VUTS = \delta\,\mathrm{id}_E$, and so the (algebraic) ideal generated by T is all of $\mathcal{B}(E)$.

As noted in [**5**], the key properties of E which we use are that every normalised block-basis is equivalent to the canonical basis of E, and that the span of each such block-basis is complemented. As shown by Zippin (see [**8**, Theorem 2.a.9]), the first of these properties actually characterises l^p, $1 \le p < \infty$, and c_0. Thus there is no obvious way to extend the above theorem, and indeed, we know of no other Banach spaces E for which $\mathcal{K}(E)$ is the only non-trivial closed ideal in $\mathcal{B}(E)$.

Theorem 4·2 is enough to show the following.

PROPOSITION 4·3. *Let I be an uncountable set, let $E = l^p(I)$, for $1 \le p < \infty$, or $E = c_0(I)$, and let $T \in \mathcal{B}(E)$ have separable range, but not be compact. Then the ideal generated by T is $\mathcal{X}(E) = \mathcal{K}_{\aleph_1}(E)$.*

Proof. Note that the ideal generated by T is certainly contained in $\mathcal{X}(E)$, as T has separable range. Let $(x_n)_{n=1}^\infty$ be a dense sequence in $T(E)$, and let $A = \bigcup_n \mathrm{supp}\,(x_n) \subseteq I$, so that A is countable and $P_A T = T$. Since T is not compact, we can choose a sequence (y_n) in $E_{[1]}$ such that $(T(y_n))$ has no convergent subsequence. Let $B = \bigcup_n \mathrm{supp}\,(y_n)$, so that B is countable, and, as $P_B(y_n) = y_n$, $P_A T P_B$ cannot be compact.

We can view $P_A T P_B$ as an operator on $l^p(\mathbb{N})$ or $c_0(\mathbb{N})$, as appropriate. Thus, by Theorem 4·2, the ideal generated by T contains an isomorphism from $P_B(E)$ to $P_A(E)$ of the form

$$S(e_{\beta(n)}) = e_{\alpha(n)} \qquad (n \in \mathbb{N}),$$

where we have enumerations $A = \{\alpha(n) : n \in \mathbb{N}\}$ and $B = \{\beta(n) : n \in \mathbb{N}\}$. We thus see that, if $C \subseteq I$ is countable, then the ideal generated by T contains P_C.

Then as above, if $R \in \mathcal{X}(E)$, then, for some countable $C \subseteq I$, we have $P_C R = R$, and thus R is in the ideal generated by T.

5. *Closed ideal structure of* $\mathcal{B}(l^p(I))$ *and* $\mathcal{B}(c_0(I))$

For the moment we can work with Banach spaces E which merely have an unconditional basis. For an infinite cardinal κ, as $\mathcal{K}_\kappa(E)$ is a closed ideal in $\mathcal{B}(E)$, we can form the quotient $\mathcal{B}(E)/\mathcal{K}_\kappa(E)$, which is a Banach algebra for the norm

$$\|T + \mathcal{K}_\kappa(E)\| = \inf\{\|T + S\| : S \in \mathcal{K}_\kappa(E)\} \qquad (T \in \mathcal{B}(E)).$$

PROPOSITION 5·1. *Let E be a Banach space with an unconditional basis $(e_i)_{i\in I}$. Let κ be an infinite cardinal and let $T \in \mathcal{B}(E)$. Then we have*

$$\begin{aligned}\|T+\mathcal{K}_\kappa(E)\| &= \inf\{\|P_{I\setminus A}T\| : A\subseteq I, |A|<\kappa\}\\ &= \inf\{\|P_{I\setminus A}TP_{I\setminus B}\| : A,B\subseteq I, |A|<\kappa, |B|<\kappa\}.\end{aligned}$$

Further, if $\mathrm{cf}(\kappa)>\aleph_0$, then we can find $A\subseteq I$ with $|A|<\kappa$ and $\|T+\mathcal{K}_\kappa(E)\| = \|P_{I\setminus A}T\|$.

Now suppose that $T\in\mathcal{K}_\kappa(E)$. Then we have:

(i) *if κ is a cardinal with $\mathrm{cf}(\kappa)>\aleph_0$, then there exists $A\subseteq I$ with $|A|<\kappa$ and $P_AT=T$;*

(ii) *if $\mathrm{cf}(\kappa)=\aleph_0$, then, for each $\epsilon>0$, there exists $A\subseteq I$ with $|A|<\kappa$ and $\|P_AT-T\|<\epsilon$.*

Proof. The second part of the proposition clearly follows from the first, as $T\in\mathcal{K}_\kappa(E)$ if and only if $\|T+\mathcal{K}_\kappa(E)\|=0$.

For the first part of the proposition, for any κ, if $A\subseteq I$ with $|A|<\kappa$ then $P_A\in\mathcal{K}_\kappa(E)$, so that

$$\|P_{I\setminus A}T\| = \|T-P_AT\| \geq \|T+\mathcal{K}_\kappa(E)\|.$$

Suppose we have $\epsilon>0$ and $S\in\mathcal{K}_\kappa(E)$ such that $\|T+S\|+\epsilon\leq\|T-P_AT\|$ for each $A\subseteq I$ with $|A|<\kappa$. We can find $Y\subseteq S(E_{[1]})$ with $|Y|<\kappa$ and such that

$$\inf\{\|S(x)-y\| : y\in Y\} < \epsilon/4 \qquad (x\in E_{[1]}).$$

Let $A=\bigcup_{y\in Y}\operatorname{supp} y$, so that $|A|\leq\aleph_0\times|Y|<\kappa$. Then $P_A(y)=y$ for each $y\in Y$, so that, for each $x\in E_{[1]}$, we have

$$\begin{aligned}\|S(x)-P_AS(x)\| &\leq \inf\{\|S(x)-y\|+\|P_A(y-S(x))\| : y\in Y\}\\ &\leq \inf\{\|S(x)-y\|+\|y-S(x)\| : y\in Y\} < \epsilon/2.\end{aligned}$$

Thus $\|S-P_AS\|\leq\epsilon/2$, and so

$$\begin{aligned}\|P_{I\setminus A}T\| &= \|P_{I\setminus A}(T+P_AS)\| \leq \|T+P_AS\| \leq \|T+S\|+\|P_AS-S\|\\ &\leq \|P_{I\setminus A}T\|-\epsilon+\epsilon/2,\end{aligned}$$

a contradiction showing that $\|T+\mathcal{K}_\kappa(E)\| = \inf\{\|P_{I\setminus A}T\| : A\subseteq I, |A|<\kappa\}$.

For $B\subseteq I$ with $|B|<\kappa$, we have that $P_B\in\mathcal{K}_\kappa(E)$, so that $T+\mathcal{K}_\kappa(E) = TP_{I\setminus B}+\mathcal{K}_\kappa(E)$, and thus immediately

$$\inf\{\|P_{I\setminus A}TP_{I\setminus B}\| : A,B\subseteq I, |A|<\kappa, |B|<\kappa\} = \|TP_{I\setminus B}+\mathcal{K}_\kappa(E)\| = \|T+\mathcal{K}_\kappa(E)\|,$$

as required.

If $\mathrm{cf}(\kappa)>\aleph_0$, then, for each $n\in\mathbb{N}$, choose $A_n\subseteq I$ with $|A_n|<\kappa$ and

$$\|P_{I\setminus A_n}T\| < \|T+\mathcal{K}_\kappa(E)\|+n^{-1}.$$

Let $A=\bigcup_n A_n$, so that $|A|<\kappa$ and, for each $n\in\mathbb{N}$,

$$\|T+\mathcal{K}_\kappa(E)\| \leq \|P_{I\setminus A}T\| = \|P_{I\setminus A}P_{I\setminus A_n}T\| \leq \|P_{I\setminus A_n}T\| < \|T+\mathcal{K}_\kappa(E)\|+n^{-1}.$$

Thus we must have $\|T+\mathcal{K}_\kappa(E)\| = \|P_{I\setminus A}T\|$.

We can now prove a converse to Proposition 3·5, at least when E has a shrinking, unconditional basis.

Proposition 5·2. *Let E have a shrinking, unconditional basis $(e_i)_{i\in I}$, let κ be an infinite cardinal, and let $T \in \mathcal{K}_\kappa(E)$. Then $T' \in \mathcal{K}_\kappa(E')$.*

Proof. As $(e_i^*)_{i\in I}$ is a basis for E', let $Q_A \in \mathcal{B}(E')$ be the analogue of $P_A \in \mathcal{B}(E)$. Then a quick calculation shows that $Q_A = P_A'$. Pick $\epsilon > 0$, and let $T \in \mathcal{K}_\kappa(E)$. By Proposition 5·1, for some $A \subseteq I$ with $|A| < \kappa$, we have $\|T - P_A T\| < \epsilon$. Thus $\|T' - T'Q_A\| = \|(T - P_A T)'\| < \epsilon$, so as $Q_A \in \mathcal{K}_\kappa(E')$, we have $\|T' + \mathcal{K}_\kappa(E')\| < \epsilon$. As $\epsilon > 0$ was arbitrary, we conclude that $T' \in \mathcal{K}_\kappa(E')$, as required.

For $T \in \mathcal{B}(E)$, let $\mathrm{ideal}(T)$ be the algebraic ideal generated by T in $\mathcal{B}(E)$, and $\overline{\mathrm{ideal}}(T)$ be its closure.

Lemma 5·3. *Let E be a Banach space with an unconditional basis $(e_i)_{i\in I}$. Suppose that for each cardinal $\kappa \geq \aleph_0$ and each $T \in \mathcal{B}(E) \setminus \mathcal{K}_\kappa(E)$, we have $\mathcal{K}_{\kappa^+}(E) \subseteq \overline{\mathrm{ideal}}(T)$. Then $\mathcal{B}(E)$ has compact ideal structure.*

Proof. Let J be a non-trivial closed ideal in $\mathcal{B}(E)$. If $J \subseteq \mathcal{K}(E)$, then $J = \mathcal{K}(E)$. Thus we may suppose that $\mathcal{K}(E) = \mathcal{K}_{\aleph_0}(E) \subsetneq J$. Let

$$X = \{\sigma : J \setminus \mathcal{K}_\sigma(E) \neq \emptyset\}$$

so that, by our assumption, if $\sigma \in X$, then $\mathcal{K}_{\sigma^+}(E) \subseteq J$. Suppose that X contains a maximal element κ, so that $\kappa > \aleph_0$. Then, as $\kappa \in X$, we have $\mathcal{K}_{\kappa^+}(E) \subseteq J$, and as κ is maximal in X, we have $J \setminus \mathcal{K}_{\kappa^+}(E) = \emptyset$, so that $J \subseteq \mathcal{K}_{\kappa^+}(E)$. Thus $J = \mathcal{K}_{\kappa^+}(E)$ as required.

If X does not contain a maximum element, then, for some limit cardinal κ, $X = \{\sigma : \sigma < \kappa\}$, and so $\kappa \notin X$, meaning that $J \subseteq \mathcal{K}_\kappa(E)$. Choose $T \in \mathcal{K}_\kappa(E)$ and $\epsilon > 0$. Then, by Proposition 5·1, we can find $A \subseteq I$ with $|A| < \kappa$ and $\|P_A T - T\| < \epsilon$. As $P_A \in \mathcal{K}_{|A|^+}(E)$, $P_A T \in J$. As $\epsilon > 0$ was arbitrary and J is closed, $T \in J$. Thus $J = \mathcal{K}_\kappa(E)$.

At this point we restrict ourselves to considering $E = l^p(I)$, for $1 \leq p < \infty$, or $E = c_0(I)$. Then, by the structure of E, if $A, B \subseteq I$ with $|A| = |B|$, then $P_B \in \mathrm{ideal}(P_A)$.

Lemma 5·4. *Let $E = l^p(I)$, for $1 \leq p < \infty$, or $E = c_0(I)$, let $\kappa \geq \aleph_0$ be a cardinal, and let $T \in \mathcal{B}(E) \setminus \mathcal{K}_\kappa(E)$. Then $\mathcal{K}_{\kappa^+}(E) \subseteq \overline{\mathrm{ideal}}(T)$ if and only if, for some $A \subseteq I$ with $|A| = \kappa$, $P_A \in \overline{\mathrm{ideal}}(T)$.*

Proof. If $P_A \in \overline{\mathrm{ideal}}(T)$ for some $A \subseteq I$ with $|A| = \kappa$, then $P_B \in \overline{\mathrm{ideal}}(T)$ for every $B \subseteq I$ with $|B| \leq |A|$. For $S \in \mathcal{K}_{\kappa^+}(E)$, by Proposition 5·1, there exists $B \subseteq I$ with $|B| \leq \kappa$ and $P_B S = S$. Thus $S \in \mathrm{ideal}(P_B) \subseteq \overline{\mathrm{ideal}}(T)$, so we see that $\mathcal{K}_{\kappa^+}(E) \subseteq \overline{\mathrm{ideal}}(T)$.

Conversely, if $\mathcal{K}_{\kappa^+}(E) \subseteq \overline{\mathrm{ideal}}(T)$, then for $A \subseteq I$ with $|A| = \kappa$, we have $P_A \in \mathcal{K}_{\kappa^+}(E)$, so that $P_A \in \overline{\mathrm{ideal}}(T)$.

Proposition 5·5. *Let $E = l^p(I)$, for $1 \leq p < \infty$, or $E = c_0(I)$. Suppose that for each cardinal $\kappa \geq \aleph_0$ and each $T \in \mathcal{B}(E) \setminus \mathcal{K}_\kappa(E)$, there exists $A \subseteq I$ with $|A| = \kappa$ and $P_A \in \overline{\mathrm{ideal}}(T)$. Then $\mathcal{B}(E)$ has perfect compact ideal structure.*

Proof. Use Proposition 3·7, and Lemma 5·4 applied with Lemma 5·3.

6. *When E has a shrinking basis*

For the moment, we shall assume only that E has a shrinking basis $(e_\alpha)_{\alpha\in I}$.

Proposition 6·1. *Let E have a shrinking basis $(e_\alpha)_{\alpha\in I}$, let $\kappa > \aleph_0$ be a cardinal, and let $T \in \mathcal{B}(E)\setminus\mathcal{K}_\kappa(E)$. Then we can find a family $(x_i)_{i\in\kappa}$ of vectors in E such that for some $\delta > 0$ we have:*

(i) *for $i \in \kappa$, we have $\|x_i\| = 1$ and $\|T(x_i)\| \geq \delta$;*

(ii) *for each $i, j \in \kappa$ with $i \neq j$, we have $\operatorname{supp} T(x_i) \cap \operatorname{supp} T(x_j) = \operatorname{supp}(x_i) \cap \operatorname{supp}(x_j) = \emptyset$.*

Proof. As $T \notin \mathcal{K}_\kappa(E)$, let $2\delta = \|T + \mathcal{K}_\kappa(E)\| > 0$. For $A \subseteq I$ with $|A| < \kappa$, as $P_A \in \mathcal{K}_\kappa(E)$, we have that $2\delta = \|T + \mathcal{K}_\kappa(E)\| \leq \|T - TP_A\| = \|TP_{I\setminus A}\|$.

A simple Zorn's Lemma argument shows that we can find a maximal family of vectors X in E such that conditions (i) and (ii) hold.

If $|X| \geq \kappa$, then we are done. Suppose, towards a contradiction, that $|X| < \kappa$, so that if we set $A = \bigcup_{x\in X}\operatorname{supp}(x)$ and $B = \bigcup_{x\in X}\operatorname{supp} T(x)$, then $|A| \leq |X| \times \aleph_0 = \max(|X|, \aleph_0) < \kappa$ and, similarly, $|B| < \kappa$. As E has a shrinking basis, we may set

$$C = \bigcup_{i\in B}\operatorname{supp} T'(e_i^*),$$

so that, again, $|C| < \kappa$. For $y \in E$, we have that $B \cap \operatorname{supp} T(y) \neq \emptyset$ if and only if, for some $i \in B$, we have $0 \neq \langle e_i^*, T(y)\rangle = \langle T'(e_i^*), y\rangle$, which implies that $C \cap \operatorname{supp}(y) \neq \emptyset$. Thus, for each $y \in E$, we have $\operatorname{supp} TP_{I\setminus C}(y) \subseteq I \setminus B$. Finally, let $D = A \cup C$, so that $|D| < \kappa$, and if $y \in E$ with $P_{I\setminus D}(y) = y$, then $\operatorname{supp} T(y) \subseteq I \setminus B$, so that by the maximality of X, we must have $\|T(y)\| < \delta\|y\|$. This implies that $\|TP_{I\setminus D}\| \leq \delta$, which is a contradiction by our choice of δ.

We can then certainly apply this proposition to $E = c_0(I)$ or $E = l^p(I)$, for $1 < p < \infty$.

Theorem 6·2. *If $E = c_0(I)$ or $E = l^p(I)$ for $1 < p < \infty$, then for a closed ideal J in $\mathcal{B}(E)$, we have $J = \mathcal{K}_\kappa(E)$ for some cardinal κ.*

Proof. We use Proposition 5·5, so let $\kappa \geq \aleph_0$ be a cardinal and $T \in \mathcal{B}(E)\setminus\mathcal{K}_\kappa(E)$. If $\kappa = \aleph_0$, we need to show that, if T is not compact, then $P_A \in \overline{\text{ideal}}(T)$ for some countable $A \subseteq I$. This follows directly from Proposition 4·3. Thus we may suppose that $\kappa > \aleph_0$. We can then apply Proposition 6·1 to find a family $(x_i)_{i\in\kappa}$ and $\delta > 0$ with properties as in the proposition.

As $(T(x_i))_{i\in\kappa}$ is a family of vectors with pairwise-disjoint support, we can find a family $(\mu_i)_{i\in\kappa} \subseteq E'$ with pairwise-disjoint support (recall that E has a shrinking basis) and such that $\langle \mu_i, T(x_j)\rangle = \delta_{ij}$, the Kronecker delta. As $\|T(x_i)\| \geq \delta$ for each $i \in \kappa$, we may suppose that $\|\mu_i\| \leq \delta^{-1}$ for each $i \in \kappa$. Let $K \subseteq I$ be some subset with $|K| = \kappa$, and let $\phi : K \to \kappa$ be a bijection. We can then define $Q, S \in \mathcal{B}(E)$ by

$$Q(x) = \sum_{j\in K} T(x_{\phi(j)})\langle \mu_{\phi(j)}, x\rangle \qquad S(x) = \sum_{j\in K} x_{\phi(j)}\langle \mu_{\phi(j)}, Q(x)\rangle \qquad (x \in E).$$

A calculation shows that, in all cases for E, $\|Q\| \leq \delta^{-1}\|T\|$ and $\|S\| \leq \delta^{-1}\|Q\| \leq \delta^{-2}\|T\|$. For $i \in \kappa$, we then have $Q(T(x_i)) = T(x_i)$, and so $ST(x_i) = x_i$.

Similarly, we can find a family $(\lambda_i)_{i\in K} \subseteq E'$ with pairwise-disjoint support and such that $\langle \lambda_i, x_{\phi(j)}\rangle = \delta_{ij}$, and $\|\lambda_i\| = 1$ for each $i \in K$. Then we may define $R, U \in \mathcal{B}(E)$ by

$$U\Big(\sum_{i\in I} a_i e_i\Big) = \sum_{i\in K} a_i x_{\phi(i)} \qquad R(x) = \sum_{j\in K} e_j\langle \lambda_j, x\rangle \qquad (x \in E),$$

and again a calculation yields that $\|R\| = 1$ and that U is an isometry onto its range. Then, for each $j \in K$, we have $R(x_{\phi(j)}) = e_j$, so that $RSTU(e_j) = RST(x_{\phi(j)}) = R(x_{\phi(j)}) = e_j$. Thus $RSTU = P_K$, and as $|K| = \kappa$, we are done.

7. *When* $E = l^1(I)$

We use a different argument to that applied in Proposition 6·1, as $l^1(I)$ does not have a shrinking basis.

LEMMA 7·1. *For an index set* I, $T \in \mathcal{B}(l^1(I))$ *and* $A \subseteq I$, *we have*

$$\|TP_A\| = \sup\{\|T(e_i)\| : i \in A\}.$$

Proof. Just note that, for $x = \sum_{i\in A} a_i e_i \in P_A(l^1(I))$, we have

$$\|T(x)\| = \Big\| \sum_{i\in A} a_i T(e_i) \Big\| \leq \sum_{i\in A} |a_i| \|T(e_i)\| \leq \|x\| \sup\{\|T(e_i)\| : i \in A\}.$$

PROPOSITION 7·2. *Let* $E = l^1(I)$ *for some index set* I. *Let* $\kappa > \aleph_0$ *be a cardinal, let* $\epsilon \in (0,1)$, *and let* $T \in \mathcal{B}(E) \setminus \mathcal{K}_\kappa(E)$ *be such that*

$$1 \geq \|T\| \geq \|T + \mathcal{K}_\kappa(E)\| \geq 1 - \epsilon.$$

Then there exists $K \subseteq I$ *with* $|K| \geq \kappa$, *and a family* $(A_i)_{i\in K}$ *of subsets of* I, *such that:*

(i) *for* $i \in K$, A_i *is countable and* $\|P_{A_i} T(e_i)\| \geq 1 - 2\epsilon$;

(ii) *for* $i, j \in K$ *with* $i \neq j$, $A_i \cap A_j = \emptyset$.

Proof. For $L \subseteq I$ and $B = (B_i)_{i\in L}$ a family of subsets of I, we say that (L, B) is *admissible* if conditions (1) and (2) are satisfied. Let X be the collection of admissible pairs; since $\|T\| \geq 1 - \epsilon > 0$, the set X is not empty by Lemma 7·1. Partially order X be setting $(L, (B_i^L)_{i\in L}) \leq (J, (B_i^J)_{i\in J})$ if and only if $L \subseteq J$ and, for each $i \in L$, $B_i^J = B_i^L$.

Let $Y \subseteq X$ be a chain, and let $L_0 = \bigcup_{(L,B^L)\in Y} L \subseteq I$. Then, for $i \in L_0$, we have $i \in L$ for some $(L, B^L) \in Y$. Set $B_i = B_i^L$. This is well-defined, for if $i \in J$ for some $(J, B^J) \in Y$, then either $(L, B^L) \leq (J, B^J)$, so that $B_i^L = B_i^J$, or $(J, B^J) \leq (L, B^L)$ and $B_i^J = B_i^L$. Let $B = (B_i)_{i\in L_0}$, so, if $i \in L_0$, B_i is countable, and $\|P_{B_i} T(e_i)\| \geq 1 - 2\epsilon$. Similarly, we can show that $(L_0, B) \in X$ and that (L_0, B) is an upper bound for Y. We can thus apply Zorn's Lemma to find a maximal admissible pair $(K, (A_i)_{i\in K})$.

If $|K| \geq \kappa$ then we are done. Otherwise, let $B = \bigcup_{i\in K} A_i$ so that $|B| \leq \aleph_0 \times |K| < \kappa$. As (K, A) is maximal, suppose that for some $i \in I \setminus K$ we have $\|P_{I\setminus B} T(e_i)\| \geq 1 - 2\epsilon$. Then set $C = (I \setminus B) \cap \operatorname{supp} T(e_i)$, so that C is countable and $\|P_C T(e_i)\| \geq 1 - 2\epsilon$. This contradicts the maximality of (K, A). Hence we see that

$$\|P_{I\setminus B} T(e_i)\| < 1 - 2\epsilon \qquad (i \in I \setminus K).$$

By Lemma 7·1, we conclude that $\|P_{I\setminus B} T P_{I\setminus K}\| \leq 1 - 2\epsilon$. By Lemma 3·6, P_B and P_K are κ-compact, so that

$$1 - 2\epsilon \geq \|P_{I\setminus B} T P_{I\setminus K}\| = \|T - TP_K - P_B T P_{I\setminus K}\| \geq \|T + \mathcal{K}_\kappa(E)\| \geq 1 - \epsilon.$$

This contradiction shows that $|K| \geq \kappa$, as required.

THEOREM 7·3. *Let* $E = l^1(I)$, *and let* J *be a closed ideal in* $\mathcal{B}(E)$. *Then* $J = \mathcal{K}_\kappa(E)$ *for some cardinal* κ.

Proof. We use Proposition 5·5, so let $\kappa \geq \aleph_0$ be a cardinal and $T \in \mathcal{B}(E) \setminus \mathcal{K}_\kappa(E)$. As in the proof of Theorem 6·2, we may suppose that $\kappa > \aleph_0$. Fix $\epsilon > 0$. By Proposition 5·1, we can find $A \subseteq I$ with $|A| < \kappa$ and $\|P_{I\setminus A}T\| \geq \|T+\mathcal{K}_\kappa(E)\| \geq \|P_{I\setminus A}T\|(1-\epsilon/4)$. Then, since $P_A \in \mathcal{K}_\kappa(E)$, we have

$$\|P_{I\setminus A}T + \mathcal{K}_\kappa(E)\| = \|T + \mathcal{K}_\kappa(E)\| \geq (1-\epsilon/4)\|P_{I\setminus A}T\|.$$

Let $T_0 = P_{I\setminus A}T\|P_{I\setminus A}T\|^{-1}$, so that

$$1 = \|T_0\| \geq \|T_0 + \mathcal{K}_\kappa(E)\| = \|P_{I\setminus A}T + \mathcal{K}_\kappa(E)\|\|P_{I\setminus A}T\|^{-1} \geq 1-\epsilon/4.$$

Apply Proposition 7·2 to T_0 to find $K \subseteq I$ with $|K| = \kappa$, and a family $(A_k)_{k\in K}$ of subsets of I, such that:

(i) for $k \in K$, A_k is countable, and $\|P_{A_k}T_0(e_k)\| \geq 1-\epsilon/2$;

(ii) for $j,k \in K$ with $j \neq k$, $A_j \cap A_k = \emptyset$.

For $k \in K$ let $v_k = P_{A_k}T_0(e_k)\|P_{A_k}T_0(e_k)\|^{-1}$, so that $\|v_k\| = 1$ and, recalling that $\|T_0\| = 1$, we also have

$$\begin{aligned}\|T_0(e_k) - v_k\| &= \Big\|P_{I\setminus A_k}T_0(e_k) + P_{A_k}T_0(e_k)\Big(1 - \|P_{A_k}T_0(e_k)\|^{-1}\Big)\Big\| \\ &= \|P_{I\setminus A_k}T_0(e_k)\| + \Big|\|P_{A_k}T_0(e_k)\| - 1\Big| \\ &= \|P_{I\setminus A_k}T_0(e_k)\| + 1 - \|P_{A_k}T_0(e_k)\| \\ &= \|T_0(e_k)\| + 1 - 2\|P_{A_k}T_0(e_k)\| \leq 1 + 1 - 2(1-\epsilon/2) = \epsilon.\end{aligned}$$

Let $F = \overline{\text{lin}}(v_k)_{k\in K}$, and define $U : F \to \overline{\text{lin}}(T_0(e_k))_{k\in K}$ by $U(v_k) = T_0(e_k)$. Then, for $x = \sum_{k\in K} a_k v_k$, we have, noting that (v_k) has pairwise-disjoint support,

$$\|U(x)\| = \Big\|\sum_{k\in K} a_k T_0(e_k)\Big\| \leq \|T_0\|\Big\|\sum_{k\in K} a_k e_k\Big\| = \|T_0\|\|x\|,$$

so that U is bounded. As (v_k) has pairwise disjoint support, we can find a projection $P : E \to F$ with $\|P\| = 1$. Then, with $x = \sum_{k\in K} a_k v_k \in F$, we have

$$\begin{aligned}\|x - PU(x)\| = \|P(x - U(x))\| \leq \|x - U(x)\| &= \Big\|\sum_{k\in K} a_k(v_k - T_0(e_k))\Big\| \\ &\leq \Big(\sup_{k\in K}\|v_k - T_0(e_k)\|\Big)\sum_{k\in K}|a_k| \leq \epsilon\|x\|.\end{aligned}$$

Thus, if $\epsilon < 1$, noting that $\text{id}_F - PU \in \mathcal{B}(F)$, we have $\|\text{id}_F - PU\| < 1$ so that PU is invertible in $\mathcal{B}(F)$.

Then we have, for $k \in K$, $PU(v_k) = PT_0(e_k)$, so that $v_k = (PU)^{-1}PT_0(e_k)$. Define $V : F \to P_K(E)$ by, for $k \in K$, $V(v_k) = e_k$, so that V is an isometry. Thus, letting $S = V(PU)^{-1}P$, we have $ST_0P_K = P_K$. Thus

$$P_K = ST_0P_K = \|P_{I\setminus A}T\|^{-1}SP_{I\setminus A}TP_K,$$

so that $P_K \in \text{ideal}(T)$, as required.

This proof is the correct analogue of Theorem 6·2, for above we showed that $\sup_k \|T(e_k) - v_k\| < 1$, whereas for the l^p and c_0 cases we would need to show that

$$\sum_{k\in K}\|T(e_k) - v_k\|^q < 1,$$

where $q^{-1} + p^{-1} = 1$ (or $q = 1$ in the c_0 case). However, if K is uncountable, then such a sum must contain all but countably many terms which are actually zero. As we are free to remove such terms (and still have K being of the same cardinality) we arrive at the conclusions of Proposition 6·1 (at least with (e_i) replaced by a family of disjointly supported unit vectors (x_i)).

To sum up, we have shown the following generalisation of the Gohberg, Markus and Feldman theorem.

THEOREM 7·4. *Let I be an infinite set, and let $E = l^p(I)$ for $1 \leq p < \infty$, or $E = c_0(I)$. Then $\mathcal{B}(E)$ has perfect compact ideal structure. That is, the closed ideals in $\mathcal{B}(E)$ form an ordered chain*

$$\{0\} \subsetneq \mathcal{K}(E) \subsetneq \mathcal{K}_{\aleph_1}(E) \subsetneq \cdots \subsetneq \mathcal{K}_{|I|}(E) \subsetneq \mathcal{K}_{|I|^+}(E) = \mathcal{B}(E).$$

8. *Generalisation*

An immediate question is whether there are any other Banach spaces E such that $\mathcal{B}(E)$ has (perfect) compact ideal structure. However, even for non-separable spaces, we are hampered by our lack of knowledge in the separable case.

PROPOSITION 8·1. *Let E be a Banach space such that $\mathcal{B}(E)$ has compact ideal structure. Suppose that F is a complemented subspace of E. Then $\mathcal{B}(F)$ has compact ideal structure.*

Proof. Let F be complemented in E with projection $P : E \to F$. Let J be a closed ideal in $\mathcal{B}(F)$, and define

$$J_0 = \overline{\text{lin}}\{RST : S \in J, T \in \mathcal{B}(E, F), R \in \mathcal{B}(F, E)\} \subseteq \mathcal{B}(E).$$

Clearly J_0 is a closed ideal in $\mathcal{B}(E)$, so that $J_0 = \mathcal{K}_\kappa(E)$ for some cardinal κ.

If $S \in J$, $T \in \mathcal{B}(E, F)$ and $R \in \mathcal{B}(F, E)$, then $PR \in \mathcal{B}(F)$ and $T|_F \in \mathcal{B}(F)$ so that $PRST|_F \in J$, as J is an ideal. Thus if $U \in J_0$ then $PU|_F \in J$. Let $\iota : F \to E$ be the inclusion map. Clearly, if $V \in J$, then $\iota VP \in J_0$.

We thus claim that $J = \mathcal{K}_\kappa(F)$, for if $V \in J$ then $\iota VP \in J_0$ so ιVP is κ-compact, and thus V is κ-compact. Conversely, if $W \in \mathcal{K}_\kappa(F)$ then ιWP is κ-compact, so that $\iota WP \in J_0$, and thus $P\iota WP|_F = W \in J$.

Hence, in practical terms, if we exhibit a Banach space E with $\mathcal{B}(E)$ having compact ideal structure, we need separable complemented subspaces of E to be isomorphic to l^p or c_0. If we look at spaces with an unconditional basis, then such spaces have a plethora of separable complemented subspaces. Indeed, in some special cases, we can show that such spaces are trivial.

PROPOSITION 8·2. *Let E be a Banach space with an unconditional basis $(e_i)_{i \in I}$ such that every subspace $P_A(E)$, for countably infinite $A \subseteq I$, is isomorphic to some l^p space $(1 \leq p < \infty)$, or to c_0. Then each separable, complemented subspace of E is isomorphic to a fixed l^p space, or c_0. Furthermore, if this fixed space is c_0, l^1 or l^2, then E is isomorphic to $c_0(I)$, $l^1(I)$ or $l^2(I)$, respectively.*

Proof. Throughout this proof, we shall write l^∞ for c_0. Then suppose that for countably infinite $A_i \subseteq I$, $P_{A_i}(E)$ is isomorphic to l^{p_i}, for $i = 1, 2$. Then let $A = A_1 \cup A_2$ so that $P_A(E)$ is isomorphic to l^p say. Then $P_{A_i}(E)$ is isomorphic to a complemented subspace of l^p, and thus must be isomorphic to l^p by [**8**, Theorem 2.a.3], as every complemented, infinite dimensional subspace of l^p is isomorphic to l^p. Thus l^{p_i} is isomorphic to l^p, and so $p_i = p$, for $i = 1, 2$.

Now let $F \subseteq E$ be a complemented, separable subspace. We can then find a countable $A \subseteq I$ with $F \subseteq P_A(E)$, so that F is isomorphic to a complemented subspace of l^p, and thus isomorphic to l^p.

Now suppose that $p = 1, 2$ or ∞ (the c_0 case). Then, by [**8**, Theorem 2.b.10], we know that each such space has exactly one unconditional basis, up to equivalence. For each countably infinite $A \subseteq I$, let $T_A : P_A(E) \to l^p$ be an isomorphism, chosen such that $\|T_A\|\|T_A^{-1}\| \leq 2d(P_A(E), l^p)$, the Banach-Mazur distance. Then it is clear that, if we take an enumeration of A, $A = \{a_n^A : n \in \mathbb{N}\}$, then the sequence $(T_A(e_{a_n^A}))$ is an unconditional basis for l^p, and thus there exists $K_A \geq 1$ such that

$$K_A^{-1} \left(\sum_{n=1}^{\infty} |b_n|^p \right)^{1/p} \leq \left\| \sum_{n=1}^{\infty} b_n T_A(e_{a_n^A}) \right\| \leq K_A \left(\sum_{n=1}^{\infty} |b_n|^p \right)^{1/p}$$

for each sequence of scalars (b_n). Then we have that, for a sequence of scalars (b_n),

$$K_A^{-1} \|T_A\|^{-1} \left(\sum_{n=1}^{\infty} |b_n|^p \right)^{1/p} \leq \left\| \sum_{n=1}^{\infty} b_n e_{a_n^A} \right\| \leq K_A \|T_A^{-1}\| \left(\sum_{n=1}^{\infty} |b_n|^p \right)^{1/p}.$$

Given an injection $f : \mathbb{N} \to I$, let $B_f \geq 1$ be the minimal constant such that

$$B_f^{-1} \left(\sum_{n=1}^{\infty} |b_n|^p \right)^{1/p} \leq \left\| \sum_{n=1}^{\infty} b_n e_{f(n)} \right\| \leq B_f \left(\sum_{n=1}^{\infty} |b_n|^p \right)^{1/p}$$

holds for all sequences of scalars (b_n). We claim that the family (B_f) is bounded. For if not, let $f_n : \mathbb{N} \to I$ be such that $B_{f_n} \geq n$, for each $n \in \mathbb{N}$. Then let $g : \mathbb{N} \to I$ be an injective function chosen so that $g(\mathbb{N}) = \bigcup_{n \in \mathbb{N}} f_n(\mathbb{N})$. Now pick $N \in \mathbb{N}$, and given a sequence of scalars (b_n), let (c_n) be a sequence of scalars such that

$$c_n = \begin{cases} b_m & : g(n) = f_N(m), \\ 0 & : \text{otherwise.} \end{cases}$$

Then, as F_N and g are injective, and the image of g contains that image of F_N, we see that

$$\left(\sum_{n=1}^{\infty} |b_n|^p \right)^{1/p} = \left(\sum_{n=1}^{\infty} |c_n|^p \right)^{1/p} \leq B_g \left\| \sum_{n=1}^{\infty} c_n e_{g(n)} \right\| = B_g \left\| \sum_{m=1}^{\infty} b_m e_{f_N(m)} \right\|,$$

and similarly

$$\left\| \sum_{m=1}^{\infty} b_m e_{f_N(m)} \right\| = \left\| \sum_{n=1}^{\infty} c_n e_{g(n)} \right\| \leq B_g \left(\sum_{n=1}^{\infty} |c_n|^p \right)^{1/p} = B_g \left(\sum_{n=1}^{\infty} |b_n|^p \right)^{1/p}.$$

As B_{f_N} is minimal, we must have $B_g \geq B_{f_N}$, which contradicts $(B_{f_N})_{N=1}^{\infty}$ being unbounded.

Hence let $M = \sup_f B_f < \infty$. Define $T : E \to l^p(I)$ by $T(e_i) = d_i$, $(d_i)_{i \in I}$ being the standard basis for $l^p(I)$. Then, if $x \in E$, we have $x = \sum_{n=1}^{\infty} a_n e_{f(n)}$ for some injection

$f : \mathbb{N} \to I$ and some sequence of scalars (a_n), so that

$$\|T(x)\| = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} \leq M \left\|\sum_{n=1}^{\infty} a_n e_{f_n}\right\| = M\|x\|.$$

We can similarly show that T has an inverse, so that E is isomorphic to $l^p(I)$ as required.

We note that the case where $1 < p < \infty$, $p \neq 2$, seems to be a good deal harder.

If E is an arbitrary non-separable Banach space, suppose that we only know that every closed ideal J of $\mathcal{B}(E)$ with $\mathcal{K}_{\aleph_1}(E) \subseteq J$ has $J = \mathcal{K}_\kappa(E)$ for some κ. Then by examining the proof of Proposition 8·1, we see that, if F is separable, we gain no information on the ideal structure of $\mathcal{B}(F)$ because then $J_0 \subseteq \mathcal{K}_{\aleph_1}(E)$. So we could ask an easier question: namely, are there more Banach spaces E such that, beyond the operators with separable range, every closed ideal is an ideal of κ-compact operators? However, this is too easy, for consider

$$E = l^1(\mathbb{N}) \oplus l^2(I)$$

for some uncountable I. A moments thought shows that this rather simple example does satisfy our conditions.

We hence conclude that the interesting questions, with regards to compact ideal structure, lie in studying the ideal structure of $\mathcal{B}(E)$ for separable Banach spaces E.

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