Commentary on "Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres"

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Abstract

We

1 Notation

The original paper assumed throughout that Hilbert spaces are separable. We shall try hard *not* to use this assumption. Exceptions are: Proposition prop:4 2.13.

We shall follow the convention that inner products are linear on the right. We write $\otimes$ for various completed tensor products, which should be clear by context (either Hilbert space, or the minimal C*-algebraic, tensor products).

Given H a Hilbert space and $\xi \in H$, define

$$\theta_\xi, \theta'_\xi \in \mathcal{B}(H, H \otimes H); \qquad \theta_\xi(\eta) = \xi \otimes \eta, \quad \theta'_\xi(\eta) = \eta \otimes \xi \qquad (\eta \in H).$$

Similarly, for $i = 1, 2, 3$, define $\theta_{i,\xi} \in \mathcal{B}(H \otimes H, H \otimes H \otimes H)$ by $\theta_{1,\xi}(\eta \otimes \zeta) = \xi \otimes \eta \otimes \zeta$, $\theta_{2,\xi}(\eta \otimes \zeta) = \eta \otimes \xi \otimes \zeta$ and $\theta_{3,\xi}(\eta \otimes \zeta) = \eta \otimes \zeta \otimes \xi$.

For $T \in \mathcal{B}(H \otimes H)$, we define $T_{12}, T_{13}, T_{23} \in \mathcal{B}(H \otimes H \otimes H)$ using the usual leg-numbering notation. Notice that $T_{12}\theta_{3,\xi} = \theta_{3,\xi}T$, $T_{13}\theta_{2,\xi} = \theta_{2,\xi}T$ and $T_{23}\theta_{1,\xi} = \theta_{1,\xi}T$. Similarly, if $\Sigma \in \mathcal{B}(H \otimes H)$ denotes the "swap map", then $T_{21} = \Sigma T_{12} \Sigma$, and so forth.

We shall also sometimes work with Hilbert C*-modules (see, for example, lance [lan]. For a Hilbert C*-module E over A, we shall write (in a non-standard way) $\mathcal{B}(E)$ for the adjointable maps on E.

Given $T \in \mathcal{B}(H \otimes H)$ and $\omega \in \mathcal{B}(H)_*$, we define the slice maps $(\omega \otimes \iota)(T)$ and $(\iota \otimes \omega)(T)$ as usual. Notice that

$$\big(\xi \big| (\iota \otimes \omega)(T)\eta\big) = \langle \theta_\xi^* T \theta_\eta, \omega \rangle, \quad \big(\xi \big| (\omega \otimes \iota)(T)\eta\big) = \langle {\theta'_\xi}^* T \theta'_\eta, \omega \rangle.$$

Given a C*-algebra A, we denote by $\tilde{A}$ the C*-algebra given by adjoining a unit, and we denote by $M(A)$ the multiplier algebra of A (see r33 [10, 3.12]). If J is a closed two-sided ideal in A, let $M(A;J) = \{m \in M(A) : mA + Am \subseteq J\}$. Clearly $M(A;J)$ is a sub-C*-algebra of $M(A)$. Restricting each element of $M(A)$ to J defines a member of $M(J)$; indeed, for $m \in M(A)$ and $a \in J$, if (e_i) is an approximate identity for A, then $ma = \lim_i (e_i m)a \in J$, and similarly $am \in J$. Thus we get a $*$-homomorphism $M(A) \to M(J)$, and so a $*$-homomorphism $M(A;J) \to M(J)$. This latter map is injective, as if $m \in M(A;J)$ with $mJ = \{0\} = Jm$, then for $a \in A$, and (f_i) an approximate identity for J, then $am = \lim a(mf_i) = 0$, as $am \in J$; similarly $ma = 0$ and so $m = 0$. Thus we can also regard $M(A;J)$ as a sub-C*-algebra of $M(J)$.

Recall that a $*$-homomorphism $\pi : A \to M(B)$ is *non-degenerate* if $\pi(e_i) \to 1$ strictly (meaning that $\lim_i \pi(e_i)b = \lim_i b\pi(e_i) = b$ for $b \in B$) for a (or equivalently, any) approximate identity (e_i) for A. This is equivalent to asking that π extends to a strictly-continuous, unital $*$-homomorphism $\pi : M(A) \to M(B)$. (This notion is termed "spécial" in r50 [12]).

Definition 1.1 (Définition 0.1)**.** *A* Hopf-C*-algebra *is a pair (A, δ) where A is a C*-algebra and $\delta : A \to M(\tilde{A} \otimes A + A \otimes \tilde{A}; A \otimes A)$ is a non-degenerate $*$-homomorphism (notice that this means that δ is a non-degenerate $*$-homomorphism $A \to M(A \otimes A)$ such that $\delta(a)(1 \otimes b), \delta(a)(b \otimes 1) \in A \otimes A$) with*

$$\begin{array}{ccc} A & \xrightarrow{\delta} & M(A \otimes A) \\ \big\downarrow{\scriptstyle \delta} & & \big\downarrow{\scriptstyle \iota\otimes\delta} \\ M(A \otimes A) & \xrightarrow{\delta\otimes\iota} & M(A \otimes A \otimes A). \end{array}$$

We call δ the coproduct *of A. We say that A is right simplifiable (or left simplifiable) if $\delta(A)(1 \otimes A)$ is linearly dense in $A \otimes A$ (respectively $\delta(A)(A \otimes 1)$). We say that A is bisimplifiable if A is left and right simplifiable.*

Be aware that this clashes with [2, 1.1]. Given a Hilbert space H, we can form the interior tensor product (see [lan, Chapter 4]) $(H \otimes A) \otimes_\delta (A \otimes A)$. Recall that this is the completion of $(H \otimes A) \otimes_{\text{alg}} (A \otimes A)/X$ where X is the linear span of elements of the form $(\xi \otimes ab) \otimes (c \otimes d) - (\xi \otimes a) \otimes \delta(b)(c \otimes d)$. A little bit of work shows that we can identify, as $A \otimes A$-modules, the spaces $(H \otimes A) \otimes_\delta (A \otimes A)$ and $H \otimes A \otimes A$ by the map $(\xi \otimes a) \otimes (c \otimes d) \mapsto \xi \otimes \delta(a)(c \otimes d)$.

Definition 1.2 (Définition 0.2)**.** *A* coaction *of a Hopf-C*-algebra on a C*-algebra B is a non-degenerate $*$-homomorphism $\delta_B : B \to M(\tilde{B} \otimes A; B \otimes A)$ such that the following diagram commutes:*

$$\begin{array}{ccc} B & \xrightarrow{\delta_B} & M(B \otimes A) \\ \big\downarrow{\scriptstyle \delta_B} & & \big\downarrow{\scriptstyle \iota\otimes\delta} \\ M(B \otimes A) & \xrightarrow{\delta_B\otimes\iota} & M(B \otimes A \otimes A). \end{array}$$

(Again, this means that $\delta_B(b)(1 \otimes a) \in B \otimes A$). A C-algebra B with a coaction δ_B of a Hopf-C*-algebra (A, δ) is an A-*algebra *if additionally δ_B is injective, and $\delta_B(B)(1 \otimes A)$ is linearly dense in $B \otimes A$.*

Definition 1.3 (Définition 0.3)**.** *Let A be a Hopf-C*-algebra. A* unitary corepresentation *of A on a Hilbert space (or Hilbert C*-module) H is a unitary $u \in \mathcal{B}(H \otimes A)$ such that $(\iota \otimes \delta)(u) = u_{12}u_{13}$; alternatively, in*

$$(H \otimes A) \otimes_\delta (A \otimes A) \cong H \otimes A \otimes A \quad \textit{we have} \quad u \otimes_\delta 1 = u_{12}u_{13}.$$

Let B be a C-algebra with a coaction δ_B of (A, δ). A* covariant representation *of (B, δ_B) is a pair (π, u) where $\pi : B \to \mathcal{B}(H)$ is a $*$-representation, and u is a unitary corepresentation of A, such that $(\pi \otimes \iota)\delta_B(b) = u(\pi(b) \otimes 1)u^*$ for each $b \in B$.*

Remember that $\mathcal{B}(H \otimes A) \cong M(\mathcal{B}_0(H) \otimes A)$, so if H is a Hilbert space, we can phrase the above without reference to Hilbert C*-modules.

Definition 1.4 (Définition 0.4)**.** *Let B be a C*-algebra with a coaction δ_B of (A, δ). A unitary $u \in M(B \otimes A)$ is a* cocycle for δ_B *if*

$$u_{12}(\delta_B \otimes \iota)(u) = (\iota \otimes \delta)(u).$$

If u is a cocycle for δ_B, the map $\delta_{B,u} : B \to M(B \otimes A); x \mapsto u\delta_B(x)u^*$ satisfies $(\delta_{B,u} \otimes \iota)\delta_{B,u} = (\iota \otimes \delta)\delta_{B,u}$, and hence is a coaction.

Finally, we recall the notion of morphism for the category of Hopf-C*-algebras.

Definition 1.5 (Définition 0.5)**.** *Let (S, δ) and (S', δ') be Hopf-C*-algebras. A* morphism *$(S, \delta) \to (S', \delta')$ is a non-degenerate $*$-homomorphism $\phi : S \to M(S')$ with $(\phi \otimes \phi)\delta = \delta'\phi$.*

2 Definitions

Consult [9] for motivations on studying the Pentagonal equation.

Definition 2.1 (Définition 1.1)**.** *A unitary* $V \in \mathcal{B}(H \otimes H)$ *is* multiplicative *if it satisfies the* pentagonal equation*:*

$$V_{12}V_{13}V_{23} = V_{23}V_{12}.$$

eg:1 **Examples 2.2** (Exemples 1.2)**.** 1. • The identity $1 \in \mathcal{B}(H \otimes H)$ is a multiplicative unitary.

eg:1.2 • If V is a multiplicative unitary and $U \in \mathcal{B}(H, H')$ is a unitary, then $W = (U \otimes U)V(U^* \otimes U^*)$ is a multiplicative unitary on H'. We say that V and W are *equivalent*.

• If V is a multiplicative unitary and $\Sigma \in \mathcal{B}(H \otimes H)$ is the swap map, then $\Sigma V^* \Sigma$ is also a multiplicative unitary. We say that V and W are *opposite* if V and $\Sigma W^* \Sigma$ are equivalent.

• If V and W are two multiplicative unitaries on H and K, respectively, then $V_{13}W_{24} \in \mathcal{B}(H \otimes K \otimes H \otimes K)$ is a multiplicative unitary on $H \otimes K$. We call this the *tensor product* of V and W, sometimes denoted (abusively) by $V \otimes W$. Notice that $V \otimes W$ and $W \otimes V$ are equivalent.

2. If G is a locally compact group with right Haar measure dg, then $V_G(\xi)(s,t) = \xi(st,t)$ is a multiplicative unitary on $L^2(G, dg)$.

3. If W is the fundamental unitary of a Kac algebra (see [3], [6] and [7]) then $V = W^*$ is a multiplicative unitary.

eg:1.4 4. If (A, δ) is a Hopf-C*-algebra, and ϕ is a right Haar measure on A (so $\phi \in A^*$ is a state with $(\phi \otimes \mu)\delta(a) = \phi(a)\mu(1)$ for $a \in A, \mu \in A^*$), then let (H, π, ξ) be the cyclic GNS construction for ϕ. If we define V_ϕ by $V_\phi(\pi(x)\xi \otimes \eta) = (\pi \otimes \pi)(\delta(x))(\xi \otimes \eta)$ for $\eta \in H$, then V_ϕ is an isometry which satisfies the pentagonal equation. If V_ϕ surjects, then it is a multiplicative unitary; this is the case of a compact quantum group in the sense of Woronowicz, [13].

5. Let (A, δ) be a Hopf-C*-algebra. The coproduct δ is a coaction of A on itself. If also (π, u) is a covariant representation of (A, δ) on a Hilbert space H. So $(\iota \otimes \delta)(u) = u_{12}u_{13}$ and $(\pi \otimes \iota)\delta(a) = u(\pi(a) \otimes 1)u^*$ for $a \in A$. Setting $V = (\iota \otimes \pi)(u)$, we see that V is a multiplicative unitary.

6. Another interpretation of the pentagonal equation is the following:

   If A is a finite-dimensional Hopf algebra, and let E be the algebra of linear maps $A \to A$. We identify E with $A^* \otimes A$, and let $v \in A^* \otimes A$ be the identity map. Define a homomorphism $L : A \to E$ by $L(a)(b) = ab$. Recall that A^* becomes an algebra for the product

$$\langle xy, a \rangle = \langle x \otimes y, \delta(a) \rangle \qquad (x, y \in A^*, a \in A).$$

   For $x \in A^*$ and $a \in A$, we let $\rho(x)(a) = (\iota \otimes x)\delta(a)$, so ρ is a homomorphism $A^* \to E$.

prop:1.1 **Proposition 2.3** (Page 431)**.** *(a) For $a \in A, x \in A^*$, write $\delta(a) = \sum_i a_i \otimes b_i$; then $\rho(x)L(a) = \sum_i L(a_i)\rho(xb_i)$.*

prop:1.2 *(b) For $a \in A$, we have that $(\rho \otimes \iota)(v)(L(a) \otimes 1) = (L \otimes \iota)(\delta(a))(\rho \otimes \iota)(v)$ in $E \otimes A$.*

prop:1.3 *(c) We have that $(\iota \otimes \delta)(v) = v_{12}v_{13}$.*

prop:1.4 (d) *In* $A^* \otimes E \otimes A$, *we have that*

$$((\iota \otimes L)(v))_{12} v_{13} ((\rho \otimes \iota)(v))_{23} = ((\rho \otimes \iota)(v))_{23} ((\iota \otimes L)(v))_{12}.$$

Proof. For (6a), given $b \in A$, we have that $\rho(x)L(a)b = \rho(x)(ab) = (\iota \otimes x)\delta(ab) = \sum_i a_i(\iota \otimes xb_i)\delta(b) = \sum_i L(a_i)\rho(xb_i)b$, as claimed.

For (6b), given $x \in A^*$, we have that $(\iota \otimes x)(v) = x$, and so $(\iota \otimes x)((\rho \otimes \iota)(v)) = \rho(x)$. Thus, using part (6a),

$$\begin{aligned}
(\iota \otimes x)((\rho \otimes \iota)(v)(L(a) \otimes 1)) = \rho(x)L(a) &= \sum_i L(a_i)\rho(xb_i) \\
&= (\iota \otimes x) \sum_i (L(a_i) \otimes b_i)(\rho \otimes \iota)(v) \\
&= (\iota \otimes x)((L \otimes \iota)(\delta(a))(\rho \otimes \iota)(v)).
\end{aligned}$$

As x was arbitrary, this shows (6b).

Now let $x, y \in A^*$ and $a \in A = A^{**}$. Then $(a \otimes \iota)(v) = a$, $(\iota \otimes x)(v) = x$ and $(\iota \otimes y)(v) = y$. Thus

$$\langle a \otimes x \otimes y, (\iota \otimes \delta)(v) \rangle = \langle x \otimes y, \delta(a) \rangle = \langle xy, a \rangle.$$

However, also

$$\langle a \otimes x \otimes y, v_{12} v_{13} \rangle = \langle (\iota \otimes x)(v)(\iota \otimes y)(v), a \rangle = \langle xy, a \rangle.$$

Thus we have shown (6c).

Finally, by (6b), we see that

$$((\rho \otimes \iota)(v))_{23} ((\iota \otimes L)(v))_{12} = (\iota \otimes L \otimes \iota)(\iota \otimes \delta)(v)((\rho \otimes \iota)(v))_{23}.$$

By (6c), this is equal to $((\iota \otimes L)(v))_{12} v_{13} ((\rho \otimes \iota)(v))_{23}$, as required to show (6d). □

Corollary 2.4 (Page 431)**.** *The operator* $V = (\rho \otimes L(v)$ *satisfies the pentagonal equation.*

If A is both unital and counital, then L and ρ inject, and we have the following.

Proposition 2.5 (Page 431)**.** *Let* $1 \in A$ *be the unit of* A, *and* $\epsilon \in A^*$ *be the unit of* A^*.

prop:2.1 (a) *If* V, *and so* v, *are invertible, then the map* $\kappa : A \to A; a \mapsto (a \otimes \iota)(v^{-1})$ *is the antipode of* A. *That is, for* $a \in A$,

$$m(\iota \otimes \kappa)\delta(a) = m(\kappa \otimes \iota)\delta(a) = \epsilon(a)1.$$

Here $m : A \otimes A \to A$ *is the multiplication map.*

prop:2.2 (b) *Conversely, if* A *has an antipode, then* v *is invertible.*

Proof. For (6a), as above, we have that $\delta(a) = (a \otimes \iota \otimes \iota)(v_{12} v_{13})$ and $(\iota \otimes \kappa)\delta(a) = (a \otimes \iota \otimes \iota)(v_{12} v_{13}^{-1})$. Thus

$$m(\iota \otimes \kappa)\delta(a) = (a \otimes m)(v_{12} v_{13}^{-1}) = (a \otimes \iota)(vv^{-1}) = (a \otimes \iota)(\epsilon \otimes 1) = \epsilon(a)1.$$

Similarly, $m(\kappa \otimes \iota)\delta(a) = (a \otimes m)(v_{12}^{-1} v_{13}) = \epsilon(a)1$. This shows (6a).

For (6b), compare with [1]. Indeed, set $u = (\iota \otimes \kappa)(v)$,

$$\begin{aligned}
vu &= (\iota \otimes m)(v_{12} u_{13}) = (\iota \otimes m(\iota \otimes \kappa))(v_{12} v_{13}) = (\iota \otimes m(\iota \otimes \kappa)\delta)(v) \\
&= (\iota \otimes \epsilon)(v) \otimes 1 = \epsilon \otimes 1.
\end{aligned}$$

So $u = v^{-1}$. □

We continue studying general multiplicative unitaries. Let H be a Hilbert space and $V \in \mathcal{B}(H \otimes H)$ a multiplicative unitary.

Definition 2.6 (Définition 1.3). *Let $\omega \in \mathcal{B}(H)_*$, and define $L(\omega), \rho(\omega) \in \mathcal{B}(H)$ by $L(\omega) = (\omega \otimes \iota)(V)$ and $\rho(\omega) = (\iota \otimes \omega)(V)$. Let*

$$A(V) = \{L(\omega) : \omega \in \mathcal{B}(H)_*\} \qquad \hat{A}(V) = \{\rho(\omega) : \omega \in \mathcal{B}(H)_*\}.$$

Then $A(V)$ and $\hat{A}(V)$ form a dual pairing:

$$\langle L(\omega), \rho(\omega')\rangle = (\omega \otimes \omega')(V) = \langle \rho(\omega'), \omega\rangle = \langle L(\omega), \omega'\rangle.$$

prop:3 **Proposition 2.7** (Proposition 1.4). *The spaces $A(V)$ and $\hat{A}(V)$ are subalgebras of $\mathcal{B}(H)$, and the spaces $A(V)H$ and $\hat{A}(V)H$ are linearly dense in H.*

Proof. Let $\omega, \omega' \in \mathcal{B}(H)_*$, and define $\psi \in \mathcal{B}(H)_*$ by define $\langle T, \psi\rangle = \langle V^*(1 \otimes T)V, \omega \otimes \omega'\rangle$ for $T \in \mathcal{B}(H)$. Then, using the pentagonal equation,

$$\begin{aligned} L(\omega)L(\omega') &= (\omega \otimes \iota)(V)(\omega' \otimes \iota)(V) = (\omega \otimes \otimes' \otimes \iota)(V_{13}V_{23}) \\ &= (\omega \otimes \otimes' \otimes \iota)(V_{12}^* V_{23} V_{12}) = (\psi \otimes \iota)(V) = L(\psi). \end{aligned}$$

Similarly, $\rho(\omega)\rho(\omega') = \rho(\psi')$ where $\langle T, \psi'\rangle = (\omega \otimes \omega')(V(T \otimes 1)V^*)$.

Given non-zero $\xi, \eta \in H$, we have that $V^*(\xi \otimes \eta) \neq 0$, and so there are $\alpha, \beta \in H$ with $\langle \xi \otimes \eta, V(\alpha \otimes \beta)\rangle \neq 0$. Thus $L(\omega_{\xi,\alpha})\beta$ is not orthogonal to η, and $\rho(\omega_{\eta,\beta})\alpha$ is not orthogonal to ξ, showing linear density of the spaces $A(V)H$ and $\hat{A}(V)H$. $\square$

Definition 2.8 (Définition 1.5). *Let V be a multiplicative unitary. We write S for the norm closure of the algebra $A(V)$, and similarly denote by $\hat{S}$ the norm closure of $\hat{A}(V)$.*

We remark that the functionals ψ which appear in the proof above are dense in $\mathcal{B}(H)_*$. It follows that $\{xy : x, y \in A(V)\}$ is dense in S, and similarly $\{xy : x, y \in \hat{A}(V)\}$ is dense in $\hat{S}$.

Proposition 2.9 (Proposition 1.6). *Let $C^*(S)$ be the C^*-algebra (in $\mathcal{B}(H)$) generated by S, and similarly for $C^*(\hat{S})$. Then V is in the von Neumann algebra generated by $C^*(\hat{S}) \otimes C^*(S)$.*

Proof. Let $T \in \mathcal{B}(H \otimes H)$. For $\omega \in \mathcal{B}(H)_*$,

$$(\iota \otimes \omega \otimes \iota)(T_{13}V_{23} - V_{23}T_{13}) = T(1 \otimes L(\omega)) - (1 \otimes L(\omega))T.$$

So T commutes with $1 \otimes S$ if and only if T_{13} commutes with V_{23}. A similar calculation shows that $\hat{S} \otimes 1$ commutes with T if and only if T_{13} commutes with V_{12}.

So if $T \in (\hat{S} \otimes S)'$ then T_{13} commutes with both V_{23} V_{12}. As $V_{13} = V_{12}^* V_{23} V_{12} V_{23}^*$, it follows that T_{13} commutes with V_{13}. So $V \in (\hat{S} \otimes S)''$ and hence certainly $V \in (C^*(\hat{S}) \otimes C^*(S))''$. $\square$

Definition 2.10 (Définition 1.7). *Let V be a multiplicative unitary. We say that V is of* compact type *if S is unital. We say that V is of* discrete type *if $\hat{S}$ is unital.*

defn:2 **Definition 2.11** (Définition 1.8). *Let V be a multiplicative unitary. A vector $e \in H$ is* fixed *if $V\theta_e = \theta_e$ (that is, $V(e \otimes \xi) = e \otimes \xi$ for all $\xi \in H$), and is* cofixed *if $V\theta'_e = \theta'_e$ (that is, $V(\xi \otimes e) = \xi \otimes e$ for all $\xi \in H$).*

Proposition 2.12 (Proposition 1.9). *Let e be a fixed (respectively cofixed) unit vector. Then $L(\omega_e) = 1$ and $\rho(\omega_e)$ is the projection onto the subspace of all fixed vectors (respectively, $\rho(\omega_e) = 1$ and $L(\omega_e)$ is the projection onto the subspace of all cofixed vectors).*

Proof. Clearly $L(\omega_e) = (\omega_e \otimes \iota)(V) = 1$. Define $\psi' \in \mathcal{B}(H)_*$ by $\psi'(T) = (e \otimes e|V(T \otimes 1)V^*(e \otimes e)) = \langle T, \omega_e \rangle$, as $V^*(e \otimes e) = e \otimes e$. By (the proof of) Proposition 2.7, $\rho(\omega_e)$ is an idempotent, and as $\|\rho(\omega_e)\| \leq 1$, it follows that $\rho(\omega_e)$ is a projection. Now, $\rho(\omega_e)\xi = xi$ if and only if $(\xi|\rho(\omega_e)\xi) = \|\xi\|^2$, that is, $(\xi \otimes e|V(\xi \otimes e) = \|\xi \otimes e\|^2$. Thus the image of $\rho(\omega_e)$ is $\{\xi \in H : V(\xi \otimes e) = \xi \otimes e\}$. However, notice that if $V(\xi \otimes e) = \xi \otimes e$, then for $\eta \in H$, the vector $\xi \otimes e \otimes \eta$ is fixed by both V_{12} and V_{23}, and hence by $V_{13} = V_{12}^* V_{23} V_{12} V_{23}^*$, showing that ξ is fixed.

The other case follows by working with $\Sigma V^* \Sigma$ instead of V. □

prop:4 **Proposition 2.13** (Proposition 1.10). *Let V be a multiplicative unitary on H, where H is now separable. Then V is of compact type (respectively, discrete type) if and only if the spaces of fixed vectors (respectively, cofixed vectors) is not zero.*

Proof. If there is a fixed vector e then $L(e) = 1$ so S is unital. Conversely, suppose that S is unital, and recall from Proposition 2.7 that S acts non-degenerately on H, so the unit of S is the identity operator on H. Thus there is $\omega \in \mathcal{B}(H)_*$ with $\|L(\omega) - 1\| < 1/2$. Fix a *faithful* normal state ψ, using that H is separable. Then $|\langle \rho(\psi), \omega \rangle| = |\langle L(\omega), \psi \rangle| > 1/2$. Set $\psi^1 = \psi$, and defined inductively $\langle x, \psi^{n+1} \rangle = \langle V(x \otimes 1)V^*, \psi \otimes \psi^n \rangle$. Set $\psi_n = \frac{1}{n} \sum_{k=1}^n \psi^k$. Thus, from Proposition 2.7, $\rho(\psi^n) = \rho(\psi)^n$. Notice that $\|\rho(\psi)\| \leq 1$ and $(1 - \rho(\psi))\rho(\psi_n) = (\rho(\psi) - \rho(\psi)^{n+1})/n$, which converges to 0 in norm.

If $T = 1 - \rho(\psi)$ is an injective operator, then T^* has dense range, and so there is $\omega' \in \mathcal{B}(H)_*$ with $\|\omega - \omega' T\| < 1/4$. As $|\langle \rho(\psi_n), \omega \rangle| = |\langle L(\omega), \psi_n \rangle| \geq 1/2$, because ψ_n is a state, we arrive at a contradiction. So T is not injective, and we can find a unit vector $e \in H$ with $\rho(\psi)(e) = e$. Then $1 = \langle \rho(\psi), \omega_e \rangle = \langle L(\omega_e), \psi \rangle$. As $\|L(\omega_e)\| \leq 1$, we have that $1 - L(\omega_e)$ is positive, and $\langle 1 - L(\omega_e), \psi \rangle = 0$. As ψ is faithful, we must have that $L(\omega_e) = 1$, as required to show that e is a fixed vector. □

We see that $1 \in \mathcal{B}(H \otimes H)$ is both compact and discrete. If V is a multiplicative unitary, then V is of compact (respectively, discrete) type if and only if $\Sigma V^* \Sigma$ is of discrete (respectively, compact) type. The tensor product of two multiplicative unitaries of compact (discrete) type is again of compact (discrete) type.

If G is a compact group, and we form V_G as in Example 2.2.1, then the function which is constant 1 is fixed by V_G. Similarly, if G is a discrete group, then the function which is 1 at the identity, and 0 elsewhere, is fixed by V_G.

In Example 2.2.4, the cyclic vector ξ is fixed.

Remarks 2.14 (Remarques 1.11).
1. Let $f \in H$ be a unit vector with $V(f \otimes f) = f \otimes f$. Then $L(\omega_f)^2 = L(\omega_f)$ and $\rho(\omega_f)^2 = \rho(\omega_f)$; as both $\|L(\omega_f)\| = \|\rho(\omega_f)\| = 1$, both $L(\omega_f)$ and $\rho(\omega_f)$ are projections.

3 Commutative multiplicative unitaries

We will now study commutative multiplicative unitaries, and show that they correspond to locally compact groups.

Let V be a multiplicative unitary on a Hilbert space H.

Definition 3.1 (Définition 2.1). *We say that V is* commutative *if V_{13} and V_{23} commute. We say that V is* cocommutative *if V_{12} and V_{13} commute.*

The multiplicative unitary V_G from Example 2.2.1 is commutative. We will show that every commutative multiplicative unitary is of this form. Notice that V is commutative (respectively, cocommutative) if and only if S (respectively, $\hat{S}$) is abelian. Also, if V is commutative, then V_{13} and V_{23}^* commute, and so $C^*(S)$ is abelian.

Theorem 3.2 (Théorèm 2.2)**.** *Let V be a commutative multiplicative unitary, and let G be the spectrum of the abelian C^*-algebra $C^*(S)$. Then G is a locally compact group and there is a Hilbert space J such that V is equivalent to the multiplicative unitary $V_G \otimes 1_{K\otimes K}$.*

4 Regular multiplicative unitaries

In this section, we define and study *regular* multiplicative unitaries and deduce the existence of a densely defined antipode.

lemma:1 **Lemma 4.1** (Lemme 3.1)**.** *Let H and K be Hilbert spaces, and let $X \subseteq \mathcal{B}(H \otimes K)$. The closures of the linear spans of*

$$\big\{(1 \otimes h)x(1 \otimes k) : h, k \in \mathcal{B}_0(K), x \in X\big\}$$

and

$$\big\{(\iota \otimes \omega)(x) \otimes k : x \in X, k \in \mathcal{B}_0(K), \omega \in \mathcal{B}(K)_*\big\},$$

agree.

Proof. For $h = \theta_{\xi,\xi'}$ and $k = \theta_{\eta,\eta'}$, and $x \in X$, we have

$$(1 \otimes h)x(1 \otimes k) = (\iota \otimes \omega_{\xi',\eta})(x) \otimes \theta_{\xi,\eta'},$$

from which the claim follows. □

Given a multiplicative unitary V, we set $\mathcal{C}(V) = \{(\iota \otimes \omega)(\Sigma V) : \omega \in \mathcal{B}(H)_*\}$.

prop:5 **Proposition 4.2** (Proposition 3.2)**.** *The space $\mathcal{C}(V)$ is a subalgebra of $\mathcal{B}(H)$. The following conditions are equivalent:*

prop:5.1 *1. The closure of $\mathcal{C}(V)$ is $\mathcal{B}_0(H)$.*

prop:5.2 *2. The closure of the linear span of $\{(x \otimes 1)V(1 \otimes y) : x, y \in \mathcal{B}_0(H)\}$ is $\mathcal{B}_0(H \otimes H)$.*

Proof. For $\omega, \omega' \in \mathcal{B}(H)_*$, we have that

$$(\iota \otimes \omega)(\Sigma V)(\iota \otimes \omega')(\Sigma V) = (\iota \otimes \omega \otimes \omega')(\Sigma_{13}V_{13}\Sigma_{12}V_{12}).$$

Now, $\Sigma_{13}V_{13}\Sigma_{12}V_{12} = \Sigma_{13}\Sigma_{12}V_{23}V_{12} = \Sigma_{23}\Sigma_{13}V_{12}V_{13}V_{23} = \Sigma_{23}V_{32}\Sigma_{13}V_{13}V_{23} = V_{23}\Sigma_{23}\Sigma_{13}V_{13}V_{23}$. Setting $\langle x, \psi\rangle = (\omega' \otimes \omega)(V\Sigma(1 \otimes x)V)$, we see that $\psi \in \mathcal{B}(H)_*$, and that

$$(\iota \otimes \omega)(\Sigma V)(\iota \otimes \omega')(\Sigma V) = (\iota \otimes \psi)(\Sigma V).$$

Thus $\mathcal{C}(V)$ is a subalgebra.

Condition (prop:5.2 2) is equivalent to the closure of the linear span of

$$\{\Sigma(x \otimes 1)V(1 \otimes y) : x, y \in \mathcal{B}_0(H)\} = \{(1 \otimes x)\Sigma V(1 \otimes y) : x, y \in \mathcal{B}_0(H)\}$$

being equal to $\mathcal{B}_0(H \otimes H)$. The result follows by Lemma lemma:1 4.1. □

As V is unitary, it is clear that the functionals ψ constructed in the proof are norm dense in $\mathcal{B}(H)_*$. Thus $\{xy : x, y \in \mathcal{C}(V)\}$ is dense in $\mathcal{C}(V)$.

Definition 4.3 (Définition 3.3)**.** *A multiplicative unitary V is* regular *if the closure of $\mathcal{C}(V)$ is $\mathcal{B}_0(H)$.*

Notice that $\mathcal{C}(\Sigma V^*\Sigma) = \mathcal{C}(V)^*$. It follows that V is regular if and only if $\Sigma V^*\Sigma$ is regular. Given two equivalent multiplicative unitaries, one is regular if and only if the other is regular.

Examples 4.4 (Exemples 3.4)**.** 1. For $\omega = \omega_{\xi,\eta}$, we have that $(\iota \otimes \omega)(\Sigma) = \theta_{\eta,\xi}$. Thus $1 \in \mathcal{B}(H \otimes H)$ is a regular multiplicative unitary.

2. A direct calculation shows that for a locally compact group G, the multiplicative unitary V_G is regular. Indeed, this is a special case of the following.

3. Suppose there ia a unitary $J : H \to \overline{H}$ with $J^*\overline{L(\omega)}J = L(\omega^*)$ for each $\omega \in \mathcal{B}(H)_*$. Let T be a Hilbert-Schmidt operator on H, so we can identify T with some vector $\tau \in \overline{H} \otimes H$. Furthermore, suppose that T is trace class, and let $\omega \in \mathcal{B}(H)_*$ be the associated functional. Define W, a unitary on $\overline{H} \otimes H$, by $W = (1 \otimes J^*)\overline{V}(1 \otimes J)$. Notice that the composition of operators WT is Hilbert-Schmidt, and so can be identified as a member of $\overline{H} \otimes H$, which is just $W(\tau)$.

   For $\xi, \eta, \alpha, \beta \in H$

$$\begin{aligned}\big(\overline{\beta} \otimes \alpha \big| W(\overline{\xi} \otimes \eta)\big) &= \big(\overline{\beta} \otimes J(\alpha) \big| \overline{V}(\overline{\xi} \otimes J(\eta))\big) = \big(V(\xi \otimes \overline{J(\eta)}) \big| \beta \otimes \overline{J(\alpha)}\big) \\ &= \big(L(\omega_{\beta,\xi})\overline{J(\eta)} \big| \overline{J(\alpha)}\big) = \big(J(\alpha) \big| \overline{L(\omega_{\beta,\xi})} J(\eta)\big) \\ &= \big(\alpha \big| L(\omega_{\xi,\beta})\eta\big) = \big(\alpha \otimes \xi \big| \Sigma V(\beta \otimes \eta\big),\end{aligned}$$

   and so

$$\big(\overline{\beta} \otimes \alpha \big| WT\big) = \big(\overline{\beta} \otimes \alpha \big| W(\tau)\big) = \big(\alpha \big| (\iota \otimes \omega)(\Sigma V)\beta\big).$$

   It follows that V is regular.

   In particular, if $V = W^*$ and W is the fundamental unitary for a Kac algebra in the sense of [3], then [3, Lemme 2.2.3], together with the preceding argument, shows that V regular.

Proposition 4.5 (Proposition 3.4.4)**.** *1. Let V be a multiplicative unitary. If V is a multiplier of $\mathcal{B}_0(H) \otimes \mathcal{B}(H)$ (or $\mathcal{B}(H) \otimes \mathcal{B}_0(H)$) then $\mathcal{C}(V) \subseteq \mathcal{B}_0(H)$.*

2. Let A be a Hopf-C^*-algebra which is unital, and right simplifiable, and which has a right Haar state ϕ which satisfies $\phi(x^*x) = 0$ if and only if $\phi(xx^*) = 0$. Let (H, π, ξ) be the cyclic GNS construction. Define $V_\phi \in \mathcal{B}(H \otimes H)$ by $V_\phi(\pi(x)\xi \otimes \eta) = (\pi \otimes \pi)\delta(x)(\xi \otimes \eta)$ for $\eta \in H$. Then V_ϕ is a regular multiplicative unitary.

Proof. For $x, y \in \mathcal{B}_0(H)$, we have that $(x \otimes 1)V \in \mathcal{B}_0(H) \otimes \mathcal{B}(H)$ and so $(x \otimes 1)V(1 \otimes y) \in \mathcal{B}_0(H) \otimes \mathcal{B}_0(H) = \mathcal{B}_0(H \otimes H)$. The result follows by the methods used in Lemma 4.1 and Proposition 4.2. The other option follows by working with $\Sigma V^*\Sigma$.

As ϕ is right invariant, V_ϕ is isometric, compare Example 2.2.4. Clearly the image of V_ϕ contains the set

$$\big\{(\pi \otimes \pi)(\delta(x)(1 \otimes y)) : x, y \in A\big\},$$

and so, as (A, δ) is right simplifiable, we conclude that V_ϕ surjects. So V_ϕ is a unitary, and a calculation shows that V_ϕ is multiplicative.

Then, for $a, b \in A$ and $\xi_0, \xi_1, \eta \in H$,

$$V_\phi(\theta_{\pi(a)\xi,\xi_0} \otimes \pi(b))(\xi_1 \otimes \eta) = V_\phi(\pi(a)\xi \otimes \pi(b)\eta)(\xi_0|\xi_1) = (\pi \otimes \pi)\delta(a)(\xi \otimes \pi(b)\eta)(\xi_0|\xi_1).$$

We can approximate $\delta(a)(1 \otimes b)$ be a sum of tensors of the form $x, y \in A$, so this is approximately

$$(\pi(x) \otimes \pi(y))(\xi \otimes \eta)(\xi_0|\xi_1) = \big(\theta_{\pi(x)\xi,\xi_0} \otimes \pi(y)\big)(\xi_1 \otimes \eta).$$

Hence V_ϕ is a multiplier of $\mathcal{B}_0(H) \otimes \pi(A)$. As A is unital, it follows that V_ϕ is a multiplier of $\mathcal{B}_0(H) \otimes \mathcal{B}(H)$, and so the first part of the proposition shows that $\mathcal{C}(V_\phi) \subseteq \mathcal{B}_0(H)$.

Then, for $\eta, \eta_1 \in H$ and $a \in A$, we have that

$$\begin{aligned}\big(\eta_1\big|(\iota \otimes \omega_{\xi,\eta})(\Sigma V_\phi)\pi(a)\xi\big) &= \big(\xi \otimes \eta_1\big|V_\phi(\pi(a)\xi \otimes \eta)\big) = \big(\xi \otimes \eta_1\big|(\pi \otimes \pi)\delta(a)(\xi \otimes \eta)\big) \\ &= \big(\eta_1\big|\pi\big((\phi \otimes \iota)\delta(a)\big)\eta\big) = \phi(a)(\eta_1|\eta) = \big(\eta_1\big|\theta_{\eta,\xi}\pi(a)\xi\big).\end{aligned}$$

So $(\iota \otimes \omega_{\xi,\eta})(\Sigma V_\phi) = \theta_{\eta,\xi}$.

To show that $\mathcal{C}(V_\phi)$ is dense in $\mathcal{B}_0(H)$, it suffices to prove that for each non-zero $\xi_1 \in H$, there is $x \in \mathcal{C}(V_\phi)$ with $(\xi|x(\xi_1)) \neq 0$. Indeed, this would show that $\{x^*(\xi) : x \in \mathcal{C}(V_\phi)\}$ is dense in H. Then, for $x \in \mathcal{C}(V_\phi)$ and $\eta \in H$, we have that $\theta_{\eta,x^*\xi} = \theta_{\eta,\xi}x \in \mathcal{C}(V_\phi)$, and thus $\mathcal{C}(V_\phi)$ is dense in $\mathcal{B}_0(H)$.

Now, for $b, c \in A$ and $\xi_1, \xi_2 \in H$, we have that

$$\begin{aligned}\big(\xi_1\big|L(\omega_{\pi(b)\xi,\pi(c)\xi})\xi_2\big) &= \big(\pi(b)\xi \otimes \xi_1\big|V_\phi(\pi(c)\xi \otimes \xi_2)\big) \\ &= \big(\xi \otimes \xi_1\big|(\pi \otimes \pi)((b^* \otimes 1)\delta(c))(\xi \otimes \xi_2)\big) = \big(\xi_1\big|\pi(d)\xi_2\big),\end{aligned}$$

where $d = (\phi \otimes \iota)((b^* \otimes 1)\delta(c)) \in A$, as $(b^* \otimes 1)\delta(c) \in A \otimes A$. Hence $L(\omega_{\pi(b)\xi,\pi(c)\xi}) = \pi(d) \in \pi(A)$. Now, $\pi(A)$ is closed in $\mathcal{B}(H)$, and so by continuity, $L(\omega) \in \pi(A)$ for all $\omega \in \mathcal{B}(H)_*$.

For $\eta, \eta_1, \eta_2 \in H$, we have that

$$\big(\xi\big|(\iota \otimes \omega_{\eta_2,\eta_1})(\Sigma V_\phi)\eta\big) = \big(\eta_2 \otimes \xi\big|V_\phi(\eta \otimes \eta_1)\big) = \big(\xi\big|L(\omega_{\eta_2,\eta})\eta_1\big).$$

Suppose that $(\xi|x(\eta)) = 0$ for all $x \in \mathcal{C}(V_\phi)$. Thus $(\xi|L(\omega_{\eta_2,\eta})\eta_1) = 0$ for all $\eta_2, \eta_1 \in H$, that is, $L(\omega_{\eta_2,\eta})^*\xi = 0$ for all $\eta_2 \in H$. However, $L(\omega_{\eta_2,\eta}) = \pi(a)$ for some $a \in A$, and so $a^*\xi = 0 \implies \phi(aa^*) = 0 \implies \phi(a^*a) = 0 \implies a\xi = 0$. Thus $L(\omega_{\eta_2,\eta})\xi = 0$ for all $\eta_2 \in H$, which shows that $V_\phi(\eta \otimes \xi) = 0$, so $\eta = 0$, as required. □

In particular, this result applies to compact quantum groups in the sense of Woronowicz, r54 [13]. Furthermore, in this case, $S = \pi(A)$.

prop:6 **Proposition 4.6** (Proposition 3.5)**.** *If V is a regular multiplicative unitary, the algebras S and $\hat{S}$ are self-adjoint.*

Proof. Let E be the linear span of

$$\big\{(\omega \otimes \omega' \otimes \iota)(\Sigma_{12}V_{23}^*V_{12}V_{13})^* : \omega, \omega' \in \mathcal{B}(H)_*\big\}.$$

As $\Sigma_{12}V_{23}^*V_{12}V_{13} = \Sigma_{12}V_{12}V_{23}^*$, we see that E is the linear span of

$$\big\{(\omega \otimes \omega' \otimes \iota)(V_{23}^*)^* : \omega, \omega' \in \mathcal{B}(H)_*\big\} = \big\{(\omega' \otimes \iota)V : \omega' \in \mathcal{B}(H)_*\big\},$$

and so the closure of E is S. Alternatively, $\Sigma_{12}V_{23}^*V_{12}V_{13} = V_{13}^*\Sigma_{12}V_{12}V_{13}$, and so

$$(\omega \otimes \omega' \otimes \iota)(\Sigma_{12}V_{23}^*V_{12}V_{13}) = (\omega \otimes \iota)(V^*(y \otimes 1)V),$$

where $y = (\iota \otimes \omega')(\Sigma V)$. From this, it follows that the norm closure of E is the norm closure of

$$\big\{(\omega \otimes \iota)(V^*(y \otimes 1)V) : \omega \in \mathcal{B}(H)_*, y \in \mathcal{B}_0(H)\big\},$$

which is clearly self-adjoint. So S is self-adjoint. The $\hat{S}$ case follows, as $\hat{S} = S(\Sigma V^*\Sigma)^*$. □

prop:7 **Proposition 4.7** (Proposition 3.6)**.** *Let V be a regular multiplicative unitary, with associated C^*-algebras S and $\hat{S}$. We have that* prop:7.1 1. $V \in M(\mathcal{B}_0(H) \otimes S)$ *and* $V \in M(\hat{S} \otimes \mathcal{B}_0(H))$;

prop:7.2 2. *The closed linear span of* $\{(x \otimes 1)V(1 \otimes y) : x \in \mathcal{B}_0(H), y \in S\}$ *is* $\mathcal{B}_0(H) \otimes S$, *and the closed linear span of* $\{(x \otimes 1)V(1 \otimes y) : x \in \hat{S}, y \in \mathcal{B}_0(H)\}$ *is* $\hat{S} \otimes \mathcal{B}_0(H)$;

prop:7.3 3. $V \in M(\hat{S} \otimes S)$;

prop:7.4 4. *The closed linear span of* $\{(x \otimes 1)V(1 \otimes y) : x \in \hat{S}, y \in S\}$ *is* $\hat{S} \otimes S$.

Proof. For $x, y \in \mathcal{B}_0(H)$ and $\omega \in \mathcal{B}(H)_*$, we have that $V(x \otimes L(y\omega)) = (\iota \otimes \omega \otimes \iota)((V_{13}V_{23})(x \otimes y \otimes 1)) = (\iota \otimes \omega \otimes \iota)((V_{12}^* V_{23} V_{12})(x \otimes y \otimes 1))$. As $V(x \otimes y) \in \mathcal{B}_0(H \otimes H)$, we see that $V(x \otimes L(y\omega))$ is in the closed linear span of

$$\{(\iota \otimes \omega \otimes \iota)((V_{12}^* V_{23})(a \otimes b \otimes 1)) : a, b \in \mathcal{B}_0(H)\}.$$

Let $\omega = \omega' c$ for some $\omega' \in \mathcal{B}(H)_*$ and $c \in \mathcal{B}_0(H)$ (we may do this, by Lemma A.1). Then

$$(\iota \otimes \omega \otimes \iota)((V_{12}^* V_{23})(a \otimes b \otimes 1)) = (\iota \otimes b\omega' \otimes 1)((1 \otimes c \otimes 1)V_{12}^*(a \otimes 1 \otimes 1)V_{23}) \in \mathcal{B}_0(H) \otimes S,$$

using Proposition 4.2(2).

Also $(x \otimes L(\omega^* y^*)^*)V = (x \otimes (y\omega \otimes \iota)(V^*))V = (\iota \otimes \omega \otimes \iota)(V_{23}^*(x \otimes y \otimes 1)V_{13})$, so using Proposition 4.2(2) is in the closed linear span of

$$\{(\iota \otimes \omega \otimes \iota)(V_{23}^*(a \otimes 1 \otimes 1)V_{12}(1 \otimes b \otimes 1)V_{13}) : a, b \in \mathcal{B}_0(H)\}.$$

Notice that $(\iota \otimes \omega \otimes \iota)(V_{23}^*(a \otimes 1 \otimes 1)V_{12}(1 \otimes b \otimes 1)V_{13}) = (\iota \otimes \omega \otimes \iota)((a \otimes 1 \otimes 1)V_{23}^* V_{12} V_{13}(1 \otimes b \otimes 1)) = (\iota \otimes b\omega \otimes \iota)((a \otimes 1 \otimes 1)V_{12}V_{23}^*)$. Writing $b\omega = \omega' c$, with $c \in \mathcal{B}_0(H)$, as $(a \otimes c)V \in \mathcal{B}_0(H \otimes H)$, we have that

$$(\iota \otimes b\omega \otimes \iota)((a \otimes 1 \otimes 1)V_{12}V_{23}^*) = (\iota \otimes \omega' \otimes \iota)((a \otimes c \otimes 1)V_{12}V_{23}^*) \in \mathcal{B}_0(H) \otimes S,$$

where here we use Proposition 4.6. This shows the first part of (1); the second part follows by working with $\Sigma V^* \Sigma$.

Let $a, b \in \mathcal{B}_0(H), \omega \in \mathcal{B}(H)_*$ and set $y = L(\omega a)$. Then

$$(b \otimes 1)V(1 \otimes y) = (\iota \otimes \omega \otimes \iota)\big((b \otimes a \otimes 1)V_{13}V_{23}\big) = (\iota \otimes \omega \otimes \iota)\big(((b \otimes a)V^* \otimes 1)V_{23}V_{12}\big).$$

Again, as V is unitary, the closed linear span of $\{(b \otimes a)V^* : a, b \in \mathcal{B}_0(H)\}$ is $\mathcal{B}_0(H) \otimes \mathcal{B}_0(H)$. To show (2) it hence suffices to show that

$$\begin{aligned}&\big\{(\iota \otimes \omega \otimes \iota)\big((a \otimes 1 \otimes 1)V_{23}V_{12}\big) : a \in \mathcal{B}_0(H), \omega \in \mathcal{B}(H)_*\}\\ &\quad = \big\{(\iota \otimes b\omega \otimes \iota)\big((a \otimes 1 \otimes 1)V_{23}V_{12}\big) : a, b \in \mathcal{B}_0(H), \omega \in \mathcal{B}(H)_*\}\end{aligned}$$

is linearly dense in $B_0(H) \otimes S$. However,

$$(\iota \otimes b\omega \otimes \iota)\big((a \otimes 1 \otimes 1)V_{23}V_{12} = (\iota \otimes \omega \otimes \iota)\big(V_{23}\big((a \otimes 1)V(1 \otimes b) \otimes 1\big)\big),$$

and so the result follows by Proposition 4.2. Similarly, the second claim of (2) follows by working with $\Sigma V^* \Sigma$.

For (3), notice that by (1), both V_{12} and V_{23} are multipliers of $\hat{S} \otimes \mathcal{B}_0(H) \otimes S$, and hence so is $V_{13} = V_{12}^* V_{23} V_{12} V_{23}^*$. Thus $V \in M(\hat{S} \otimes S)$, as claimed.

For (4), it suffices to show that the closed linear span of

$$\big\{(x \otimes a \otimes 1)V_{13}(1 \otimes b \otimes y) : a, b \in \mathcal{B}_0(H), x \in \hat{S}, y \in S\big\}$$

is $\hat{S} \otimes \mathcal{B}_0(H) \otimes S$. As $V_{13} = V_{12}^* V_{23} V_{12} V_{23}^*$, as $V^* \in M(\hat{S} \otimes \mathcal{B}_0(H)$ and $V \in M(\mathcal{B}_0(H) \otimes S)$, and as V is unitary, we equivalently can show that the closed linear span of

$$\big\{(x \otimes a \otimes 1) V_{23} V_{12} (1 \otimes b \otimes y) : a, b \in \mathcal{B}_0(H), x \in \hat{S}, y \in S\big\}$$

is $\hat{S} \otimes \mathcal{B}_0(H) \otimes S$. Notice that

$$(x \otimes a \otimes 1) V_{23} V_{12} (1 \otimes b \otimes y) = \big(x \otimes (a \otimes 1) V (1 \otimes y)\big)\big(V(1 \otimes b) \otimes 1\big),$$

and so by (2), we get the closed linear span of

$$\begin{aligned} &\big\{\big(x \otimes c \otimes z\big) V_{12} (1 \otimes b \otimes 1) : b, c \in \mathcal{B}_0(H), z \in S, x \in \hat{S}\big\} \\ &\quad = \big\{(1 \otimes c \otimes z)\big((x \otimes 1) V (1 \otimes b) \otimes 1\big) : b, c \in \mathcal{B}_0(H), z \in S, x \in \hat{S}\big\}, \end{aligned}$$

which again by (2) is the closed linear span of

$$\big\{(1 \otimes c \otimes z)(x \otimes b \otimes 1) : b, c \in \mathcal{B}_0(H), z \in S, x \in \hat{S}\big\},$$

which is of course $\hat{S} \otimes \mathcal{B}_0(H) \otimes S$, as required. □

corr:1 **Corollary 4.8** (Corollaire 3.7)**.** *Let V be a regular multiplicative unitary, and let $S, \hat{S}$ be the associated C^*-algebras. Then:*

corr:1.1 1. *The closed linear spans of $\{V(x \otimes 1) V^* (1 \otimes y) : x, y \in S\}$ and $\{V(x \otimes 1) V^* (y \otimes 1) : x, y \in S\}$ are both equal to $S \otimes S$;*

corr:1.2 2. *The closed linear spans of $\{V^*(1 \otimes x) V (1 \otimes y) : x, y \in \hat{S}\}$ and $\{V^*(1 \otimes x) V (y \otimes 1) : x, y \in \hat{S}\}$ are both equal to $\hat{S} \otimes \hat{S}$;*

Proof. For $a \in \mathcal{B}_0(H), \omega \in \mathcal{B}(H)_*$ and $y \in S$,

$$\begin{aligned} V(L(a\omega) \otimes 1) V^* (1 \otimes y) &= (\omega \otimes \iota \otimes \iota)\big(V_{23} V_{12} (a \otimes 1 \otimes 1) V_{23}^* (1 \otimes 1 \otimes y)\big) \\ &= (\omega \otimes \iota \otimes \iota)\big(V_{12} V_{13} (a \otimes 1 \otimes y)\big). \end{aligned}$$

By Proposition 4.7(1) we see that

$$\begin{aligned} \overline{\text{lin}}\{V(x \otimes 1) V^* (1 \otimes y) : x, y \in S\} &= \overline{\text{lin}}\{(\omega \otimes \iota \otimes \iota)(V_{12}(a \otimes 1 \otimes y) : a \in \mathcal{B}_0(H), y \in S\} \\ &= \overline{\text{lin}}\{(\omega \otimes \iota)(V(a \otimes 1)) : a \in \mathcal{B}_0(H)\} \otimes S = S \otimes S. \end{aligned}$$

Now consider

$$\begin{aligned} V(L(\omega a) \otimes 1) V^* (y \otimes 1) &= (\omega \otimes \iota \otimes \iota)\big(V_{23} (a \otimes 1 \otimes 1) V_{12} V_{23}^* (1 \otimes y \otimes 1)\big) \\ &= (\omega \otimes \iota \otimes \iota)\big((a \otimes 1 \otimes 1) V_{12} V_{13} (1 \otimes y \otimes 1)\big). \end{aligned}$$

Thus, now using Proposition 4.7(2),

$$\begin{aligned} &\overline{\text{lin}}\{V(x \otimes 1) V^* (y \otimes 1) : x, y \in S\} \\ &\quad = \overline{\text{lin}}\{(\omega \otimes \iota \otimes \iota)\big(((a \otimes 1) V (1 \otimes y) \otimes 1) V_{13}\big) : a \in \mathcal{B}_0(H), y \in S\} \\ &\quad = \overline{\text{lin}}\{(\omega \otimes \iota \otimes \iota)\big((a \otimes y \otimes 1) V_{13}\big) : a \in \mathcal{B}_0(H), y \in S\} = S \otimes S. \end{aligned}$$

This shows (1), and then (2) follows by working with $\Sigma V^* \Sigma$. □

thm:1 **Theorem 4.9** (Théorème 3.8). *Let V be a regular multiplicative unitary, and let $S, \hat{S}$ be the associated C*-algebras. We may define a coproduct δ on S by $\delta(x) = V(x \otimes 1)V^*$, and then (S, δ) becomes a bisimplifiable Hopf-C*-algebra. We may define a coproduct $\hat{\delta}$ on $\hat{S}$ by $\hat{\delta}(x) = V^*(1 \otimes x)V$, and then $(\hat{S}, \hat{\delta})$ becomes a bisimplifiable Hopf-C*-algebra.*

Proof. By Corollary 4.8(1) it follows that δ is indeed a $*$-homomorphism $S \to M(S \otimes S)$ such that $\delta(S)(1 \otimes S)$ and $\delta(S)(S \otimes 1)$ are (dense) subsets of $S \otimes S$; this also shows that (S, δ) is bisimplifiable. That δ is coassociative follows as

$$(\iota \otimes \delta)\delta(x) = V_{23}V_{12}(x \otimes 1 \otimes 1)V_{12}^*V_{23}^* = V_{12}V_{13}V_{23}(x \otimes 1 \otimes 1)V_{23}^*V_{13}^*V_{12}^* = (\delta \otimes \iota)\delta(x),$$

as required. Let (u_i) be a bounded approximate identity for S, and let $x, y \in S$, so with $\tau = \delta(x)(1 \otimes y) \in S \otimes S$,

$$\delta(u_i)\tau = \delta(u_i x)(1 \otimes y) \to \delta(x)(1 \otimes y) = \tau.$$

By Corollary 4.8(1), such τ are dense, and so δ is non-degenerate. The results for $\hat{S}$ follow from working with $\Sigma V^* \Sigma$. ∎

prop:8 **Proposition 4.10** (Proposition 3.9). *The map $\kappa : A(V) \to S; (\omega \otimes \iota)(V) \mapsto (\omega \otimes \iota)(V^*)$ is a well-defined algebra antihomomorphism, called the* antipode.

Proof. We have that $(\omega \otimes \iota)(V^*) = L(\omega^*)^* \in S$ by Proposition 4.6. If $L(\omega) = 0$ then

$$0 = \langle L(\omega), \omega' \rangle = \langle \rho(\omega'), \omega \rangle = \langle x, \omega \rangle \qquad (\omega' \in \mathcal{B}(H)_*, x \in \hat{S}),$$

the last equality following by density. As $\hat{S}$ is self-adjoint, also $\langle x, \omega^* \rangle = \overline{\langle x^*, \omega \rangle} = 0$ for all $x \in \hat{S}$, and so $\langle L(\omega^*), \omega' \rangle = \langle \rho(\omega'), \omega^* \rangle = 0$ for all $\omega' \in \mathcal{B}(H)_*$. Thus $L(\omega^*) = 0$, and so κ is well-defined.

As in the proof of Proposition 2.7, given $\omega, \omega' \in \mathcal{B}(H)_*$, if $\psi \in \mathcal{B}(H)_*$ is defined by $\langle T, \psi \rangle = \langle V^*(1 \otimes T)V, \omega \otimes \omega' \rangle$ then $L(\omega)L(\omega') = L(\psi)$. Then $\langle T, \psi^* \rangle = \overline{\langle V^*(1 \otimes T^*)V, \omega \otimes \omega' \rangle} = \langle V^*(1 \otimes T)V, \omega^* \otimes (\omega')^* \rangle$ and so $L(\psi^*) = L(\omega^*)L((\omega')^*)$. Thus $\kappa(L(\omega)L(\omega')) = L(\psi^*)^* = L((\omega')^*)^* L(\omega^*)^* = \kappa(L(\omega'))\kappa(L(\omega))$ and so κ is an antihomomorphism as required. ∎

Definition 4.11 (Définition 3.10). *A multiplicative unitary V is* biregular *if it is regular, and if $\{(\omega \otimes \iota)(\Sigma V) : \omega \in \mathcal{B}(H)_*\}$ is dense in $\mathcal{B}_0(H)$.*

defn:1 **Remark 4.12** (Remarques 3.11(a)). Let W be the fundamental unitary associated to a Kac-von Neumann algebra, see [3]. Set $V = W^*$ and let $\hat{\Delta}$ be the modular operator associated with the dual Haar weight $\hat{\phi}$ on the dual Kac algebra $\hat{M}$. Following [3, 2.1.5(a)] it follows that $\hat{A}(V)$ generates $\hat{M}$ as a von Neumann algebra; the same is true of S. Then [3, corollaire 3.1.10] shows that the restriction of $\hat{\phi}$ to S^+, say ψ, defines a normal semi-finite weight on S. By [4, Lemme I.1], we have that $V^*(1 \otimes \hat{\Delta})V = \hat{\Delta} \otimes \hat{\Delta}$. Thus, for $\omega \in M_*$ and all $t \in \mathbb{R}$, we have that $L(\hat{\Delta}^{it}\omega) = \hat{\Delta}^{it} L(\omega) \hat{\Delta}^{-it}$ and so the modular automorphism group (σ_t) of $\hat{M}$ restricts to S to give a norm-continuous group of automorphisms. It is now easy to verify that S together with κ and ψ gives a Kac C*-algebra in the sense of [12].

Remark 4.13 (Remarques 3.11(b)). Let V be a regular multiplicative unitary. For $\omega \in \mathcal{B}(H)_*$, as in the proof above, we see that $L(\omega) = 0$ if and only if ω induces the zero functional on S. As S is a non-degenerate C*-algebra of H (by Proposition 2.7) we see that if $\omega \geq 0$, then ω is zero on S only if $\omega = 0$. (This follows, as let (e_α) be a bounded approximate identity in S. Non-degeneracy implies that $e_\alpha \to 1$ strongly, and so $\|\omega\| = \langle 1, \omega \rangle = \lim_\alpha \langle e_\alpha, \omega \rangle$.) Similarly, if $\omega \geq 0$ and $\rho(\omega) = 0$, then $\omega = 0$.

For $x \in S$ and $\omega, \omega' \in S^*$, define

$$x * \omega = (\omega \otimes \iota)\delta(x), \quad \omega * x = (\iota \otimes \omega)\delta(x), \quad \omega * \omega' = (\omega \otimes \omega') \circ \delta.$$

By Lemma A.1, we may suppose that $\omega = \omega_0 a_0$ for some $\omega_0 \in S^*, a_0 \in S$. Then $x * \omega = (\omega \otimes \iota)((a_0 \otimes 1)\delta(x)) \in S$, as $(a_0 \otimes 1)\delta(x) \in S \otimes S$. Similarly $\omega * x \in S$.

Suppose now $x \geq 0$ and $\omega \geq 0$ and that $\omega * x = 0$. If $\omega \neq 0$, write $x = y^* y$ for some $y \in S$, and let (π, H, ξ) be the cyclic GNS construction for ω. Then

$$0 = (\iota \otimes \omega)\delta(x) = (\iota \otimes \omega_\xi)(\iota \otimes \pi)(V(y^* y \otimes 1)V^*),$$

and so $(y \otimes 1)(\iota \otimes \pi)(V^*)(\cdot \otimes \xi) = 0$. In particular, for $a \in \mathcal{B}_0(H), b \in S$, also

$$0 = (y \otimes \pi(b))(\iota \otimes \pi)(V^*)(a(\cdot) \otimes \xi) = (y \otimes 1)(\iota \otimes \pi)\big((1 \otimes b)V^*(a \otimes 1)\big)(\cdot \otimes \xi).$$

By Proposition 4.7(2), this shows that

$$0 = (y \otimes 1)(c \otimes \pi(d))(\cdot \otimes \xi) \qquad (c \in \mathcal{B}_0(H), d \in S).$$

It follows that $y = 0$, so $x = 0$. In conclusion, $x \geq 0, \omega \geq 0, \omega * x = 0 \implies x = 0$ or $\omega = 0$.

Remark 4.14 (Remarques 3.11(c))**.** We say that (A, δ) is *right reduced* (respectively, *left reduced*) if for non-zero $\omega \in A_+^*, x \in A_+$ also $\omega * x$ (respectively, $x * \omega$) is non-zero. We have just shown that (S, δ) arising from a regular multiplicative unitary is right reduced; similarly $\hat{S}$ will be left reduced.

prop:9 **Proposition 4.15** (Proposition 3.11.1)**.** *Let (A, δ) be right (respectively left) reduced. Then:*

prop:9.1 1. *For non-zero $\omega, \omega' \in A_+^*$ with ω faithful, and for non-zero $x \in A_+$, we have that $\omega * \omega'$ (respectively $\omega' * \omega$) is faithful, and $x * \omega$ (respectively $\omega * x$) is strictly positive (meaning that $\langle \mu, x * \omega \rangle > 0$ for all states μ, or that the right ideal generated by $x * \omega$ is all of A).*

prop:9.2 2. *If A is unital and separable, then it admits a right (respectively, left) faithful Haar state.*

Proof. We prove the assertions in the right reduced case; the left reduced case follows by replacing δ with $\sigma\delta$ where $\sigma : A \otimes A \to A \otimes A$ is the swap map. For non-zero $y \in A_+$,

$$\langle \omega * \omega', y \rangle = \langle \omega, \omega' * y \rangle \neq 0,$$

as $\omega' * y \neq 0$ and ω is faithful. Similarly, for a state μ,

$$\langle \mu, x * \omega \rangle = \langle \omega * \mu, x \rangle \neq 0,$$

by using the previous calculation. To show (2), we use the following lemma. □

lem:1 **Lemma 4.16** (Lemme 3.11.2)**.** *With (A, δ) being unital and right reduced, let ω be a faithful state. Then:*

lem:1.1 1. *If $x \in A$ with $x * \omega = x$, then $x \in \mathbb{C}1$;*

lem:1.2 2. *There is a state ϕ with $\omega * \phi = \phi * \omega = \phi$ (compare [13]).*

lem:1.3 3. *Such ϕ is also a faithful right Haar state.*

Proof. As $(x * \omega)^* = ((\omega \otimes \iota)\delta(x))^* = (\omega \otimes \iota)\delta(x^*) = x^* * \omega$, for (1) we may suppose that $x = x^*$. Notice that $1 * \omega = (\omega \otimes \iota)\delta(1) = 1$. So for $\lambda \in \mathbb{R}$, if $x - \lambda \geq 0$ is positive and non-zero, then by Proposition 4.15(1) we have that $(x - \lambda) * \omega = x - \lambda$ is strictly positive. Taking λ to be the minimum of the spectrum of x shows that $x \in \mathbb{R}1$ as claimed.

For (2) let ϕ be a weak*-limit of the Cesaro means of $\omega^n = \omega * \omega * \cdots * \omega$ (n times). Then ϕ is a state, and clearly $\phi * \omega = \omega * \phi = \phi$.

For (3), for $x \in A$ we have that $(x * \phi) * \omega = x * (\phi * \omega) = x * \phi$ and so by (1) $x * \phi$ is a scalar. But then $x * \phi = (x * \phi) * \omega = (\omega \otimes \iota)\delta(x * \phi) = \langle \omega, x * \phi \rangle 1 = \langle \phi, x \rangle 1$ so ϕ is a right Haar state. As $\phi = \omega * \phi$, by Proposition 4.15(1), ϕ is faithful. □

5 Multiplicative unitaries of compact type, and Woronowicz C*-algebras

In this section, we depart from the original paper, and study the relationship between Compact Quantum Groups (in the sense of [wor], a paper not published at the time) and multiplicative unitaries of compact type. Compact Quantum Groups have subsumed the theory of Matrix Pseudogroups as a special case, and an added advantage is that the resulting proofs are easier in some cases.

Firstly, let (A, δ) be a compact quantum group. That is, A is unital and (A, δ) is bisimplifiable. Then [wor] shows that (A, δ) admits a unique Haar state ϕ. By Example 2.2(4) we construct a multiplicative unitary V on the GNS space for ϕ. By Proposition 4.5(2) V is regular (we note that the condition here, that $\phi(x^*x) = 0$ if and only if $\phi(xx^*) = 0$ is quite involved to prove– see [wor, ???]). The C*-algebra S is simply $\pi(A)$, and the coproduct on S is the natural quotient of δ. As S is thus unital, V is of compact type.

[Do we want to give a self-contained (sketch/account) of all of this? It might be rather involved...]

We now start with a multiplicative unitary V on H of compact type which admits a non-zero fixed vector $E \in H$ (see Definition 2.11). If H is separable, then by Proposition 2.13 such a fixed vector automatically exists. Let $\phi = \omega_e \in \mathcal{B}(H)_*$.

defn:3 **Definition 5.1** (Définition 4.3). *For $\xi \in H$ define $\lambda_\xi \in \mathcal{B}(H)$ by $\lambda_\xi = (\theta'_e)^* V^* \theta_\xi$. That is, for $\eta, \eta' \in H$, $(\lambda_\xi(\eta)|\eta') = (V(\xi \otimes \eta)|\eta' \otimes e)$.*

prop:13 **Proposition 5.2** (Proposition 4.4). *For $\xi \in H$, we have that $(\lambda_\xi \otimes 1)V = V(\lambda_\xi \otimes 1)$.*

Proof. We have that $\lambda^*_\xi \otimes 1 = \theta^*_{1,\xi} V_{12} \theta_{2,e}$. Thus

$$\begin{aligned} V(\lambda^*_\xi \otimes 1) &= V\theta^*_{1,\xi} V_{12}\theta_{2,e} = \theta^*_{1,\xi} V_{23} V_{12} \theta_{2,e} = \theta^*_{1,\xi} V_{12} V_{13} V_{23} \theta_{2,e} \\ &= \theta^*_{1,\xi} V_{12} V_{13} \theta_{2,e} \quad \text{as } e \text{ is fixed, so } V\theta_e = \theta_e \\ &= \theta^*_{1,\xi} V_{12} \theta_{2,e} V = (\lambda^*_\xi \otimes 1)V. \end{aligned}$$

□

6 Constructions with Woronowicz C*-algebras

sec:5

7 Irreducible multiplicative unitaries

prop:10 **Proposition 7.1** (Proposition 6.1). *Let V be a multiplicative unitary on H and let $U \in \mathcal{B}(H)$ be a unitary with $U^2 = 1$ such that $\hat{V} = \Sigma(U \otimes 1)V(U \otimes 1)\Sigma$ and $\tilde{V} = (U \otimes U)\hat{V}(U \otimes U)$ are both multiplicative. Then the following formulae hold:*

rop:10.1 1. $V_{12}(1 \otimes U \otimes 1)V_{23}(1 \otimes U \otimes 1) = (1 \otimes U \otimes 1)V_{23}(1 \otimes U \otimes 1)V_{13}V_{12}$;

rop:10.2 2. $\hat{V}_{23}V_{12}V_{13} = V_{13}\hat{V}_{23}$;

rop:10.3 3. $\tilde{V}_{12}V_{13} = V_{13}V_{23}\tilde{V}_{12}$;

rop:10.4 4. *the unitaries $\Sigma_{23}\hat{V}_{23}V_{23}$ and V_{12} commute;*

rop:10.5 5. *the unitaries $V_{12}\tilde{V}_{12}\Sigma_{12}$ and V_{23} commute.*

Proof. We have that

$$\Sigma_{13}\hat{V}_{12}\Sigma_{13} = (1 \otimes U \otimes 1)V_{23}(1 \otimes U \otimes 1),$$
$$\Sigma_{13}\hat{V}_{23}\Sigma_{13} = (U \otimes 1 \otimes 1)V_{12}(U \otimes 1 \otimes 1),$$

That $\hat{V}$ is multiplicative means that

$$\begin{aligned}\hat{V}_{12}\hat{V}_{13}\hat{V}_{23} &= \hat{V}_{12}\Sigma_{13}(U \otimes 1 \otimes 1)V_{13}(U \otimes 1 \otimes 1)\Sigma_{13}\hat{V}_{23} \\ &= \Sigma_{13}(1 \otimes U \otimes 1)V_{23}(U \otimes U \otimes 1)V_{13}V_{12}(U \otimes 1 \otimes 1)\Sigma_{13} = \hat{V}_{23}\hat{V}_{12},\end{aligned}$$

that is

$$(U \otimes U \otimes 1)V_{23}(1 \otimes U \otimes 1)V_{13}V_{12}(U \otimes 1 \otimes 1) = (U \otimes 1 \otimes 1)V_{12}(1 \otimes U \otimes 1)V_{23}(U \otimes U \otimes 1).$$

Then (1) follows.

Applying Σ_{23} to the left and right of (1) gives (2). Using Σ_{12} instead gives (3), once we notice that $\tilde{V} = \Sigma(1 \otimes U)V(1 \otimes U)\Sigma$.

As V is multiplicative, (2) gives that $\hat{V}_{23}V_{23}V_{13} = V_{13}\hat{V}_{23}\hat{V}_{23}$ and applying Σ_{23} on the left gives (4). A similar argument applied to (3) gives (5). □

Definition 7.2 (Définition 6.2)**.** *A multiplicative unitary V is* irreducible *is there is a unitary $U \in \mathcal{B}(H)$ with:*

1. *$U^2 = 1$ and $(\Sigma(1 \otimes U)V)^3 = 1$;*
2. *the unitaries $\hat{V} = \Sigma(U \otimes 1)V(U \otimes 1)\Sigma$ and $\tilde{V} = (U \otimes U)\hat{V}(U \otimes U)$ are both multiplicative.*

Notice that clearly $\tilde{V}$ is multiplicative if and only if $\hat{V}$ is multiplicative. That $(\Sigma(1 \otimes U)V)^3 = 1$ is equivalent to $\hat{V}V\tilde{V} = (U \otimes 1)\Sigma$. Finally, observe that U being unitary with $U^2 = 1$ is equivalent to U being self-adjoint and unitary.

Proposition 7.3 (Proposition 6.3)**.** *Let V be a multiplicative unitary which is regular and irreducible. Then $\{xy : x \in S, y \in \hat{S}\}$ is linearly dense in $\mathcal{B}_0(H)$.*

Proof. Notice that $\Sigma\tilde{V}^* = (1 \otimes U^*)V^*(1 \otimes U^*)\Sigma = (1 \otimes U^*)\Sigma(\Sigma V^*\Sigma)(U^* \otimes 1)$ and so

$$\mathcal{C}(\Sigma\tilde{V}^*) = \{(\iota \otimes \omega)((1 \otimes U^*)\Sigma(\Sigma V^*\Sigma)(U^* \otimes 1)) : \omega \in \mathcal{B}(H)_*\} = \mathcal{C}(\Sigma V^*\Sigma)U^* = \mathcal{C}(V)^*U^*,$$

which equals $\mathcal{B}_0(H)$ as V is regular. Hence also $\{(\iota \otimes \omega)((U \otimes 1)\Sigma\tilde{V}^*) : \omega \in \mathcal{B}(H)_*\}$ is dense in $\mathcal{B}_0(H)$. As V is irreducible, $(U \otimes 1)\Sigma\tilde{V}^* = \hat{V}V$, and so $\{(\iota \otimes \omega)(\hat{V}V) : \omega \in \mathcal{B}(H)_*\}$ is dense in $\mathcal{B}_0(H)$. As $\hat{S}$ acts irreducibly on H, also $\{(\iota \otimes \omega)(\hat{V}V)y : \omega \in \mathcal{B}(H)_*, y \in \hat{S}\}$ is linearly dense in $\mathcal{B}_0(H)$.

Now, $(\iota \otimes \omega)(\hat{V}V)y = (\iota \otimes \omega)(\hat{V}V(y \otimes 1))$ and as V is a unitary multiplier of $\hat{S} \otimes \mathcal{B}_0(H)$ (by Proposition 4.7(1)) it follows that

$$\{(\iota \otimes \omega)(\hat{V}(y \otimes 1)) : \omega \in \mathcal{B}(H)_*, y \in \hat{S}\}$$

is linearly dense in $\mathcal{B}_0(H)$. As $(\iota \otimes \omega)(\hat{V}(y \otimes 1)) = (U\omega U \otimes \iota)(V)y = L(U\omega U)y$ the result follows. □

Definition 7.4 (Définition 6.4)**.** *A* Kac system *is a triple (H, V, U) where H is a Hilbert space, V is a biregular multiplicative unitary (see Definition 4.12) and U is a unitary verifing that V is also irreducible.*

lem:2 **Lemma 7.5** (Définition 6.5)**.** *Let (H, V, U) be a Kac system. Then:*

lem:2.1 1. $(H, \Sigma V^*\Sigma, U)$ *and* $(H, \hat{V}, U)$ *are Kac systems;*

lem:2.2 2. *The unitaries* V_{12} *and* $\tilde{V}_{23}$ *commute;*

lem:2.3 3. *The unitaries* V_{23} *and* $\hat{V}_{12}$ *commute.*

Proof. By definition, V is biregular if and only if $\mathcal{C}(V) = \{(\iota \otimes \omega)(\Sigma V) : \omega \in \mathcal{B}(H)_*\}$ is dense in $\mathcal{B}_0(H)$ and $\{(\omega \otimes \iota(\Sigma V) : \omega \in \mathcal{B}(H)_*\} = \{(\iota \otimes \omega(V\Sigma) : \omega \in \mathcal{B}(H)_*\} = \{(\iota \otimes \omega(\Sigma\hat{V}) : \omega \in \mathcal{B}(H)_*\} = \mathcal{C}(\hat{V})$ is dense in $\mathcal{B}_0(H)$. That is, V is biregular if and only if V and $\hat{V}$ are regular.

So set $W = \Sigma V^*\Sigma$, so

$$\hat{W} = \Sigma(U \otimes 1)\Sigma V^*\Sigma(U \otimes 1)\Sigma = (1 \otimes U)V^*(1 \otimes U) = \Sigma\tilde{V}^*\Sigma.$$

Similarly $\tilde{W} = \Sigma\hat{V}^*\Sigma$. Then (1) follows.

As $\hat{V}V\tilde{V} = (U \otimes 1)\Sigma$ we see that $\tilde{V}_{23}^* = \Sigma_{23}(1 \otimes U \otimes 1)\hat{V}_{23}V_{23} = (1 \otimes 1 \otimes U)(\Sigma\hat{V}V)_{23}$ which commutes with V_{12} by Proposition 7.1(4). Hence also $\tilde{V}_{23}$ commutes with V_{12}, giving (2). Similarly, Proposition 7.1(5) shows (3). □

Definition 7.6 (Définition 6.6)**.** *We say that* $(H, \hat{V}, U)$ *is the* dual Kac system *to* (H, V, U)*, and that* $(H, \Sigma V^*\Sigma, U)$ *is the* opposite Kac system *to* (H, V, U)*. Two Kac systems* (H, V, U) *and* (H', V', U') *are* isomorphic *if there is a unitary* $w \in \mathcal{B}(H, H')$ *with* $(w \otimes w)V = V'(w \otimes w)$ *and* $wU = U'w$*. We also say that* (H', V', U') *is* dual to (H, V, U) *if it is isomorphic to* $(H, \hat{V}, U)$*.*

Notice that the Kac systems $(H, \hat{V}, U)$ and $(H, \tilde{V}, U)$ are isomorphic (by U).

Definition 7.7 (Définition 6.7)**.** *Let* (H, V, U) *be a Kac system. For* $\omega \in \mathcal{B}(H)_*$*, we write*

$$\lambda(\omega) = L_{\hat{V}}(\omega) = (\omega \otimes \iota)(\hat{V}), \quad R(\omega) = \rho_{\tilde{V}}(\omega) = (\iota \otimes \omega)(\tilde{V}).$$

[Note: At this point, the original paper overloads notation, and seems to write L for both the map $\mathcal{B}(H)_* \to S \subseteq \mathcal{B}(H)$, and also for the (trivial) representation of S on $\mathcal{B}(H)$. Then λ is now both a map $\mathcal{B}(H)_* \to U\hat{S}U$, and also the representation $\hat{S} \to \mathcal{B}(H)$ given by $y \mapsto UyU$. We have tried to avoid doing this, and continue to view S and $\hat{S}$ as concrete subalgebras of $\mathcal{B}(H)$.]

lem:3 **Proposition 7.8.** *(Proposition 6.8) We have that:*

lem:3.1 1. $\lambda(\omega) = U\rho(\omega)U$ *and* $R(\omega) = UL(\omega)U$*;*

lem:3.2 2. *For all* $\omega, \omega' \in \mathcal{B}(H)_*$*, the operators* $\rho(\omega)$ *and* $\lambda(\omega')$ *commute, and also* $L(\omega)$ *and* $R(\omega')$ *commute;*

lem:3.3 3. *For* $x \in S, y \in \hat{S}$ *we have that*

$$\delta(x) = \hat{V}^*(1 \otimes x)\hat{V}, \qquad (U \otimes U)\hat{\delta}(y)(U \otimes U) = \hat{V}(UyU \otimes 1)\hat{V}^*.$$

Proof. For (1) we simply calculate that

$$\lambda(\omega) = (\omega \otimes \iota)(\Sigma(U \otimes 1)V(U \otimes 1)\Sigma) = U(\iota \otimes \omega)(V)U = U\rho(\omega)U,$$

the other case following similarly.

For (2) we see that

$$\rho(\omega)\lambda(\omega') = (\iota \otimes \omega)(V)(\omega' \otimes \iota)(\hat{V}) = (\omega' \otimes \iota \otimes \omega)(V_{23}\hat{V}_{12}),$$

and so the result follows from Lemma 7.5(3). The other case uses Lemma 7.5(2).

Let $\omega \in \mathcal{B}(H)_*$ and set $x = L(\omega)$. Then

$$\begin{aligned}\delta(x) &= V((\omega \otimes \iota)(V) \otimes 1)V^* = (\omega \otimes \iota \otimes \iota)(V_{23}V_{12}V_{23}^*) = (\omega \otimes \iota \otimes \iota)(V_{12}V_{13}) \\ &= (\omega \otimes \iota \otimes \iota)(\hat{V}_{23}^* V_{13} \hat{V}_{23}) = \hat{V}^*(1 \otimes x)\hat{V},\end{aligned}$$

where we have used that V is multiplicative, and also Proposition 7.1(2). Then the first part of (3) follows as such x are dense in S. Similarly, using Proposition 7.1(3) shows that

$$\hat{\delta}(y) = \tilde{V}(y \otimes 1)\tilde{V}^* \qquad (y \in \hat{S}).$$

Then the second part of (3) follows immediately. □

prop:11 **Proposition 7.9** (Proposition 6.9). *Let V be a mutliplicative unitary on H, and let $U \in \mathcal{B}(H)$ be a unitary with $U^2 = 1$, and such that V_{12} and $\tilde{V}_{23}$ commute, and $\hat{V}_{12}$ and V_{23} commute. Then:*

rop:11.1 1. *If the set $\{\rho(\omega)L(\omega') : \omega, \omega' \in \mathcal{B}(H)_*\}$ is linearly dense in $\mathcal{B}_0(H)$, then V is regular;*

rop:11.2 2. *If $\hat{V}$ is multiplicative, and both $(S \cup \hat{S})' = \mathbb{C}1$ and $(S \cup U\hat{S}U)' = \mathbb{C}1$, then $(1 \otimes U)\Sigma\hat{V}V\tilde{V} \in \mathbb{C}1$.*

Proof. We first prove (1). Let $\omega, \omega' \in \mathcal{B}(H)_*$, set $x = (\iota \otimes \omega)(\Sigma V) \in \mathcal{C}(V)$ and set $s = UL(\omega')U = R(\omega') = (\iota \otimes \omega')(\tilde{V})$. As V_{12} and $\tilde{V}_{23}$ commute, it follows that $(1 \otimes s)V = V(1 \otimes s)$ and so

$$sx = (\iota \otimes \omega)((s \otimes 1)\Sigma V) = (\iota \otimes \omega)(\Sigma V(1 \otimes s)) = (\iota \otimes s\omega)(\Sigma V) \in \mathcal{C}(V).$$

As $A(V)H$ is linearly dense in H (by Proposition 2.7) it follows that $\mathcal{C}(V)$ has the same closure as the linear span of $UA(V)U\mathcal{C}(V)$.

Similarly, setting $t = U\rho(\omega')U = (\omega' \otimes \iota)(\hat{V})$ and using that $\hat{V}_{12}$ and V_{23} commute will show that $\mathcal{C}(V)U\hat{A}(V)U$ has closed linear span equal to the closure of $\mathcal{C}(V)$.

We hence see that $\mathcal{C}(V)^2$ has closed linear span equal to $\overline{\text{lin}}\mathcal{C}(V)U\hat{A}(V)\hat{A}(V)U\mathcal{C}(V)$. As remarked after Proposition 4.2, $\mathcal{C}(V)^2$ is linearly dense in $\mathcal{C}(V)$. By hypothesis, $\hat{A}(V)\hat{A}(V)$ is linearly dense in $\mathcal{B}_0(H)$. As V is unitary, it is easy to see that $\mathcal{C}(V)H$ and $\mathcal{C}(V)^*H$ are linearly dense in H. It follows that $\mathcal{C}(V)U\hat{A}(V)\hat{A}(V)U\mathcal{C}(V)$ is linearly dense in $\mathcal{B}_0(H)$, and so the same is true of $\mathcal{C}(V)$ showing that V is regular.

For (2), set $W = (1 \otimes U)\Sigma\hat{V}V\tilde{V}$. As V_{12} commutes with $\tilde{V}_{23}$, and as we can now apply Proposition 7.1(4), we conclude that V_{12} and W_{23} commute. Applying Proposition 7.1(4) to V, and noting that $\hat{\hat{V}} = V$, we see that $\tilde{V}_{12}$ and $\Sigma_{23}V_{23}\tilde{V}_{23}$ commute. As $\hat{V}_{12}$ and V_{23} commute, also $\tilde{V}_{12}$ and $(1 \otimes U \otimes U)V_{23}(1 \otimes U \otimes 1)$ commute. As $W = (U \otimes U)V(U \otimes 1)\Sigma V\tilde{V}$, we conclude that $\tilde{V}_{12}$ and W_{23} commute. So W will commute with $(x \otimes 1)$ for all x of the form $(\omega \otimes \iota)(V)$ and of the form $(\omega \otimes \iota)(\tilde{V}) = (\omega \otimes \iota)(\Sigma(1 \otimes U)V(1 \otimes U)\Sigma) = (\iota \otimes U\omega U)(V)$, that is, for all $x \in S \cup \hat{S}$.

If we replace V by $\hat{V}$ in the argument of the previous paragraph, then as $\hat{\hat{V}} = (U \otimes U)V(U \otimes U)$ and $\tilde{\hat{V}} = V$, we see that $X = (1 \otimes U)\Sigma(U \otimes U)V(U \otimes U)\hat{V}V$ commutes with $1 \otimes x$ for all x of the form $(\omega \otimes \iota)(\hat{V}) = U\rho(\omega)U$ and of the form $(\iota \otimes \omega)(\hat{V}) = L(U\omega U)$. That is, for all $x \in S \cup U\hat{S}U$. As $X = \Sigma(U \otimes 1)W(U \otimes 1)\Sigma$, we conclude that W commutes with $1 \otimes x$ for all $x \in S \cap U\hat{S}U$. Thus $W \in \mathbb{C}1$ as required. □

Corollary 7.10 (Corollaire 6.10). *Let V be a multiplicative unitary and let $U \in \mathcal{B}(H)$ be a unitary with $U^2 = 1$. Form $\hat{V}, \tilde{V}$ as before, and suppose that $\hat{V}$ is multiplicative, that V_{12} commutes with $\tilde{V}_{23}$, and that $\hat{V}_{12}$ commutes with V_{23}. If the closed linear span of $\{xUyU : x \in S, y \in \hat{S}\}$ is $\mathcal{B}_0(H)$, then $\tilde{V}$ and $\hat{V}$ are regular.*

Proof. Apply the previous proposition to $\hat{V}$. □

Examples 7.11 (Exemples 6.11)**.** 1. The multiplicative unitary $1 \in \mathcal{B}(H \otimes H)$ is not irreducible unless $H = \mathbb{C}$, as $(\Sigma(1 \otimes U))^3 = \Sigma(U \otimes 1)$.

2. Let G be a locally compact group, equipped with the right Haar measure. Define a unitary U on $L^2(G)$ by $(U\xi)(t) = \Delta^{1/2}(t)\xi(t^{-1})$, where Δ is the modular function for the Haar measure. Then $(L^2(G), V_G, U)$ is a Haar system (with $V_G\xi(s,t) = \xi(st,t)$ as in Examples 2.2). Indeed, we showed in Examples 4.4 that V_G is regular. Then $\Sigma(1 \otimes U)V_G\xi(s,t) = V_G\xi(t,s^{-1})\Delta^{1/2}(s) = \xi(ts^{-1},s^{-1})\Delta^{1/2}(s)$, and it follows that $(\Sigma(1 \otimes U)V_G)^3 = 1$. Then $\hat{V}_G\xi(s,t) = \xi(s,s^{-1}t)\Delta^{1/2}(s)$ and direct calculation shows this to be multiplicative and regular.

3. Let (A,δ) be a compact quantum group and form (H,V,U) as in Section 6. **TO FINISH!**

4. Let W be the fundamental unitary of Kac-von Neumann algebra (see [3]). Let $V = W^*$ and set $U = J\hat{J} = \hat{J}J$ (see [11]). As $\hat{V}$ is the fundamental unitary associated with the dual Kac-von Neumann algebra, it is regular. It's a result of [11], and Proposition 7.9, that $(1 \otimes U)\Sigma\hat{V}V\tilde{V}$ is a scalar, and in fact, it's not hard to show that $(1 \otimes U)\Sigma\hat{V}V\tilde{V} = 1$. Thus (H,V,U) is a Kac system.

Remark 7.12. (Remarque 6.12)

1. Let (H,U,V) be a Kac system. As $\hat{\hat{V}} = \tilde{\tilde{V}} = (U \otimes U)V(U \otimes U)$ we have that $(1 \otimes U)\Sigma\hat{V}V\tilde{V} = \hat{\hat{V}}\hat{V}V(1 \otimes U)\Sigma$. It follows that $\hat{V}V\tilde{V} = \hat{\hat{V}}\hat{V}V = (U \otimes 1)\Sigma$ and so $\hat{V}V\tilde{V} = \hat{\hat{V}}\hat{V}V = \tilde{V}\hat{\hat{V}}\hat{V} = V\tilde{V}\tilde{\tilde{V}}$.

2. The operator $\mathcal{R} = V(U \otimes 1)V(U \otimes 1)$ satisfies the Yang-Baxter equation: $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$.

3. Some comments about [5].

8 Multiplicative unitaries and Takesaki-Takai biduality

Fix a Kac system (H,V,U).

Definition 8.1. *(Définition 7.1) Let δ_A be a coaction of S (or $\hat{S}$) on a C*-algebra A. Write π_L and π_R (respectively, $\hat{\pi}_\lambda$ and $\hat{\pi}_\rho$) for the representations of A on the Hilbert C*-module $A \otimes H$ defined by*

$$\pi_L = (\iota \otimes \iota) \circ \delta_A, \qquad \pi_R = (\iota \otimes U(\cdot)U) \circ \delta_A,$$

respectively,

$$\hat{\pi}_\lambda = (\iota \otimes U(\cdot)U) \circ \delta_A, \qquad \hat{\pi}_\rho = (\iota \otimes \iota)\delta_A.$$

Denote by $A \times \hat{S}$ (respectively $A \times S$) the crossed product *of A by S (respectively, $\hat{S}$), which is the C*-algebra generated by $\{\pi_L(a)(1 \otimes \rho(\omega)) : a \in A, \omega \in \mathcal{B}(H)_*\}$ (respectively, $\{\hat{\pi}_\lambda(a)(1 \otimes L(\omega)) : a \in A, \omega \in \mathcal{B}(H)_*\}$) inside $\mathcal{B}(A \otimes H)$.*

Here $U(\cdot)U$ is the $*$-homomorphism $S \to \mathcal{B}(H); x \mapsto UxU$ (the notation π_R being inspired by Proposition 7.8). [The odd notation is due to the fact that we are concretely viewing S as a subalgebra of $\mathcal{B}(H)$; whereas the original paper has by this point started using L to denote the inclusion map $S \to \mathcal{B}(H)$, and so forth; see the comment before Proposition 7.8.]

In fact, it is not really necessary to work with $A \otimes H$. Instead, we could work in $M(A \otimes \mathcal{B}_0(H))$, noticing that clearly $M(A \otimes S)$ and $M(A \otimes \hat{S})$ are subalgebras of $M(A \otimes \mathcal{B}_0(H))$. Then we can form $A \times S$ and $A \times \hat{S}$ inside $M(A \otimes \mathcal{B}_0(H))$.

lem:4 **Lemma 8.2.** *(Lemme 7.2, see* r23 *[8]) The crossed product $A\times\hat S$ (or $A\times S$) is the closed linear span of $\{\pi_L(a)(1\otimes\rho(\omega)) : a\in A, \omega\in\mathcal{B}(H)_*\}$ (respectively, $\{\hat\pi_\lambda(a)(1\otimes L(\omega)) : a\in A,\omega\in\mathcal{B}(H)_*\}$).*

Proof. We give a proof for $A\times\hat S$; the proof for $A\times S$ follows by working with $\hat V$ in place of V. We need to show that, for $a\in A$ and $\omega\in\mathcal{B}(H)_*$, we have that $(1\otimes\rho(\omega))\pi_L(a)$ is in the closed linear span of $\{\pi_L(a)(1\otimes\rho(\omega)) : a\in A,\omega\in\mathcal{B}(H)_*\}$. Let $\tilde\pi$ be the representation of A on the Hilbert C*-module $A\otimes H\otimes H$ defined by

$$\tilde\pi = (\pi_L\otimes\iota)\circ\delta_A = (\iota\otimes\delta)\circ\delta_A,$$

which follows as δ_A is a coaction. As $\delta_A(\cdot) = V(\cdot\otimes 1)V^*$, we see that $\tilde\pi(\cdot) = V_{23}\delta_A(\cdot)_{12}V_{23}^*$, and so

$$(1\otimes\rho(\omega))\pi_L(a) = (\iota\otimes\iota\otimes\omega)(V_{23}\pi_L(a)_{12}) = (\iota\otimes\iota\otimes\omega)(\tilde\pi(a)V_{23}).$$

Writing $\omega=\omega' s$ for some $\omega'\in\mathcal{B}(H)_*$ and $s\in S$, we obtain

$$(1\otimes\rho(\omega))\pi_L(a) = (\iota\otimes\iota\otimes\omega')\big((\pi_L\otimes\iota)\big((1\otimes s)\delta_A(a)\big)V_{23}\big).$$

Now, $(1\otimes s)\delta_A(a)\in A\otimes S$ and so we can approximate it by a linear span of elements of the form $b\otimes t$. However, then observe that

$$(\iota\otimes\iota\otimes\omega')\big((\pi_L\otimes\iota)(b\otimes t)V_{23}\big) = \pi_L(b)(1\otimes\rho(\omega' t)).$$

The result follows. □

The previous lemma shows that for each $a\in A$, we have that $\pi_L(a)\in M(A\times\hat S)$ (by the definition of $A\times\hat S$, we see that $\pi_L(a)$ is a left multiplier, and the lemma shows that it is also a right multiplier). Denote by π the resulting $*$-homomorphism $A\to M(A\times\hat S)$. This is non-degenerate, as clearly $\pi(A)(A\times\hat S)$ is dense in $A\times\hat S$. Similar remarks apply to $A\times S$, leading to a non-degenerate $*$-homomorphism $\hat\pi : A\to A\times S$. Similarly, for $x\in\hat S$, the map $1\otimes x\in M(A\times\hat S)$, leading to a non-degenerate $*$-homomorphism $\hat\theta:\hat S\to M(A\times\hat S)$. We also obtain $\theta : S\to M(A\times S)$.

Denote by $\Psi_{L,\rho}$ and $\Psi_{R,\lambda}$ the representations of $A\times\hat S$ on $A\otimes H$ defined by

$$\Psi_{L,\rho}\big(\pi(a)\hat\theta(x)\big) = \pi_L(a)(1\otimes x),\quad \Psi_{R,\lambda}\big(\pi(a)\hat\theta(x)\big) = \pi_R(a)(1\otimes UxU)\qquad (a\in A, x\in\hat S).$$

[Again, chasing the definitions shows that $\Psi_{L,\rho}$ is just the identity representation.] Similarly define representations $\hat\Psi_{\lambda,L}$ and $\hat\Psi_{\rho,R}$ of $A\times S$ on $A\otimes H$ by

$$\hat\Psi_{\lambda,L}\big(\hat\pi(a)\theta(y)\big) = \hat\pi_\lambda(a)(1\otimes y),\quad \hat\Psi_{\rho,R}\big(\hat\pi(a)\theta(y)\big) = \hat\pi_\rho(a)(1\otimes UyU)\qquad (a\in A, y\in S).$$

Definition 8.3. *(Définition 7.3) Let δ_A be a coaction of S (respectively, $\hat S$) on A. The* dual coaction *of $\hat S$ (respectively, S) on $A\times\hat S$ (respectively $A\times S$) by*

$$\delta_{A\times\hat S} : A\times\hat S\to M(A\times\hat S\otimes\hat S);\quad \pi(a)\hat\theta(x)\mapsto(\pi(a)\otimes 1)(\hat\theta\otimes\iota)\hat\delta(x)\qquad (a\in A, x\in\hat S).$$

$$\delta_{A\times S} : A\times S\to M(A\times S\otimes S);\quad \hat\pi(a)\theta(x)\mapsto(\hat\pi(a)\otimes 1)(\theta\otimes\iota)\delta(x)\qquad (a\in A, x\in S).$$

Notice that for $y=\hat\theta(x)=1\otimes x$, we have that

$$\tilde V_{23}(y\otimes 1)\tilde V_{23}^* = 1\otimes\tilde V(x\otimes 1)\tilde V^* = 1\otimes\hat\delta(x),$$

thanks to (the proof of) Proposition lem:3 7.8. For $y=\pi(a)=\delta(a)=V^*(a\otimes 1)V$, we have that

$$\tilde V_{23}(y\otimes 1)\tilde V_{23}^* = \tilde V_{23}V_{12}^*(a\otimes 1\otimes 1)V_{12}\tilde V_{23}^* = V_{12}^*\tilde V_{23}(a\otimes 1\otimes 1)\tilde V_{23}^*V_{12} = \delta(a)\otimes 1,$$

where here we used Lemma lem:2 lem:2.2 7.5(2). As such elements y generate $A\times\hat S$, it follows that $\delta_{A\times\hat S}(\cdot) = \tilde V_{23}(\cdot\otimes 1)\tilde V_{23}^*$, and so $\delta_{A\times\hat S}$ is well-defined and a $*$-homomorphism. Similar remarks apply to $\delta_{A\times S}$.

A Useful results

The following is an assortment of results which are used implicitly by Baaj and Skandalis. We prove (sketch) proofs to aid the reader.

lem:ap1 **Lemma A.1.** *Let A be a C^*-algebra. Then $A^* = \{a\mu : a \in A, \mu \in A^*\} = \{\mu a : a \in A, \mu \in A^*\}$. Let A act faithfully on a Hilbert space H. Then $\mathcal{B}(H)_* = \{a\omega : a \in A, \omega \in \mathcal{B}(H)_*\} = \{\omega a : a \in A, \omega \in \mathcal{B}(H)_*\}$.*

Proof. We firstly claim that $\{a\mu : a \in A, \mu \in A^*\}$ is linearly dense in A^*– this follows by a GNS argument, see mnw [mnw, Appendix A]. Then the Cohen Factorisation Theorem shows that actually $A^* = \{a\mu : a \in A, \mu \in A^*\} = \{\mu a : a \in A, \mu \in A^*\}$. Indeed, given $\lambda \in A^*$ and $\epsilon > 0$, we can find $a \in A$ with $\|a\| \leq 1$ and $\mu \in A^*$ with $a\mu = \lambda$ and $\|\mu - \lambda\| < \epsilon$.

That A acts non-degenerately on H means, again using the Cohen Factorisation Theorem, that $H = \{a(\xi) : a \in A, \xi \in H\}$. It follows that $\{a\omega : a \in A, \omega \in \mathcal{B}(H)_*\}$ is linearly dense in $\mathcal{B}(H)_*$, so the result again follows by Cohen Factorisation. □

References

r1 [1]

r2 [2]

r6 [3]

r7 [4]

r11 [5]

r13 [6]

r17 [7]

r23 [8]

r30 [9]

r33 [10]

r38 [11]

r50 [12] Ref 50

r54 [13] Ref 54

The following are extra bibliographic entries not in the original paper.

lance [lan] Lance's Hilbert C*-module book.

mnw [mnw] Masuda et al. "C^*-algebraic framework for Quantum Groups".

woro [wor] Woronowicz's Compact Quantum Groups paper.