Primer on Interpolation Spaces

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Abstract

We introduce real interpolation spaces and describe their properties. Our aim is to summarise, in English, the standard results of Peetre, Lyons and Beauzamy which are most comprehensively accessable in French only.

Contents

1 Introduction

An *interpolation space* is, roughly speaking, a way of producing a Banach space which is intermediate between two other Banach spaces. They have uses in, for example, classical analysis, but my interest (and hence the theme of this note) is in applications of interpolation spaces to abstract functional analysis: for example, the celebrated result [2] of Davis, Figiel, Johnson and Pełczyński on factoring weakly compact operators. We shall mainly follow the book [1], also using the useful reference [3, Section 2.g].

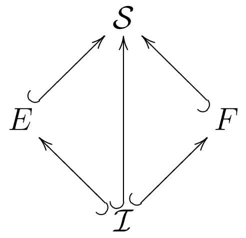
We now fix some notation and general concepts. Let E and F be normed spaces, and suppose that E and F are subspaces of some Hausdorff topological vector space X. In this case, we say that (E, F) is a *compatible couple*. Then we have vector spaces

$$\mathcal{I} = E \cap F \subseteq X, \quad \mathcal{S} = \{x \in X : \exists e \in E, f \in F, x = e + f\},$$

the *intersection* and *sum* spaces. We may clearly assume that $X = \mathcal{S}$. Let E and F have norms $\|\cdot\|_E$ and $\|\cdot\|_F$ respectively, and norm $\mathcal{I}$ and $\mathcal{S}$ by

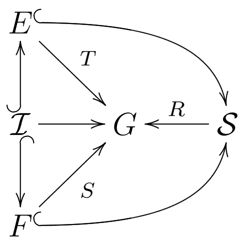
$$\|x\|_{\mathcal{I}} = \max(\|x\|_E, \|x\|_F) \qquad (x \in \mathcal{I}),$$
$$\|x\|_{\mathcal{S}} = \inf\{\|e\|_E + \|f\|_F : x = e + f\} \qquad (x \in \mathcal{S}).$$

Then, if E and F are Banach spaces, it is an easy check to show that $(\mathcal{S}, \|\cdot\|_{\mathcal{S}})$ and $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ are also Banach spaces; we shall henceforth assume that E and F are indeed Banach spaces. In this case, we have a commuting diagram of norm-decreasing maps:



It is clear that the maps $\mathcal{I} \to E$ and $\mathcal{I} \to F$ are injections. Suppose that $x \in E$ is such that $\|x\|_{\mathcal{S}} = 0$, so that there are sequences $(e_n) \subseteq E$ and $(f_n) \subseteq F$ with $e_n + f_n = x$ for all n, and $\|e_n\|_E + \|f_n\|_F \to 0$. Then $f_n = x - e_n \in \mathcal{I}$ for each n, $e_n \to 0$ in E, $f_n \to 0$ in F, and $\|f_n\|_E = \|x - e_n\|_E \to \|x\|_E$. For $n, m \in \mathbb{N}$, $\|f_n - f_m\|_{\mathcal{I}} = \max(\|f_n - f_m\|_F, \|e_n - e_m\|_E)$, so (f_n) is Cauchy in $\mathcal{I}$, tending to a limit $y \in \mathcal{I}$ say. However, $\|f_n - y\|_F \leq \|f_n - y\|_{\mathcal{I}} \to 0$, so that $y = 0$, implying that $0 = \lim_n \|f_n\|_E = \|x\|$, so that $x = 0$. Hence the map $E \to \mathcal{S}$ is injective, and by symmetry, so is the map $F \to \mathcal{S}$.

Let $\mathcal{B}(E, F)$ be a Banach space of bounded linear operators between E and F. Suppose that G is a Banach space and that we have $T \in \mathcal{B}(E, G)$ and $S \in \mathcal{B}(F, G)$ such that T and S agree on $\mathcal{I}$. Then there exists $R \in \mathcal{B}(\mathcal{S}, G)$ which extends T and S:



Firstly, we define $R(x) = T(e) + S(f)$ for $x = e + f \in \mathcal{S}$. This is well-defined, for if $x = e_1 + f_1$, then $e - e_1 = f_1 - f \in \mathcal{I}$, so that $T(e) - T(e_1) = S(f_1) - S(f)$, and hence $T(e) + S(f) = T(e_1) + S(f_1)$. Then, for $x \in \mathcal{S}$, we see that

$$\|R(x)\| = \inf\{\|T(e) + S(f)\| : x = e + f, e \in E, f \in F\} \leq \max(\|T\|, \|S\|)\|x\|_{\mathcal{S}},$$

so that R is bounded, as required.

These properties show that $\mathcal{I}$ and $\mathcal{S}$ are actually interpolation spaces: in some sense, there are the biggest and smallest interpolation spaces between E and F, a fact we shall make rigourous later. When necessary, we write $\mathcal{I}(E, F)$ and $\mathcal{S}(E, F)$.

For a Banach space E, let E' be its dual, and we write $\langle \mu, x \rangle = \mu(x)$ for $\mu \in E'$ and $x \in E$. Then we have the canonical isometry $\kappa_E : E \to E''$ defined by $\langle \kappa_E(x), \mu \rangle = \langle \mu, x \rangle$ for $x \in E$ and $\mu \in E'$.

2 Lions-Peetre interpolation method

We shall now introduce a class of interpolation spaces first studied by Lions and Peetre in [4]. Fix real numbers ξ_0 and ξ_1, and let $p \in [1, \infty]$. For normed spaces $(E_0, \|\cdot\|_0)$ and $(E_1, \|\cdot\|_1)$ such that (E_0, F_0) forms a compatible couple, we consider the collection of (mesaure classes of) functions $f : \mathbb{R} \to \mathcal{I}(E_0, E_1)$ such that

$$\|e^{\xi_0 t} f(t)\|^p_{L^p(E_0)} := \int_{\mathbb{R}} \|e^{\xi_0 t} f(t)\|_0^p \, \mathrm{d}t < \infty, \quad \|e^{\xi_1 t} f(t)\|^p_{L^p(E_1)} := \int_{\mathbb{R}} \|e^{\xi_1 t} f(t)\|_1^p \, \mathrm{d}t < \infty,$$

where, of course, a further condition on f is that these functions are integrable. If $p = \infty$, these conditions are implied to mean

$$\text{ess-sup}_{t\in\mathbb{R}} \|e^{\xi_0 t} f(t)\|_0 < \infty, \quad \text{ess-sup}_{t\in\mathbb{R}} \|e^{\xi_1 t} f(t)\|_1 < \infty.$$

We now (and henceforth) suppose that $\xi_0 < 0$ and $\xi_1 > 0$. Suppose, further, that $\int_{\mathbb{R}} f(t)\ \mathrm{d}t$ converges in $\mathcal{I}$. Then

$$\begin{aligned}
\int_{\mathbb{R}} \|f(t)\|_{\mathcal{S}}\ \mathrm{d}t &\le \int_0^\infty \|f(t)\|_1\ \mathrm{d}t + \int_{-\infty}^0 \|f(t)\|_0\ \mathrm{d}t \\
&\le \Big(\int_0^\infty e^{-\xi_1 tq}\ \mathrm{d}t\Big)^{1/q}\Big(\int_0^\infty \|e^{\xi_1 t} f(t)\|_1^p\ \mathrm{d}t\Big)^{1/p} \\
&\qquad + \Big(\int_{-\infty}^0 e^{-\xi_0 tq}\ \mathrm{d}t\Big)^{1/q}\Big(\int_{-\infty}^0 \|e^{\xi_0 t} f(t)\|_0^p\ \mathrm{d}t\Big)^{1/p} \\
&\le \Big(\frac{1}{\xi_1 q}\Big)^{1/q} \|e^{\xi_1 t} f(t)\|_{L^p(E_1)} + \Big(\frac{1}{\xi_0 q}\Big)^{1/q} \|e^{\xi_0 t} f(t)\|_{L^p(E_0)},
\end{aligned}$$

where $q^{-1} + p^{-1} = 1$. Thus $\int_{\mathbb{R}} f(t)\ \mathrm{d}t$ converges in $\mathcal{S}$.

We let $S = S(p; \xi_0, E_0; \xi_1, E_1)$ denote (for $\xi_0 < 0, \xi_1 > 0$) the collection

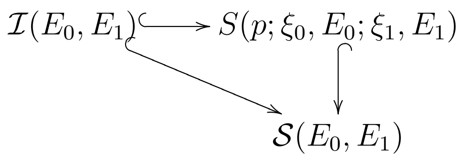
$$\Big\{x \in \mathcal{S} : x = \int_{\mathbb{R}} f(t)\ \mathrm{d}t, \|e^{\xi_0 t} f(t)\|_{L^p(E_0)}, \|e^{\xi_1 t} f(t)\|_{L^p(E_1)} < \infty\Big\}.$$

We norm this space by setting

$$\|x\|_S = \inf\Big\{\max\big(\|e^{\xi_0 t} f(t)\|_{L^p(E_0)}, \|e^{\xi_1 t} f(t)\|_{L^p(E_1)}\big) : x = \int_{\mathbb{R}} f(t)\ \mathrm{d}t\Big\},$$

and we can check that S becomes a Banach space in this norm.

We claim that we have a factorisation of following form



For $x \in \mathcal{I}(E_0, E_1)$ we let $f : \mathbb{R} \to \mathcal{I}(E_0, E_1)$ be defined by $f(t) = x$ for $0 \le t \le 1$, and $f(t) = 0$ otherwise. Then clearly $\int_{\mathbb{R}} f(t)\ \mathrm{d}t = x$. We also see that, if

$$\alpha_0 = \Big(\frac{e^{\xi_0 p} - 1}{\xi_0 p}\Big)^{1/p}, \quad \alpha_1 = \Big(\frac{e^{\xi_1 p} - 1}{\xi_1 p}\Big)^{1/p},$$

then

$$\|x\|_S \le \max\Big(\alpha_0\|x\|_0, \alpha_1\|x\|_1\Big) \le \max(\alpha_0, \alpha_1)\|x\|_{\mathcal{I}}.$$

Thus the map from $\mathcal{I}(E_0, E_1)$ to $S(p; \xi_0, E_0; \xi_1, E_1)$ is bounded by $\max(\alpha_0, \alpha_1)$. Now suppose that $x = \int_{\mathbb{R}} f(t)\ \mathrm{d}t$ for some representative f. Then

$$\|x\|_{\mathcal{S}} \le \int_{\mathbb{R}} \|f(t)\|_{\mathcal{S}}\ \mathrm{d}t \le \Big(\frac{1}{\xi_1 q}\Big)^{1/q} \|e^{\xi_1 t} f(t)\|_{L^p(E_1)} + \Big(\frac{1}{\xi_0 q}\Big)^{1/q} \|e^{\xi_0 t} f(t)\|_{L^p(E_0)},$$

so we see that

$$\|x\|_{\mathcal{S}} \le \Big(\Big(\frac{1}{\xi_1 q}\Big)^{1/q} + \Big(\frac{1}{\xi_0 q}\Big)^{1/q}\Big)\|x\|_S.$$

Thus the map from $S(p; \xi_0, E_0; \xi_1, E_1)$ to $\mathcal{S}(E_0, E_1)$ is also bounded.

Proposition 2.1. *Let* $\theta = \xi_0(\xi_0 - \xi_1)^{-1} \in (0,1)$. *Then*

$$\|x\|_{S(p;\xi_0,E_0;\xi_1,E_1)} = \inf\Big\{ \big\|e^{\xi_0 t} f(t)\big\|_{L^p(E_0)}^{1-\theta} \big\|e^{\xi_1 t} f(t)\big\|_{L^p(E_1)}^{\theta} : x = \int_{\mathbb{R}} f(t)\ \mathrm{d}t \Big\}.$$

Proof. This is [1, Chapter 1, Section 2, Proposition 1]. Notice that as $\xi_0 < 0$ and $\xi_1 > 0$, $\theta > 0$, and that $\xi_0 > \xi_0 - \xi_1$, so that $\theta < 1$. We claim that it is obvious that

$$\|x\|_{S(p;\xi_0,E_0;\xi_1,E_1)} \geq \inf\Big\{ \big\|e^{\xi_0 t} f(t)\big\|_{L^p(E_0)}^{1-\theta} \big\|e^{\xi_1 t} f(t)\big\|_{L^p(E_1)}^{\theta} : x = \int_{\mathbb{R}} f(t)\ \mathrm{d}t \Big\}.$$

This follows, as for $a, b > 0$ and $\theta \in (0,1)$, we have that $\max(a,b) \geq a^{1-\theta} b^{\theta}$.

Conversely, let $x = \int_{\mathbb{R}} f(t)\ \mathrm{d}t$. By the translation invariance of lebesgue measure, for $\tau \in \mathbb{R}$, we also have that $x = \int_{\mathbb{R}} f(t+\tau)\ \mathrm{d}t$. Thus

$$\begin{aligned}\|x\|_S &\leq \inf_\tau \max\Big(\big\|e^{\xi_0 t} f(t+\tau)\big\|_{L^p(E_0)}, \big\|e^{\xi_1 t} f(t+\tau)\big\|_{L^p(E_1)} \Big) \\ &= \inf_\tau \max\Big(e^{-\xi_0 \tau}\big\|e^{\xi_0 t} f(t)\big\|_{L^p(E_0)}, e^{-\xi_1 \tau}\big\|e^{\xi_1 t} f(t)\big\|_{L^p(E_1)} \Big).\end{aligned}$$

Then choose τ such that

$$\alpha := e^{-\xi_0 \tau}\big\|e^{\xi_0 t} f(t)\big\|_{L^p(E_0)} = e^{-\xi_1 \tau}\big\|e^{\xi_1 t} f(t)\big\|_{L^p(E_1)},$$

which we may do, as $\xi_0 < 0, \xi_1 > 0$. A calculation yields that

$$\alpha = \big\|e^{\xi_0 t} f(t)\big\|_{L^p(E_0)}^{1-\theta} \big\|e^{\xi_1 t} f(t)\big\|_{L^p(E_1)}^{\theta},$$

completing the proof. □

Corollary 2.2. *There exists a constant* $C > 0$ *(depending only on* ξ_0, ξ_1 *and* p*) such that*

$$\|x\|_{S(p;\xi_0,E_0;\xi_1,E_1)} \leq C \|x\|_0^{1-\theta} \|x\|_1^{\theta} \qquad (x \in \mathcal{I}(E_0,E_1)).$$

Proof. This is [1, Chapter 1, Section 2, Corollaire de la Proposition 1]. Let $x \in \mathcal{I}(E_0,E_1)$, so that we can represent x by $f(t) = \phi(t)x$, where $\phi : \mathbb{R} \to \mathbb{R}$ is measurable, has compact support, and has integral 1. Thus

$$\|x\|_S \leq \|x\|_0^{1-\theta}\|x\|_1^{\theta} \Big(\int_{\mathbb{R}} e^{\xi_0 t p} |\phi(t)|^p\ \mathrm{d}t \Big)^{\frac{1-\theta}{p}} \Big(\int_{\mathbb{R}} e^{\xi_1 t p} |\phi(t)|^p\ \mathrm{d}t \Big)^{\frac{\theta}{p}},$$

which completes the proof. □

Suppose that we have another compatible (F_0, F_1), and that $T : \mathcal{S}(E_0,E_1) \to \mathcal{S}(F_0,F_1)$ is a linear map. Suppose that T, restricted to E_0, is a bounded linear operator to F_0, with norm $\|T\|_0$, and similarly for E_1 to F_1 with norm $\|T\|_1$.

Proposition 2.3. *The operator* T *is a bounded linear operator from* $S(p;\xi_0,E_0;\xi_1,E_1)$ *to* $S(p;\xi_0,F_0;\xi_1,F_1)$ *with norm less than or equal to* $\|T\|_0^{1-\theta}\|T\|_1^{\theta}$.

Proof. This is [1, Chapter 1, Section 2, Proposition 2]. Let $x \in S(p;\xi_0,E_0;\xi_1,E_1)$ have representation $x = \int_{\mathbb{R}} f(t)\ \mathrm{d}t$, so that $T(x)$ has representation $\int_{\mathbb{R}} T(f(t))\ \mathrm{d}t$. Thus, by Proposition 2.1,

$$\begin{aligned}\|T(x)\|_S &\leq \big\|e^{\xi_0 t} T(f(t))\big\|_{L^p(F_0)}^{1-\theta} \big\|e^{\xi_1 t} T(f(t))\big\|_{L^p(F_1)}^{\theta} \\ &\leq \|T\|_0^{1-\theta}\|T\|_1^{\theta} \big\|e^{\xi_0 t} f(t)\big\|_{L^p(E_0)}^{1-\theta} \big\|e^{\xi_1 t} f(t)\big\|_{L^p(E_1)}^{\theta},\end{aligned}$$

which completes the proof. □

As in the introduction, if we have an operator from $T : \mathcal{I}(E_0, E_1) \to \mathcal{I}(F_0, F_1)$ which admits extensions to operators $T_i : E_i \to F_i$, for $i = 0, 1$, then T extends uniquely to an operator $\tilde{T} : \mathcal{S}(E_0, E_1) \to \mathcal{S}(F_0, F_1)$, and the above proposition gives the estimate $\|T_0\|^{1-\theta}\|T_1\|^{\theta}$ for the norm of the operator $\tilde{T} : S(p; \xi_0, E_0; \xi_1, E_1) \to S(p; \xi_0, F_0; \xi_1, F_1)$.

2.1 Varying the parameters

We shall now study how varying p, ξ_0 and ξ_1 affect the interpolation space S. Throughout, (E_0, E_1) shall be a compatible couple, and $\theta = \xi_0(\xi_0 - \xi_1)^{-1} \in (0, 1)$.

Proposition 2.4. *For $\lambda \neq 0$, the vector spaces $S(p; \xi_0, E_0; \xi_1, E_1)$ and $S(p; \lambda\xi_0, E_0; \lambda\xi_1, E_1)$ are equal, and the norms satisfy*

$$\|x\|_{S(p;\xi_0,E_0;\xi_1,E_1)} = \lambda^{1-1/p}\|x\|_{S(p;\lambda\xi_0,E_0;\lambda\xi_1,E_1)} \qquad (x \in S(p; \xi_0, E_0; \xi_1, E_1)).$$

Proof. This is [1, Chapter 1, Section 3, Proposition 1]. Let $x \in S(p; \xi_0, E_0; \xi_1, E_1)$ have representation $x = \int_{\mathbb{R}} f(t)\ \mathrm{d}t$, and define f_λ by $f_\lambda(t) = \lambda f(\lambda t)$ for $t \in \mathbb{R}$. By the homogeneity of the Lebesgue integral, $x = \int_{\mathbb{R}} f_\lambda(t)\ \mathrm{d}t$, so that

$$\begin{aligned}\|x\|_{S(p;\lambda\xi_0,E_0;\lambda\xi_1,E_1)} &\leq \left\|e^{\lambda\xi_0 t} f_\lambda(t)\right\|^{1-\theta}_{L^p(E_0)} \left\|e^{\lambda\xi_1 t} f_\lambda(t)\right\|^{\theta}_{L^p(E_1)} \\ &\leq \lambda^{\frac{p-1}{p}} \left\|e^{\xi_0 t} f(t)\right\|^{1-\theta}_{L^p(E_0)} \left\|e^{\xi_1 t} f(t)\right\|^{\theta}_{L^p(E_1)}.\end{aligned}$$

We complete the proof by replacing λ by λ^{-1}. □

Given ξ_0 and ξ_1, let $\lambda = (\xi_1 - \xi_0)^{-1}$ (note that $\xi_1 - \xi_0 > 0$) so that $\lambda\xi_0 = -\theta$ and $\lambda\xi_1 = 1 - \theta$. Hence the interpolation space $S(p; \xi_0, E_0; \xi_1, E_1)$ is equivalent (by factor $(\xi_1 - \xi_0)^{1/p-1}$) to the space $S(p; -\theta, E_0; 1 - \theta, E_1)$. We denote the resulting family of isomorphic interpolation spaces by $(E_0, E_1)_{\theta,p}$ (that is, we consider all spaces of the form $S(p; \xi_0, E_0; \xi_1, E_1)$ where $\xi_0 < 0, \xi_1 > 0$ and $\theta(\xi_0 - \xi_1) = \xi_0$). It is common to isometrically associate $(E_0, E_1)_{\theta,p}$ with $S(p; -\theta, E_0; 1 - \theta, E_1)$ (is this true???)

Proposition 2.5. *For $\theta \in (0, 1)$ and $p \leq q$, the natural map*

$$(E_0, E_1)_{\theta,p} \to (E_0, E_1)_{\theta,q}$$

is a continuous injection.

Proof. This is [1, Chapter 1, Section 3, Proposition 2]. Pick $\xi_0 < 0$ and $\xi_1 > 0$ with $\theta = \xi_0(\xi_0 - \xi_1)^{-1}$. For $x \in S(p; \xi_0, E_0; \xi_1, E_1)$ with representation $x = \int_{\mathbb{R}} f(t)\ \mathrm{d}t$, let $\phi : \mathbb{R} \to \mathbb{R}$ have compact support and integral 1, and consider the convolution $g(t) = \int_{\mathbb{R}} f(t-s)\phi(s)\ \mathrm{d}s$. Again, we have that $x = \int_{\mathbb{R}} g(t)\ \mathrm{d}t$, so if r satisfies $r^{-1} = 1-(p^{-1}-q^{-1})$, then

$$\begin{aligned}\left\|e^{\xi_0 t} g(t)\right\|_{L^q(E_0)} &\leq \left\|e^{\xi_0 t}\phi(t)\right\|_{L^r} \left\|e^{\xi_0 t} f(t)\right\|_{L^p(E_0)}, \\ \left\|e^{\xi_1 t} g(t)\right\|_{L^q(E_1)} &\leq \left\|e^{\xi_1 t}\phi(t)\right\|_{L^r} \left\|e^{\xi_1 t} f(t)\right\|_{L^p(E_1)}.\end{aligned}$$

Consequently, the norm of the injection is bounded above by

$$\inf\left\{\left\|e^{\xi_0 t}\phi(t)\right\|^{1-\theta}_{L^r} \left\|e^{\xi_1 t}\phi(t)\right\|^{\theta}_{L^r} : \int \phi(t)\ \mathrm{d}t = 1, \phi \text{ has compact support}\right\}.$$

□

We consequently have a family of spaces which lie between E_0 and E_1. We can verify that, for $x \in \mathcal{I}(E_0, E_1)$,

$$\lim_{\xi_0 \to 0} \|x\|_{S(p;\xi_0,E_0;1,E_1)} = \|x\|_0, \qquad \lim_{\xi_1 \to 0} \|x\|_{S(p;-1,E_0;\xi_1,E_1)} = \|x\|_1.$$

2.2 Discrete definitions

We can consider discrete analogues of the above definitions, which are often easier to perform calculations with. We consider sequences $(x_n)_{n\in\mathbb{Z}}$ in $\mathcal{I}(E_0, E_1)$ such that

$$\left\|(e^{\xi_0 n}x_n)\right\|_{l^p(E_0)} := \Big(\sum_n \left\|e^{\xi_0 n}x_n\right\|_0^p\Big)^{1/p} < \infty,$$
$$\left\|(e^{\xi_1 n}x_n)\right\|_{l^p(E_1)} := \Big(\sum_n \left\|e^{\xi_1 n}x_n\right\|_1^p\Big)^{1/p} < \infty,$$

where, as before, $\xi_0 < 0$, $\xi_1 > 0$, $p \in [1, \infty]$ and (E_0, E_1) is a compatible couple. Then we denote by $s_1(p; \xi_0, E_0; \xi_1, E_1)$ the space of $x \in \mathcal{S}(E_0, E_1)$ such that for some $(x_n)_{n\in\mathbb{Z}} \subseteq \mathcal{I}(E_0, E_1)$ satisfying the above, we have that $x = \sum_n x_n$ with convergence in $\mathcal{S}$. We give s_1 the norm

$$\|x\|_{s_1} = \inf\Big\{\max\Big(\left\|(e^{\xi_0 n}x_n)\right\|_{l^p(E_0)}, \left\|(e^{\xi_1 n}x_n)\right\|_{l^p(E_1)}\Big) : x = \sum_n x_n\Big\}.$$

Notice that if $(x_n) \subseteq \mathcal{I}$ is such that $\left\|(e^{\xi_0 n}x_n)\right\|_{l^p(E_0)} < \infty$ and $\left\|(e^{\xi_1 n}x_n)\right\|_{l^p(E_1)} < \infty$, then

$$\begin{aligned}
\sum_n \|x_n\|_{\mathcal{S}} &\leq \sum_{n=-\infty}^{-1} e^{-\xi_0 n}\|e^{\xi_0 n}x_n\|_0 + \sum_{n=0}^{\infty} e^{-\xi_1 n}\|e^{\xi_1 n}x_n\|_1 \\
&\leq \Big(\sum_{n=-\infty}^{-1} e^{-\xi_0 nq}\Big)^{1/q}\left\|(e^{\xi_0 n}x_n)\right\|_{l^p(E_0)} + \Big(\sum_{n=0}^{\infty} e^{-\xi_1 nq}\Big)^{1/q}\left\|(e^{\xi_1 n}x_n)\right\|_{l^p(E_1)} \\
&= \Big(\frac{e^{\xi_0 q}}{1-e^{\xi_0 q}}\Big)^{1/q}\left\|(e^{\xi_0 n}x_n)\right\|_{l^p(E_0)} + \Big(\frac{1}{1-e^{-\xi_1 q}}\Big)^{1/q}\left\|(e^{\xi_1 n}x_n)\right\|_{l^p(E_1)}
\end{aligned}$$

where $q^{-1} = 1 - p^{-1}$. Thus certainly $\sum_n x_n$ converges in $\mathcal{S}$.

Proposition 2.6. *The spaces $S(p; \xi_0, E_0; \xi_1, E_1)$ and $s_1(p; \xi_0, E_0; \xi_1, E_1)$ are naturally isomorphic.*

Proof. This is [1, Chapter 1, Section 4, Proposition 1]. Let $x \in S(p; \xi_0, E_0; \xi_1, E_1)$ have representation $x = \int_{\mathbb{R}} f(t)\ \mathrm{d}t$, and let $x_n = \int_n^{n+1} f(t)\ \mathrm{d}t$ for each $n \in \mathbb{Z}$. Then $x = \sum_n x_n$ in $\mathcal{S}$, $x_n \in \mathcal{I}$ for each n, and

$$\begin{aligned}
\left\|(e^{\xi_0 n}x_n)\right\|_{l^p(E_0)}^p &= \sum_n \left\|e^{\xi_0 n}\int_n^{n+1} f(t)\ \mathrm{d}t\right\|_0^p \leq \sum_n \int_n^{n+1} \left\|e^{\xi_0 n}f(t)\right\|_0^p\ \mathrm{d}t \\
&\leq e^{-\xi_0 p}\int_{\mathbb{R}} \left\|e^{\xi_0 t}f(t)\right\|_0^p\ \mathrm{d}t,
\end{aligned}$$

as $\xi_0 < 0$. Similarly,

$$\left\|(e^{\xi_1 n}x_n)\right\|_{l^p(E_1)} \leq \left\|e^{\xi_1 t}f(t)\right\|_{L^p(E_1)}.$$

Consequently $\|x\|_{s_1} \leq e^{-\xi_0}\|x\|_S$.

Conversely, let $x \in s_1$ with $x = \sum_n x_n$. Define $f : \mathbb{R} \to \mathcal{I}$ be setting $f(t) = x_n$ for $n \leq t < n+1$. Then $x = \int_{\mathbb{R}} f(t)$ and

$$\left\|e^{\xi_0 t}f(t)\right\|_{L^p(E_0)}^p = \sum_n \int_n^{n+1} \left\|e^{\xi_0 t}x_n\right\|_0^p\ \mathrm{d}t = \sum_n \|x_n\|_0^p e^{\xi_0 np}\frac{e^{\xi_0 p}-1}{\xi_0 p} \leq \left\|(e^{\xi_0 n}x_n)\right\|_{l^p(E_0)}^p,$$

and similarly, we can show that

$$\|e^{\xi_1 t} f(t)\|_{L^p(E_1)} = \Big(\frac{e^{\xi_1 p}-1}{\xi_1 p}\Big)^{1/p} \|(e^{\xi_1 n} x_n)\|_{l^p(E_1)} \leq e^{\xi_1} \|(e^{\xi_1 n} x_n)\|_{l^p(E_1)}.$$

Consequently,

$$\|x\|_S \leq \exp\Big(\frac{\xi_0 \xi_1}{\xi_0 - \xi_1}\Big) \|x\|_{s_1}.$$

Note: I am not sure where this last inequality comes from, but it is in [1]. □

Consider now sequences $(x_n^0)_{n\in\mathbb{Z}} \subseteq E_0$ and $(x_n^1)_{n\in\mathbb{Z}} \subseteq E_1$ such that

$$\|(e^{\xi_0 n} x_n^0)\|_{l^p(E_0)} < \infty, \qquad \|(e^{\xi_1 n} x_n^1)\|_{l^p(E_1)} < \infty.$$

Suppose also that $x_n^0 + x_n^1 = x \in \mathcal{S}$ for each $n \in \mathbb{Z}$. We denote by $s_2(p; \xi_0, E_0; \xi_1, E_1)$ the collection of such x with the norm

$$\|x\|_{s_2} = \inf\Big\{ \max\Big(\|(e^{\xi_0 n} x_n^0)\|_{l^p(E_0)}, \|(e^{\xi_1 n} x_n^1)\|_{l^p(E_1)} \Big) : x = x_n^0 + x_n^1 \ (n \in \mathbb{Z}) \Big\}.$$

Proposition 2.7. *The spaces s_1 and s_2 are naturally isomorphic. To be precise, for $x \in s_1$,*

$$(1+e^{\xi_1})^{-1} \|x\|_{s_1} \leq \|x\|_{s_2} \leq \max\Big(\frac{1}{1-e^{\xi_0}}, \frac{1}{1-e^{-\xi_1}}\Big) \|x\|_{s_1}.$$

Proof. This is [1, Chapter 1, Section 4, Proposition 2]. Let $x \in s_1$ be such that $x = \sum_n x_n$. Then let

$$y_n^0 = \sum_{k\geq 0} x_{n-k}, \qquad y_n^1 = \sum_{k<0} x_{n-k} \qquad (n \in \mathbb{Z}),$$

so that $y_n^0 + y_n^1 = \sum_k x_k = x$ for each n. Then, by the triangle inequality

$$\begin{aligned}\|(e^{\xi_0 n} y_n^0)\|_{l^p(E_0)} &= \Big\| \Big(\sum_{k\geq 0} e^{\xi_0 k} e^{\xi_0 (n-k)} x_{n-k} \Big) \Big\|_{l^p(E_0)} \\ &\leq \sum_{k\geq 0} e^{\xi_0 k} \|(e^{\xi_0(n-k)} x_{n-k})\|_{l^p(E_0)} = \frac{1}{1-e^{\xi_0}} \|(e^{\xi_0 n} x_n)\|_{l^p(E_0)},\end{aligned}$$

and similarly,

$$\begin{aligned}\|(e^{\xi_1 n} y_n^1)\|_{l^p(E_1)} &= \Big\| \Big(\sum_{k<0} e^{\xi_1 k} e^{\xi_1 (n-k)} x_{n-k} \Big) \Big\|_{l^p(E_1)} \\ &\leq \sum_{k<0} e^{\xi_1 k} \|(e^{\xi_1(n-k)} x_{n-k})\|_{l^p(E_1)} = \frac{1}{1-e^{-\xi_1}} \|(e^{\xi_1 n} x_n)\|_{l^p(E_1)},\end{aligned}$$

Hence we conclude that

$$\|x\|_{s_2} \leq \max\Big(\frac{1}{1-e^{\xi_0}}, \frac{1}{1-e^{-\xi_1}}\Big) \|x\|_{s_1}.$$

Conversely, if $x \in s_2$ and $((y_n^0), (y_n^1))$ represents x, then for each $k \in \mathbb{Z}$, let

$$x_k = y_k^0 - y_{k-1}^0 = y_{k-1}^1 - y_k^1,$$

so that $x_k \in \mathcal{I}$ for each k. Then $\sum_{n\leq 0} x_n$ converges in $\mathcal{S}$, as $(e^{\xi_0 n} y_n^0) \in l^p(E_0)$, so that for $n < 0$, $\|y_n^0\|_{\mathcal{S}} \leq \|y_n^0\|_0 \leq e^{\xi_n} \|y_n^0\|_0 \to 0$ as $n \to -\infty$. Similarly, $\sum_{n>0} x_n$ converges in $\mathcal{S}$, and hence we have

$$\sum_n x_n = \sum_{n\leq 0} x_n + \sum_{n>0} x_n = y_0^0 + y_0^1 = x.$$

We then see that

$$\begin{aligned}\big\|(e^{\xi_0 n}x_n)\big\|_{l^p(E_0)} &= \Big(\sum_n \|e^{\xi_0 n}(y_n^0 - y_{n-1}^0)\|_0^p\Big)^{1/p} \\ &\le \big\|(e^{\xi_0 n}y_n^0)\big\|_{l^p(E_0)} + e^{\xi_0}\big\|(e^{\xi_0(n-1)}y_{n-1}^0)\big\|_{l^p(E_0)} \\ &= (1+e^{\xi_0})\big\|(e^{\xi_0 n}y_n^0)\big\|_{l^p(E_0)},\end{aligned}$$

and similarly,

$$\big\|(e^{\xi_1 n}x_n)\big\|_{l^p(E_1)} \le (1+e^{\xi_1})\big\|(e^{\xi_1 n}y_n^1)\big\|_{l^p(E_1)}.$$

Thus we conclude that

$$\|x\|_{s_1} \le (1+e^{\xi_1})\|x\|_{s_2}.$$

□

Proposition 2.8. *For $x \in s_1$, we have*

$$\begin{aligned}&\inf\Big\{\big\|(e^{\xi_0 n}x_n)\big\|_{l^p(E_0)}^{1-\theta}\big\|(e^{\xi_1 n}x_n)\big\|_{l^p(E_1)}^{\theta} : \sum_n x_n = x\Big\} \le \|x\|_{s_1} \\ &\qquad \le \exp\Big(\frac{\xi_0\xi_1}{\xi_0-\xi_1}\Big)\inf\Big\{\big\|(e^{\xi_0 n}x_n)\big\|_{l^p(E_0)}^{1-\theta}\big\|(e^{\xi_1 n}x_n)\big\|_{l^p(E_1)}^{\theta} : \sum_n x_n = x\Big\}, \\ &\inf\Big\{\big\|(e^{\xi_0 n}y_n^0)\big\|_{l^p(E_0)}^{1-\theta}\big\|(e^{\xi_1 n}y_n^1)\big\|_{l^p(E_1)}^{\theta} : y_n^0 + y_n^1 = x\ (n\in\mathbb{Z})\Big\} \le \|x\|_{s_2} \\ &\qquad \le \exp\Big(\frac{\xi_0\xi_1}{\xi_0-\xi_1}\Big)\inf\Big\{\big\|(e^{\xi_0 n}y_n^0)\big\|_{l^p(E_0)}^{1-\theta}\big\|(e^{\xi_1 n}y_n^1)\big\|_{l^p(E_1)}^{\theta} : y_n^0 + y_n^1 = x\ (n\in\mathbb{Z})\Big\}\end{aligned}$$

Proof. This is [1, Chapter 1, Section 4, Proposition 3]. As in the proof of Proposition 2.1, the first inequality is simple. For the second, we note that if $x = \sum_n x_n$, then also $x = \sum_n x_{n+k}$ for each $k \in \mathbb{Z}$, so that

$$\|x\|_{s_1} \le \max\Big(e^{-\xi_0 k}\big\|(e^{\xi_0 n}x_n)\big\|_{l^p(E_0)}, e^{-\xi_1 k}\big\|(e^{\xi_1 n}x_n)\big\|_{l^p(E_1)}\Big) = \beta(k),$$

say. As in the proof of Proposition 2.1, we can choose $t \in \mathbb{R}$ such that

$$\alpha := e^{-\xi_0 t}\big\|(e^{\xi_0 n}x_n)\big\|_{l^p(E_0)} = e^{-\xi_1 t}\big\|(e^{\xi_1 n}x_n)\big\|_{l^p(E_1)} = \big\|(e^{\xi_0 n}x_n)\big\|_{l^p(E_0)}^{1-\theta}\big\|(e^{\xi_1 n}x_n)\big\|_{l^p(E_1)}^{\theta} \ge 0.$$

Let $\lfloor t\rfloor, \lceil t\rceil \in \mathbb{Z}$ be such that $\lfloor t\rfloor \le t < 1 + \lfloor t\rfloor$ and $t \le \lceil t\rceil < t+1$. Then

$$\begin{aligned}e^{\xi_1(t-\lfloor t\rfloor)}\alpha &= e^{-\xi_1\lfloor t\rfloor}\big\|(e^{\xi_1 n}x_n)\big\|_{l^p(E_1)} = \beta(\lfloor t\rfloor), \\ e^{\xi_0(t-\lceil t\rceil)}\alpha &= e^{-\xi_0\lceil t\rceil}\big\|(e^{\xi_0 n}x_n)\big\|_{l^p(E_0)} = \beta(\lceil t\rceil).\end{aligned}$$

Notice that

$$\sup_{s\in[0,1]}\min(e^{\xi_1 s}, e^{-\xi_0(1-s)}) = \max\Big(\sup_{0\le s\le\theta} e^{\xi_1 s},\ \sup_{\theta<s\le 1} e^{-\xi_0(1-s)}\Big) = e^{\xi_1\theta} = e^{-\xi_0(1-\theta)},$$

so, as $(t-\lfloor t\rfloor) - (t-\lceil t\rceil) = 1$, we see that

$$\|x\|_{s_1} \le \inf_k \beta(k) = \min\big(\beta(\lfloor t\rfloor), \beta(\lceil t\rceil)\big) = \min\big(e^{\xi_1(t-\lfloor t\rfloor)}, e^{\xi_0(t-\lceil t\rceil)}\big)\alpha \le e^{\xi_1\theta}\alpha,$$

as required.

For proof for s_2 proceeds in an analogous manner. □

For a compatible couple (E_0, E_1) let B_i be the canonical image of the closed unit ball of E_i in $\mathcal{S}(E_0, E_1)$. Then let

$$U_n = e^{-\xi_0 n} B_0 + e^{-\xi_1 n} B_1 \subseteq \mathcal{S} \qquad (n \in \mathbb{Z}),$$

so that U_n is an absolutely convex subset of $\mathcal{S}$. Then let ψ_n be the *gauge* of U_n, that is,

$$\psi_n(x) = \inf\{t > 0 : x \in tU_n\} \qquad (x \in \mathcal{S}).$$

It is simple to verify that ψ_n is an equivalent norm on $\mathcal{S}$: indeed, $\psi_n(x) \le \max(e^{\xi_0 n}, e^{\xi_1 n})\|x\|_{\mathcal{S}}$ and $\|x\|_{\mathcal{S}} \le (e^{-\xi_0 n} + e^{-\xi_1 n})\psi_n(x)$ for each $x \in \mathcal{S}$. We then let $s(p; \xi_0, E_0; \xi_1, E_1)$ be the space of $x \in \mathcal{S}$ such that

$$\|x\|_s := \Big(\sum_{n\in\mathbb{Z}} \psi_n(x)^p\Big)^{1/p} < \infty.$$

Proposition 2.9. *The spaces S and s are naturally isomorphic; more specifically,*

$$2^{-1/p}\|x\|_{s_2} \le \|x\|_s \le 2^{1/p}\|x\|_{s_2} \qquad (x \in s),$$

the result following as s_2 and S are isomorphic.

Proof. This is [1, Chapter 1, Section 4, Proposition 4]. For $x \in s_2$, we have that

$$\begin{aligned}\|x\|_{s_2} &\le \inf\Big\{\Big(\sum_{n\in\mathbb{Z}} \|e^{\xi_0 n} y_n^0\|_0^p + \|e^{\xi_1 n} y_n^1\|_1^p\Big)^{1/p} : x = y_n^0 + y_n^1 \ (n \in \mathbb{Z})\Big\} \\ &= \Big(\sum_{n\in\mathbb{Z}} \inf\big\{\|e^{\xi_0 n} y_n^0\|_0^p + \|e^{\xi_1 n} y_n^1\|_1^p : x = y_n^0 + y_n^1\big\}\Big)^{1/p} \le 2^{1/p}\|x\|_{s_2}.\end{aligned}$$

However, we also have that

$$\begin{aligned}\psi_n(x) &= \inf\big\{t > 0 : \exists\, y \in B_0, z \in B_1, x = t(e^{-\xi_0 n} y + e^{-\xi_1 n} z)\big\} \\ &= \inf\big\{t > 0 : \exists\, y \in E_0, z \in E_1, \|y\|_0 \le te^{-\xi_0 n}, \|z\|_1 \le te^{-\xi_1 n}, x = y + z\big\} \\ &= \inf\big\{\max\big(\|e^{\xi_0 n} y\|_0, \|e^{\xi_1 n} z\|_1\big) : y \in E_0, z \in E_1, x = y + z\big\},\end{aligned}$$

so that

$$\|x\|_s \le \Big(\sum_{n\in\mathbb{Z}} \inf\big\{\|e^{\xi_0 n} y_n^0\|_0^p + \|e^{\xi_1 n} y_n^1\|_1^p : x = y_n^0 + y_n^1\big\}\Big)^{1/p} \le 2^{1/p}\|x\|_s,$$

completing the proof. □

2.3 Aside on notation

In modern literature, the following notation (see [3]) is more commonly used. For $a, b > 0$, we define

$$k(x, a, b) = \inf\big\{a\|x_0\|_0 + b\|x_1\|_1 : x = x_0 + x_1\big\} \qquad (x \in \mathcal{S}(E_0, E_1)).$$

Then, for example, the norm on $\mathcal{S}$ is $k(\cdot, 1, 1)$, while the norm on s is seen to be equivalent to the norm

$$\|x\| = \Big(\sum_{n\in\mathbb{Z}} k(x, e^{\xi_0 n}, e^{\xi_1 n})^p\Big)^{1/p} \qquad (x \in s),$$

as ψ_n is clearly equivalent to $k(\cdot, e^{\xi_0 n}, e^{\xi_1 n})$.

There are more complicated examples of interpolation spaces based on unconditional bases in Banach spaces (for example, the base space for s_1 and s_2 is $l^p(\mathbb{Z})$).

2.4 When E_0 embeds into E_1

Suppose that (E_0, E_1) is a compatible couple, that E_0 is actually a subspace of E_1, and that for some constant C, we have that $\|x\|_1 \leq C\|x\|_0$ for each $x \in E_0$. We denote this by $E_0 \hookrightarrow E_1$. Then $\mathcal{I}(E_0, E_1) = E_0$ with equivalent norms: $\|x\|_0 \leq \|x\|_{\mathcal{I}} \leq C\|x\|_0$ for $x \in E_0$. Similarly, $\mathcal{S}(E_0, E_1) = E_1$ with equivalent norms: $C^{-1}\|x\|_1 \leq \|x\|_{\mathcal{S}} \leq \|x\|_1$. We hence see that each of the spaces $(E_0, E_1)_{\theta,p}$ lie between E_0 and E_1, and the injection $E_0 \to E_1$ factors through $(E_0, E_1)_{\theta,p}$.

Proposition 2.10. *We define the following norms on* $(E_0, E_1)_{\theta,p}$*:*

$$\|x\|_{S^+} = \inf\Big\{ \max\Big(\Big(\int_0^\infty \|e^{\xi_0 t} f(t)\|_0^p \,\mathrm{d}t\Big)^{1/p}, \Big(\int_0^\infty \|e^{\xi_1 t} f(t)\|_1^p \,\mathrm{d}t\Big)^{1/p}\Big) : x = \int_0^\infty f(t)\,\mathrm{d}t\Big\},$$

$$\|x\|_{s_1^+} = \inf\Big\{ \max\Big(\Big(\sum_{n=0}^\infty \|e^{\xi_0 n} x_n\|_0^p\Big)^{1/p}, \Big(\sum_{n=0}^\infty \|e^{\xi_1 n} x_n\|_1^p\Big)^{1/p}\Big) : x = \sum_{n=0}^\infty x_n\Big\},$$

$$\|x\|_{s_2^+} = \inf\Big\{ \max\Big(\Big(\sum_{n=0}^\infty \|e^{\xi_0 n} x_n^0\|_0^p\Big)^{1/p}, \Big(\sum_{n=0}^\infty \|e^{\xi_1 n} x_n^1\|_1^p\Big)^{1/p}\Big) : x = x_n^0 + x_n^1 \ (n \geq 0)\Big\},$$

$$\|x\|_{s^+} = \Big(\sum_{n=0}^\infty \psi_n(x)^p\Big)^{1/p}.$$

All of these define equivalent norms on $(E_0, E_1)_{\theta,p}$.

Proof. See [1, Chapter 1, Section 5, Proposition 1]. □

Proposition 2.11. *Let* $0 < \theta_1 < \theta_2 < 1$ *and* $1 \leq p_1, p_2 \leq \infty$*. Then* $(E_0, E_1)_{\theta_1,p_1}$ *is a subspace of* $(E_0, E_1)_{\theta_2,p_2}$ *and the injection map is continuous.*

Proof. See [1, Chapter 1, Section 5, Proposition 2]. □

2.5 Influence of the intersection $\mathcal{I}$

Let (E_0, E_1) be a compatible couple, and denote by $\underline{E_0}$ the closure of the image of $\mathcal{I}(E_0, E_1)$ in E_0; similarly $\underline{E_1}$. Then $(\underline{E_0}, \underline{E_1})$ is a compatible couple, and clearly $\mathcal{I}(\underline{E_0}, \underline{E_1}) = \mathcal{I}(E_0, E_1)$, while $\mathcal{S}(\underline{E_0}, \underline{E_1}) \to \mathcal{S}(E_0, E_1)$ is a norm-decreasing map.

Proposition 2.12. *Let* $\xi_0 < 0, \xi_1 > 0$ *and* $p \in [1, \infty]$*. Then* $s_1(p; \xi_0, E_0; \xi_1, E_1)$ *is isometrically isomorphic to* $s_1(p; \xi_0, \underline{E_0}; \xi_1, \underline{E_1})$*. The same holds for* S, s_2 *and* s*.*

Proof. This is [1, Chapter 2, Section 1, Proposition 1]. As $\mathcal{I}(E_0, E_1) = \mathcal{I}(\underline{E_0}, \underline{E_1})$, the result holds for s_1 and S.

Clearly the map $s_2(p; \xi_0, \underline{E_0}; \xi_1, \underline{E_1}) \to s_2(p; \xi_0, E_0; \xi_1, E_1)$ is norm-decreasing. If $x \in s_2(p; \xi_0, E_0; \xi_1, E_1)$ has representation $((x_n^0), (x_n^1))$, then for each $n \in \mathbb{Z}$,

$$x_n = x_n^0 - x_{n-1}^0 = x_{n-1}^1 - x_n^1 \in \mathcal{I}(E_0, E_1).$$

Also, as $\|(e^{\xi_0 n} x_n^0)\|_{l^p(E_0)} < \infty$, we see that $\|x_n^0\|_0 \to 0$ as $n \to -\infty$, and similarly $\|x_n^1\|_1 \to 0$ as $n \to \infty$. Thus, for $N > 0$ very large, we have that

$$x_n^0 = x_{-N}^0 + \sum_{k=1-N}^n x_k \qquad (n\mathbb{Z}),$$

so that x_n^0 can be approximately arbitrarily well by a member of $\mathcal{I}$, implying that $x_n^0 \in \underline{E_0}$, for each n. Similarly, $x_n^1 \in \underline{E_1}$ for each n, which completes the proof for s_2.

The argument for s follows in an entirely similar manner, using the same techniques as in the proof of Proposition 2.9. □

Consequently, we can always assume that $\mathcal{I}$ is dense in both E_0 and E_1 (loosely speaking, this means that E_0 is dense in E_1 and E_1 is dense in E_0).

Corollary 2.13. *Suppose that $E_0 \hookrightarrow E_1$, and let F be the closure of E_0 in E_1. Then $S(p;\xi_0,E_0;\xi_1,E_1) = S(p;\xi_0,E_0;\xi_1,F)$, and similarly for s_1, s_2 and s.*

Proof. This is [1, Chapter 2, Section 1, Corollaire 1]. This follows as, identifying E_0 as a subspace of E_1, clearly $E_0 \cap E_1 = E_0$ and so $\mathcal{I}(E_0,E_1) = E_0$ algebraically, and $\mathcal{I}(E_0,E_1) = (E_0,F)$. Furthermore, $\underline{E_0} = E_0$ and $\underline{E_1} = F$, completing the proof. □

Proposition 2.14. *For each $\theta \in (0,1)$ and $p \in [1,\infty]$, we have that $\mathcal{I}$ is dense in $(E_0,E_1)_{\theta,p}$ with regards the norm $\|\cdot\|_{\mathcal{S}}$. If $p \neq \infty$, then $\mathcal{I}$ is dense in $(E_0,E_1)_{\theta,p}$ with respect to the norm on $(E_0,E_1)_{\theta,p}$.*

Proof. This is [1, Chapter 2, Section 1, Proposition 2]. Notice that as density is invariant under equivalent norms, we are free to work with, say, s_1 for some ξ_0,ξ_1 giving θ. Let $x \in s_1(p;\xi_0,E_0;\xi_1,E_1)$ have representation $x = \sum_{n\in\mathbb{Z}} x_n$, so that $x = \sum_{M\to\infty}\sum_{|n|\leq M} x_n$ in $\mathcal{S}$, where the partial sums lie in $\mathcal{I}$ as required.

Now suppose that $p \neq \infty$. Then we see that

$$\Big\| \sum_{|n|\leq M} x_n \Big\|_{s_1} = \Big\| x - \sum_{|n|>M} x_n \Big\|_{s_1}$$
$$\leq \max\Big(\Big(\sum_{|n|>M} \|e^{\xi_0 n} x_n\|_0^p \Big)^{1/p}, \Big(\sum_{|n|>M} \|e^{\xi_1 n} x_n\|_1^p \Big)^{1/p} \Big),$$

which tends to 0 as $M \to \infty$. □

2.6 Properties of the injection $s \to \mathcal{S}$

We shall work with the space $s = s(p;\xi_0,E_0;\xi_1,E_1)$ for convenience; all the results in this and the next section are isomorphic in natural, and so apply to any of the equivalent norms on $(E_0,E_1)_{\theta,p}$. Let us recall some standard facts. For a sequence of Banach spaces $(E_n)_{n\in\mathbb{Z}}$ and $p \in [1,\infty)$, we let

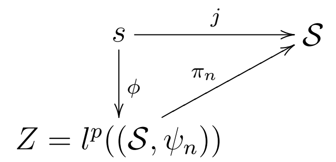
$$l^p\Big(\bigoplus_{n\in\mathbb{Z}} E_n\Big) = l^p(E_n) = \Big\{ (x_n)_{n\in\mathbb{Z}} : x_n \in E_n\,(n\in\mathbb{Z}), \|(x_n)\| = \Big(\sum_{n\in\mathbb{Z}} \|x_n\|^p\Big)^{1/p} < \infty \Big\}.$$

Then $l^p(E_n)' = l^q(E_n')$ where $p^{-1}+q^{-1}=1$, and so if $p \neq 1$, $l^p(E_n)'' = l^p(E_n'')$. For a Banach space E and a closed subspace F, there is a natural map $E' \to F'$ with kernel

$$F^\circ = \{\mu \in E' : \langle \mu, x\rangle = 0\,(x \in F)\},$$

and it is easily checked that the induced map $E'/F^\circ \to F'$ is an isometric isomorphism.

For a compatible couple (E_0,E_1) (we assume, as we may, that $\mathcal{I}$ is dense in both) we denote by j the map $s(p;\xi_0,E_0;\xi_1,E_1) \to \mathcal{S}(E_0,E_1)$. Notice that s is isometrically a subspace of $Z = l^p((\mathcal{S},\psi_n))$, indeed, define $\phi : s \to Z$, $\phi(x) = (\cdots, j(x), j(x), \cdots)$, so that ϕ is an isometry onto its range. For $n \in \mathbb{Z}$, let $\pi_n : Z \to \mathcal{S}$ be the n-th coordinate projection, so that π_n is continuous. Then, for any $n \in \mathbb{Z}$, $\pi_n \circ \phi = j$,



From now on, we suppose that $1 < p < \infty$, so that $l^p((\mathcal{S}, \phi)n))'' = l^p((\mathcal{S}, \phi_n^{**}))$, where ϕ_n^* be the dual norm to ϕ_n, so that $(\mathcal{S}, \phi_n)' = (\mathcal{S}', \phi_n^*)$, and $\phi_n^{**} = (\phi_n^*)^*$.

Lemma 2.15. *For $p \in (1, \infty)$, the map $\phi'' : s'' \to Z'' = l^p((\mathcal{S}, \phi_n^{**}))$ is defined by $\phi''(\Phi) = (\cdots, j''(\Phi), j''(\Phi), \cdots)$ for $\Phi \in s''$. Thus $j'' = \pi_n'' \circ \phi''$ for each $n \in \mathbb{Z}$.*

Proof. As ϕ is an isometry, it is standard that $\phi' : l^q((\mathcal{S}', \phi_n^*)) \to s'$ factors to give an isometric isomorphism

$$\phi' : l^q((\mathcal{S}', \phi_n^*))/\phi(s)^\circ \to s',$$

so that $\phi'' : s'' \to \phi(s)^{\circ\circ} \subseteq l^p((\mathcal{S}, \phi_n^{**}))$ is also an isometric isomorphism. Now, for $\mu = (\mu_n) \in l^q((\mathcal{S}', \phi_n^*))$, we see that

$$\langle \phi'(\mu), x \rangle = \sum_{n \in \mathbb{Z}} \langle \mu_n, j(x) \rangle = \sum_{n \in \mathbb{Z}} \langle j'(\mu_n), x \rangle \qquad (x \in s).$$

Thus, for $\Phi \in s''$, we have

$$\langle \phi''(\Phi), \mu \rangle = \sum_{n \in \mathbb{Z}} \langle \Phi, j'(\mu_n) \rangle = \langle (\cdots, j''(\Phi), j''(\Phi), \cdots), (\mu_n) \rangle \qquad (\mu \in l^q((\mathcal{S}', \phi_n^*))),$$

as required. $\square$

Proposition 2.16. *When $1 < p < \infty$, the map j'' is an injection, and $(j'')^{-1}(\mathcal{S}) = s$ (where we identify $\mathcal{S}$ with its image in $\mathcal{S}''$, and the same for s).*

Proof. This is [1, Chapter 2, Section 2, Proposition 1]. As noted in the Lemma, ϕ'' is an isometry onto its range, $\phi(s)^{\circ\circ}$, and so j'' must be injective. We then see that

$$\begin{aligned} (\phi'')^{-1}(\kappa_Z(\phi(s))) &= \big\{\Phi \in s'' : \exists x \in s,\ \phi''(\Phi) = \kappa_Z(\phi(x))\big\} \\ &= \big\{\Phi \in s'' : \exists x \in s,\ (\cdots, j''(\Phi), j''(\Phi), \cdots) = \kappa_Z(\cdots, j(x), j(x), \cdots)\big\} \\ &= \big\{\Phi \in s'' : \exists x \in s,\ j''(\Phi) = \kappa_{\mathcal{S}}(j(x))\big\}. \end{aligned}$$

As j'' is injective, we see that $(\phi'')^{-1}(\kappa_Z(\phi(s))) = \kappa_s(s)$. We then see that

$$\begin{aligned} (j'')^{-1}(\kappa_{\mathcal{S}}(\mathcal{S})) &= \{\Phi \in s'' : j''(\Phi) \in \mathcal{S}\} = \{\Phi \in s'' : \pi_0''(\phi''(\Phi)) \in \mathcal{S}\} \\ &= \{\Phi \in s'' : \phi''(\Phi) \in \kappa_Z(Z)\} = \{\Phi \in s'' : \phi''(\Phi) \in \kappa_Z(Z) \cap \phi(s)^{\circ\circ}\} \\ &= \{\Phi \in s'' : \phi''(\Phi) \in \phi(s)\} = \kappa_s(s). \end{aligned}$$

This follows, as for $\phi''(\Phi) = (j''(\Phi))$, we see that $\pi_0''(\phi''(\Phi)) = j''(\Phi) \in \mathcal{S}$ if and only if $\phi''(\Phi) = (j''(\Phi)) \in Z$; also $\kappa_Z(Z) \cap \phi(s)^{\circ\circ} = \phi(s)$, as $\phi(s)$ is closed in Z. Thus $(j'')^{-1}(\mathcal{S}) = s$, as required. $\square$

For a Banach space E, let B_E be the closed unit ball of E.

Proposition 2.17. *For $p \in (1, \infty)$, the map j'' is a homeomorphism between $B_{s''}$ with weak*-topology, and $j''(B_{s''})$ with the weak*-topology on $\mathcal{S}''$.*

Proof. This is [1, Chapter 2, Section 2, Proposition 2]. Notice that j'' is an injection and the adjoint of an operator (namely j'), so that it is weak*-continuous. As $B_{s''}$ is compact, $j''(B_{s''})$ is compact, and hence closed, in $\mathcal{S}''$. The result then follows from the standard result in topology that a bijective continuous map from a compact space to a Hausdorff space has a continuous inverse. □

Corollary 2.18. *For $p \in (1, \infty)$, the map j is a homeomorphism between B_s with the weak-topology, and $j(B_s) \subseteq \mathcal{S}$ with the weak-topology.*

Proof. This is [1, Chapter 2, Section 2, Corollaire 1]. The map κ_s takes B_s into a dense subset of $B_{s''}$ (this is Goldstein's theorem), and is continuous with respect to the weak-topology on B_s and the weak*-topology on $B_{s''}$. Let $K = j''(B_{s''})$ with the weak*-topology, so that $j'' : B_{s''} \to K$ is a homeomorphism. From above, we know that $(j'')^{-1}(\kappa_{\mathcal{S}}(\mathcal{S})) = \kappa_s(s)$, so that $(j'')^{-1}(K \cap \kappa_{\mathcal{S}}(\mathcal{S})) = \kappa_s(B_s) = B_{s''} \cap \kappa_s(s)$. Hence j'' is a homeomorphism between $B_{s''} \cap \kappa_s(s)$ and $K \cap \kappa_{\mathcal{S}}(\mathcal{S})$, which implies that j is a homeomorphism between B_s and $j(B_s)$ with respect to the weak-topology (as $j''(B_{s''}) \cap \kappa_{\mathcal{S}}(\mathcal{S}) = j(B_s)$). □

Corollary 2.19. *For $p \in (1, \infty)$, the weak*-closure of $j(B_s)$ in $\mathcal{S}''$ is $j''(B_{s''})$.*

Proof. This is [1, Chapter 2, Section 2, Corollaire 2]. As $j''(B_{s''}) \cap \mathcal{S}$ is dense in $j''(B_{s''})$, this result follows from the fact that $(j'')^{-1}(j''(B_{s''}) \cap \mathcal{S}) = B_s$. □

Proposition 2.20. *For $p \in (1, \infty)$, the space s is reflexive if and only if $j : s \to \mathcal{S}$ is weakly-compact.*

Proof. This is [1, Chapter 2, Section 2, Proposition 3]. If s is reflexive, then j is weakly-compact. Conversely, if j is weakly-compact, then $j(B_s)$ is relatively-weakly-compact and weakly-homeomorphic to B_s, implying that B_s is weakly-compact, so that s is reflexive. □

There are further interesting properties of s discussed in [1, Chapter 2, Section 2].

2.7 When E_0 is a subspace of E_1: factorisation theorems

Again, we shall consider the special case when $E_0 \hookrightarrow E_1$.

Proposition 2.21. *When $E_0 \hookrightarrow E_1$, the space s (for $1 < p < \infty$) is reflexive and if only if the inclusion $E_0 \to E_1$ is weakly-compact.*

Proof. This is [1, Chapter 2, Section 3, Proposition 1] (see also [2]). Recall that $\mathcal{S} = E_1$ with equivalent norm, and that the space s factors the inclusion $i : E_0 \to E_1$. Hence, if if s is reflexive, then i is weakly-compact. We shall now show the converse, so suppose that i is weakly-compact. Let W be the image of B_{E_0} in E_1, and let C be the image of B_s in $\mathcal{S} = E_1$. Then W is relatively weakly-compact in E_1, and hence $\overline{W}$ is weakly-compact, so that $\kappa_{E_1}(\overline{W})$ is weak*-closed in E_1'' (as κ_{E_1} is weakly-weak*-continuous, and the continuous image of a compact set is compact, and hence closed).

Notice that

$$C = \Big\{ x \in E_1 : \sum_{n \in \mathbb{Z}} \psi_n(x)^p \leq 1 \Big\},$$

so that as $\psi_n(x) = \inf \big\{ t > 0 : \exists\, y \in W, z \in B_{E_1}, x = t(e^{-\xi_0 n} y + e^{-\xi_1 n} z) \big\}$, we see that

$$C \subseteq 2\Big(e^{-\xi_0 n} W + e^{-\xi_1 n} B_{E_1} \Big) \qquad (n \in \mathbb{Z}),$$

so in particular,

$$\kappa_{E_1}(C) \subseteq 2\Big(e^{-\xi_0 n}\kappa_{E_1}(\overline{W}) + e^{-\xi_1 n}B_{E_1''}\Big) \qquad (n\in\mathbb{Z}),$$

where the set on the right is weak*-closed in E_1''. From Corollary 2.19, we know that $j''(B_{s''}) \subseteq \mathcal{S}'' = E_1''$ is equal to the weak*-closure of $\kappa_{E_1}(j(B_s)) = \kappa_{E_1}(C)$. Thus we see that, as $\xi_1 > 0$,

$$\begin{aligned} j''(B_{s''}) &\subseteq 2\bigcap_{n>0}\Big(e^{-\xi_0 n}\kappa_{E_1}(\overline{W}) + e^{-\xi_1 n}B_{E_1''}\Big) \\ &\subseteq \bigcap_{n>0}\Big(\kappa_{E_1}(E_1) + 2e^{-\xi_1 n}B_{E_1''}\Big) = \kappa_{E_1}(E_1). \end{aligned}$$

From Proposition 2.16, $(j'')^{-1}(\kappa_{E_1}(E_1)) = \kappa_s(s)$, so we see that $B_{s''} \subseteq \kappa_s(s)$, that is, s is reflexive. □

Again, [1] contains futher interesting results, which we shall now summarise.

Proposition 2.22. *When $E_0 \hookrightarrow E_1$, the unit ball B_s is relatively weak*-sequentially compact in s'' if and only if B_{E_0} is relatively weak*-sequentially compact in E_1''.*

Proof. This is [1, Chapter 2, Section 3, Proposition 2]. □

Proposition 2.23. *When $E_0 \hookrightarrow E_1$, the space s contains an isomorphic copy of l^1 if and only if E_0 and E_1 contain an isomorphic copy of l^1.*

Proof. This is [1, Chapter 2, Section 3, Proposition 3]. □

3 Dual spaces and reiteration

In [1, Chapter 4], the author concentrates on the interpolation functor S, which leads naturally to a consideration of the dual of $L^p(E_0)$, and hence to technical issues like the Radon-Nikodym property. Instead, we shall consider the functors s, s_1 and s_2, which are easier to work with (as noted at the end of [1, Chapter 4, Section 1]).

Throughout this section, (E_0, E_1) shall be a compatible couple. We shall assume (as we may, by Section 2.5) that $\mathcal{I}(E_0,E_1)$ is dense in E_0 and E_1, so that $\iota_i : \mathcal{I}(E_0,E_1) \to E_i$ has dense range, and hence $\iota_i' : E_i' \to \mathcal{I}(E_0,E_1)'$ is norm-decreasing and injective, for $i = 0,1$. As vector spaces, we can hence view E_i' as a subspace of $\mathcal{I}(E_0,E_1)'$, for $i=0,1$, showing that (E_0', E_1') is a compatible couple.

We hence see that

$$\mathcal{I}(E_0',E_1') = \Big\{\mu\in E_0' : \exists\,\lambda\in E_1', \langle\mu,x\rangle = \langle\lambda,x\rangle \ (x\in E_0\cap E_1)\Big\},$$

with norm $\|\mu\|_{\mathcal{I}} = \max(\|\mu\|_0, \|\lambda\|_1)$, which makes sense, as λ is necessarily unique, given that $E_0\cap E_1$ is dense in both E_0 and E_1. Similarly, we see that

$$\mathcal{S}(E_0',E_1') = \Big\{\mu\in\mathcal{I}(E_0,E_1)' : \mu = \mu_0+\mu_1, \mu_0\in E_0', \mu_1\in E_1'\Big\},$$

with the usual norm.

Define a map $\beta : \mathcal{I}(E_0',E_1') \to \mathcal{S}(E_0,E_1)'$ as follows. Let $\mu\in\mathcal{I}(E_0',E_1')$, so that μ is represented by a pair (μ_0,μ_1), where $\mu_0\in E_0'$, $\mu_1\in E_1'$, and μ_0 and μ_1 agree on

$E_0 \cap E_1$. Then we let $\langle \beta(\mu), x_0 + x_1 \rangle = \langle \mu_0, x_0 \rangle + \langle \mu_1, x_1 \rangle$ for $x_0 \in E_0$ and $x_1 \in E_1$. This is well-defined, as μ_0 and μ_1 agree on $E_0 \cap E_1$, and clearly β is linear. Furthermore,

$$|\langle \beta(\mu), x_0 + x_1 \rangle| \leq \|\mu_0\|_0 \|x_0\|_0 + \|\mu_1\|_1 \|x_1\|_1 \leq \|\mu\|_{\mathcal{I}} (\|x_0\|_0 + \|x_1\|_1),$$

so that $|\langle \beta(\mu), x_0 + x_1 \rangle| \leq \|\mu\|_{\mathcal{I}} \|x_0 + x_1\|_{\mathcal{S}}$, and hence β is norm-decreasing. Conversely, let $\mu \in \mathcal{S}(E_0, E_1)'$, so that we can consider μ acting on E_0 or E_1 by restriction, and hence consider μ as a member of E_0' or E_1'. We then see that

$$\begin{aligned}\|\mu\| &= \sup\{|\langle \mu, x + y \rangle| : x \in E_0, y \in E_1, \|x\|_0 + \|y\|_1 \leq 1\} \\ &= \sup\{|\langle \mu, x \rangle| + |\langle \mu, y \rangle| : x \in E_0, y \in E_1, \|x\|_0 + \|y\|_1 \leq 1\} = \max\big(\|\mu\|_0, \|\mu\|_1\big).\end{aligned}$$

We conclude that $\mathcal{S}(E_0, E_1)' = \mathcal{I}(E_0', E_1')$ isometrically.

Similarly, we have a natural inclusion $\mathcal{S}(E_0', E_1') \to \mathcal{I}(E_0, E_1)'$. Let $\mu_i \in E_i'$ for $i = 0, 1$, so that

$$\begin{aligned}\|\mu_0 + \mu_1\|_{\mathcal{I}(E_0,E_1)'} &= \sup\big\{|\langle \mu_0 + \mu_1, x \rangle| : x \in E_0 \cap E_1, \|x\|_0 \leq 1, \|x\|_1 \leq 1\big\} \\ &\leq \sup\big\{|\langle \mu_0, x \rangle| + |\langle \mu_1, y \rangle| : x, y \in E_0 \cap E_1, \|x\|_0 \leq 1, \|y\|_1 \leq 1\big\} \\ &\leq \|\mu_0\|_0 + \|\mu_1\|_1.\end{aligned}$$

Thus the map $\mathcal{S}(E_0', E_1') \to \mathcal{I}(E_0, E_1)'$ is norm-decreasing.

The interested reader will find the above calculation much easier to perform in the special case when $E_0 \hookrightarrow E_1$.

We shall now work with $s = s(p; \xi_0, E_0; \xi_1, E_1)$, where we shall assume that $1 \leq p < \infty$. As before, s embeds isometrically into the Banach space $l^p(\mathcal{S}, \psi_n)$. Hence if ψ_n^* denotes the dual norm to ψ_n, then

$$s' = l^q\Big(\bigoplus_{n\in\mathbb{Z}} (\mathcal{S}', \psi_n^*)\Big)/s^\circ,$$

where $p^{-1} + q^{-1} = 1$. That is, we have a natural map $l^q((\mathcal{S}', \psi_n^*)) \to s'$ given by

$$\langle (\mu_n), x \rangle = \sum_{n\in\mathbb{Z}} \langle \mu_n, x \rangle \qquad (x \in s, (\mu_n) \in l^q((\mathcal{S}', \psi_n^*))),$$

which is a surjection. We hence see that

$$\|\mu\|_{s'} = \inf\Big\{\Big(\sum_{n\in\mathbb{Z}} \psi_n^*(\mu_n)^q\Big)^{1/q} : \mu = \sum_{n\in\mathbb{Z}} \mu_n\Big\} \qquad (\mu \in s'),$$

where we may restrict to finite sums if one is worried about convergence. For $n \in \mathbb{Z}$ and $\mu \in \mathcal{S}(E_0, E_1)' = \mathcal{I}(E_0', E_1')$, we have

$$\begin{aligned}\psi_n^*(\mu) &= \sup\big\{|\langle \mu, x_0 + x_1 \rangle| : \|e^{\xi_0 n} x_0\|_0 \leq 1, \|e^{\xi_1 n} x_1\|_1 \leq 1\big\} \\ &= \sup\big\{|\langle \mu, x_0 \rangle| + |\langle \mu, x_1 \rangle| : \|x_i\|_i \leq e^{-\xi_i n} \ (i = 0, 1)\big\} \\ &= e^{-\xi_0 n} \|\mu\|_0 + e^{-\xi_1 n} \|\mu\|_1.\end{aligned}$$

Consequently,

$$\begin{aligned}\|\mu\|_{s'} &= \inf\Big\{\Big(\sum_{n\in\mathbb{Z}} \big(e^{-\xi_0 n} \|\mu_n\|_0 + e^{-\xi_1 n} \|\mu_n\|_1\big)^q\Big)^{1/q} : \mu = \sum_{n\in\mathbb{Z}} \mu_n\Big\} \\ &= \inf\Big\{\Big(\sum_{n\in\mathbb{Z}} \big(e^{\xi_0 n} \|\mu_n\|_0 + e^{\xi_1 n} \|\mu_n\|_1\big)^q\Big)^{1/q} : \mu = \sum_{n\in\mathbb{Z}} \mu_n\Big\} \qquad (\mu \in s'),\end{aligned}$$

which is clearly equivalent to the norm $s_1(q; \xi_0, E_0'; \xi_1, E_1')$. We hence see that $(E_0, E_1)_{\theta,p} = (E_0', E_1')_{\theta,q}$ with equivalence of norms (this is [1, Chapter 4, Section 1, Proposition 2]).

4 Appendix on integration in Banach spaces

We need very little on the theorem of integration in Banach spaces. Let E be a Banach space, and let $(\Omega, \mathcal{B}, \mu)$ be a (σ-finite if we wish) measure space. That is, Ω is a set, $\mathcal{B}$ is a σ-algebra of subsets of $\mathcal{B}$, and μ is a positive measure. For $A \in \mathcal{B}$, let χ_A be the characteristic function of A. Let $\tilde{L}^p(E,\mu)$ be the vector space of *step-functions*. We think of this as formal sums of the form

$$\sum_{j=1}^n x_j \chi_{A_j},$$

where $n \geq 1$, (x_j) is a finite sequence in E, and (A_j) is a finite collection in $\mathcal{B}$ (we can always arrange for the (A_j) to be pairwise-disjoint if we so wish). Technically, we restrict to the case when each A_j has finite measure. Equivalently, this is the collection of measurable functions $f : \Omega \to E$ which take finitely-many values, and which are non-zero on a set of finite measure. We norm $\tilde{L}^p(E,\mu)$ in the standard way:

$$\|f\|_{\tilde{L}^p(E,\mu)} = \Big(\int \|f(t)\|_E^p \, \mathrm{d}\mu(t) \Big)^{1/p} = \Big(\sum_{j=1}^n \|x_j\|^p \mu(A_j) \Big)^{1/p} \qquad \Big(f = \sum_{j=1}^n x_j \chi_{A_j} \Big).$$

Then a standard check shows that $\tilde{L}^p(E,\mu)$ is a normed vector space. We simply let $L^p(E,\mu)$ be the Banach space completion of $\tilde{L}^p(E,\mu)$. As in the main text, we denote by $L^p(E)$ the space $L^p(E,\mu)$ where μ is Lebesgue measure on $\mathbb{R}$.

There is a natural map $\int : \tilde{L}^p(E,\mu) \to E$ given by integration:

$$\int f = \int f(t) \, \mathrm{d}\mu(t) = \sum_{j=1}^n x_j \mu(A_j) \qquad \Big(f = \sum_{j=1}^n x_j \chi_{A_j} \Big).$$

Then we see that

$$\Big\| \int f \Big\| = \Big\| \sum_{j=1}^n x_j \mu(A_j) \Big\| \leq \sum_{j=1}^n \|x_j\| \mu(A_j),$$

so if $p = 1$, this mapping is norm decreasing, and hence extends to $L^1(E,\mu)$. If $(\Omega, \mathcal{B}, \mu)$ is a *finite* measure space, then

$$\Big\| \int f \Big\| \leq \Big(\sum_{j=1}^n \|x_j\|^p \Big)^{1/p} \Big(\sum_{j=1}^n \mu(A_j)^q \Big)^{1/q} \leq \mu(\Omega) \|f\|_{\tilde{L}^p(E,\mu)},$$

so that, again, this map extends to a bounded map $L^p(E,\mu) \to E$. Finally, for general $(\Omega, \mathcal{B}, \mu)$ there is no bounded map $l^p(E,\mu) \to E$ (simply consider $E = \mathbb{C}$ to see this). Notice, however, that we do have a well-defined (but unbounded) operator $\int : L^p(E,\mu) \cap L^1(E,\mu) \to E$, which shall be sufficient for our purposes.

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