Purely infinite algebras and ultrapowers

Matthew Daws

UCLan

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The plan

- I want to think about general Banach Algebras, and compare and contrast to C^*-algebras
- Particularly interested in the classification of idempotents/projections; and
- the Ultrapower construction.
- I'm going to assume some/much of this is new to at least some of the audience.

Ultrafilters: motivation

From elementary Analysis we know that compactness is equivalent to every sequence having a convergent subsequence (in a metric space; more generally work with nets).

But for example,

$$(1, 2, 1, 2, 1, 2, 1, 2, \cdots)$$

has subsequences which converge to 1 or to 2. The sequence

$$(3, 4, 3, 3, 4, 3, 3, 3, 4, 3, 3, 3, 3, 4, \cdots)$$

has subsequences which converge to 3 or to 4. How to choose?

If we now (pointwise addition) add the sequences, we get

$$(4, 6, 4, 5, 5, 5, 4, 5, \cdots)$$

We want to pick the subsequence which gives the limit which is the sum of the limits we chose before.

How can we *consistently* choose?

A bit of set theory

A *filter* $\mathcal{F}$ on a set I is a non-empty collection of subsets of I with:

1. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;
2. If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.
3. $\emptyset \not\in \mathcal{F}$ (this ensures $\mathcal{F} \neq 2^I$).

Interpretations:

- Subsets of $\mathcal{F}$ are "big";
- We are allowed to "choose" sets in $\mathcal{F}$.

Convergence

Example

The *Fréchet Filter* is the collection of all cofinite subsets of I; that is $A \in \mathcal{F}$ if and only if $I \setminus A$ is finite.

Let $\mathcal{F}$ be the Fréchet Filter on $\mathbb{N}$. Consider the condition on a (scalar) sequence (a_n) that

$$\forall\, \epsilon > 0, \quad \{n : |a_n| < \epsilon\} \in \mathcal{F}.$$

This is clearly equivalent to $\lim_{n\to\infty} a_n = 0$.

Definition

A sequence (a_n) *converges along* $\mathcal{F}$ to a if

$$\forall\, \epsilon > 0, \quad \{n : |a_n - a| < \epsilon\} \in \mathcal{F}.$$

Ultrafilters

The collection of filters on a set I is partially ordered by inclusions. Zorn's Lemma ensures that there are maximal filters, which are called *ultrafilters*.

Lemma

A filter $\mathcal{U}$ on I is an ultrafilter if and only if for each $A \subseteq I$ either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

- For example, for $i_0 \in I$ the *principle ultrafilter at i_0* is $\{A \subseteq I : i_0 \in A\}$.
- Use Zorn's Lemma to find a maximal filter which contains the Fréchet Filter. This ultrafilter is not principle.

Convergence and Ultrafilters

Fix an ultrafilter $\mathcal{U}$. If $(a_i)_{i \in I}$ is a bounded sequence in $\mathbb{R}$ then a compactness argument shows that (a_i) does converge along $\mathcal{U}$.

- This provides a "consistent choice".
- For example, given two bounded sequences (a_i) and (b_i),

$$\lim_{i \to \mathcal{U}} (a_i + b_i) = \lim_{i \to \mathcal{U}} a_i + \lim_{i \to \mathcal{U}} b_i.$$

How might we deal with sequences in a *Banach space* where (in infinite dimensions) we don't have compactness. The (slightly vague) idea is to "enlarge" the space we work with.

Ultrapowers

Given a Banach space E let $\ell^\infty(E)$ be the space of bounded sequences in E with pointwise operations, and the sup norm.
For any filter $\mathcal{F}$ define

$$N(\mathcal{F}) = \Big\{(x_n) \in \ell^\infty(E) : \lim_{n\to\mathcal{F}} \|x_n\| = 0\Big\}.$$

Recall that $\lim_{n\to\mathcal{F}} x_n = 0$ means

$$\forall\, \epsilon > 0, \quad \{n : \|x_n\| < \epsilon\} \in \mathcal{F}.$$

- Easy to see that $N(\mathcal{F})$ is a subspace.
- Also $N(\mathcal{F})$ is closed (using uniform convergence in $\ell^\infty(E)$).

So we may define the quotient space

$$(E)_{\mathcal{F}} = \ell^\infty(E)/N(\mathcal{F}).$$

Ultrapowers

Definition

Let $\mathcal{U}$ be a non-principle ultrafilter (on $\mathbb{N}$). The *ultrapower* of a Banach space E is

$$(E)_{\mathcal{U}} = \ell^\infty(E)/N(\mathcal{U}).$$

Equivalently, we define a semi-norm on $\ell^\infty(E)$ by

$$\|(x_n)\| = \lim_{n\to\mathcal{U}} \|x_n\|.$$

- Then $(E)_{\mathcal{U}}$ is simply $\ell^\infty(E)$ quotiented by the null space of this semi-norm.
- So we tend to confuse elements of $(E)_{\mathcal{U}}$ with elements of $\ell^\infty(E)$.
- We always have a map $E \to (E)_{\mathcal{U}}; x \mapsto (x)$ which is an isometry.
- This is surjective exactly when E is finite-dimensional.

Ultrapowers of Hilbert spaces

Consider defining a sesquilinear form on $(H)_\mathcal{U}$ by

$$\big((a_n)\big|(b_n)\big) = \lim_{n\to\mathcal{U}} (a_n|b_n).$$

- This is well-defined as if $(a_n) = 0$ in the quotient $(H)_\mathcal{U}$ then $\lim_{n\to\mathcal{U}} \|a_n\| = 0$ and so $\lim_{n\to\mathcal{U}}(a_n|b_n) = 0$ for any (b_n), using the Cauchy-Schwarz inequality.
- Clearly sesquilinear.
- Recover the existing norm on $(H)_\mathcal{U}$.
- So $(H)_\mathcal{U}$ is a Hilbert space.

This wouldn't work with other filters.

- If we use the Fréchet Filter, then we quotient by the sequences which (in the usual sense) tend to 0.
- So $(H)_\mathcal{F} = \ell^\infty(H)/c_0(H)$, which is not a Hilbert Space.

Algebras

Let A be a Banach algebra, and consider $(A)_{\mathcal{U}}$.

- Define a product on $(A)_{\mathcal{U}}$ by
$$(a_n) \cdot (b_n) = (a_n b_n).$$
This is of course well-defined.
- If A is a C^*-algebra then there is an involution on $(A)_{\mathcal{U}}$ given by
$$(a_n)^* = (a_n^*).$$
This satisfies the C^*-condition.
- If A is represented on a Hilbert space H, then $(A)_{\mathcal{U}}$ is represented on $(H)_{\mathcal{U}}$.

Unital algebras

This is joint work with Bence Horváth. Fix a Banach algebra A.

Question

When is $(A)_{\mathcal{U}}$ unital?

- If A is unital, under the diagonal embedding $A \to (A)_{\mathcal{U}}$, the unit becomes a unit for $(A)_{\mathcal{U}}$.
- Conversely, let $e \in (A)_{\mathcal{U}}$ be a unit. This has a representative $(e_n) \in \ell^\infty(A)$, which satisfies

$$\lim_{n\to\mathcal{U}} \|e_n a_n - a_n\| = 0, \quad \lim_{n\to\mathcal{U}} \|a_n e_n - a_n\| = 0 \qquad ((a_n) \in \ell^\infty(A)).$$

  Let's *pretend* this was original convergence.
- By picking (a_n) suitably, this shows that, for example,

$$\lim_n \sup\{\|e_n a - a\| : a \in A, \|a\| \leq 1\} = 0.$$

Unital algebras cont.

$$\lim_n \sup\{\|e_n a - a\|, \|ae_n - a\| : a \in A, \|a\| \leq 1\} = 0.$$

- Extract a subsequence (e_n) with $\|e_n a - a\|, \|ae_n - a\| \leq \frac{1}{n}\|a\|$ for $a \in A$.
- We also know that e.g. $\|e_n\| \leq K$ say.
- Thus $\|e_n - e_m\| \leq \|e_n - e_n e_m\| + \|e_n e_m - e_m\| \leq K(\frac{1}{m} + \frac{1}{n})$.
- So (e_n) is Cauchy in A, so converges in A, say to e. Clearly e is a unit.

The proper argument, with ultrafilters, is similar, just with more bookkeeping.

Metric model theory

[Health warning: I am not a model theorist!]

There is a way to study “metric objects”, like Banach algebras, from the perspective of model theory:

- The “language” takes account of uniform continuity;
- We replace binary-valued true/false with, say, values in the interval $[0, 1]$;
- We use sup and inf in place of $\exists, \forall$.

There is a notion of *ultrapower* here, which agrees with our definition. There is a version of Łoś’s Theorem which tells us that “formulae” hold in the structue if and only if they hold in an ultrapower.

- The tricky thing is how to “axiomatise” the properties we are interested in.

Axiomatising unital algebras

Proposition

A Banach algebra A is unital if and only if

$$\inf_{e\in B_1} \sup_{a\in B_1} \max(\|ea-a\|, \|ae-a\|) = 0,$$

where B_1 is the unit ball of A.

Proof.

As before, extract a Cauchy sequence (e_n). □

We can then apply Łoś's Theorem to this. Moral is that the hard work is in using the "language" to "axiomatise" the properties we are interested in.

Idempotents and equivalence

Let A be a (Banach) algebra.

Definition

$p \in A$ is an *idempotent* if $p^2 = p$.
Two idempotents p, q are *equivalent*, written $p \sim q$, if there are $a, b \in A$ with $p = ab$ and $q = ba$.

[If $q \sim r$, say $q = cd, r = dc$, then $p = p^2 = abab = aqb = (ac)(db)$ and $(db)(ac) = dqc = dcdc = r^2 = r$ so $p \sim r$.]

For example, with $A = \mathbb{M}_n \cong \mathcal{B}(\mathbb{C}^n)$:

- idempotents correspond to direct sums $\mathbb{C}^n = V \oplus W = \operatorname{Im}(p) \oplus \ker(p)$;
- equivalence looks at the *dimension* of V.

Finiteness

Definition

Let A be a unital algebra. A is *Dedekind finite* if $p \sim 1$ implies $p = 1$.

- So $\mathbb{M}_n$ is Dedekind finite, via dimension.
- A Banach algebra like $\mathcal{B}(\ell^p)$ is not, as there are proper subspaces of ℓ^p isomorphic to ℓ^p.

For C^*-algebras

For C^*-algebras:

- We typically only consider self-adjoint idempotents $p = p^* = p^2$, called *projections*.
- The equivalence we typically use is *Murray–von Neumann equivalence*, which is that $p = u^*u$ and $q = uu^*$. This implies that u is a partial isometry. We write $p \approx q$.

These are actually the same concepts as we have defined.

- For any idempotent p there is a projection q with $p \sim q$. In fact, we can choose q with $pq = q$ and $qp = p$.
- If p, q are projections with $p \sim q$ then also $p \approx q$.
- Suppose A is a Dedekind-finite C^*-algebra. If $p^2 = p \sim 1$ then there is a projection q with $q \sim p$, so also $q \sim 1$ so $q \approx 1$ so $q = 1$. Then $1 = q = pq = p$, so A is Dedekind-finite in our sense.

Properly infinite

Definition

A is *properly infinite* if there are idempotents $p \sim 1$ and $q \sim 1$ which are orthogonal: $pq = qp = 0$.

- Again, $\mathcal{B}(\ell^p)$ is properly infinite.
- Again, in a C^*-algebra, if we work only with projections, we get equivalent definitions.

Theorem

Let A be a simple unital C^*-algebra. The following are equivalent:

1. *A is infinite (that is, not (Dedekind) finite);*
2. *A is properly infinite;*
3. *A contains a left invertible element which is not invertible.*

Purely infinite

Definition

A is *purely infinite* if $A \not\cong \mathbb{C}$ and for $a \neq 0$ there are $b, c \in A$ with $bac = 1$.

Theorem (Ara, Goodearl, Pardo)

Let A be a simple algebra. TFAE:

- *A is purely infinite;*
- *every non-zero right ideal of A contains an infinite idempotent.*

Theorem

If A is a C^*-algebra, equivalently:

- *every non-zero hereditary C^*-subalgebra contains an infinite projection.*

To ultrapowers

Motivated by Łoś's Theorem, and previous work, we seek "norm control".

Definition

For a unital Banach algebra A, for $a \neq 0$, define

$$C_{pi}(a) = \inf \big\{ \|b\| \|c\| : bac = 1 \big\}.$$

- Thus A is purely infinite if $C_{pi}(a) < \infty$ for each $a \neq 0$.
- We always have

$$\frac{1}{\|a\|} \leq C_{pi}(a), \qquad C_{pi}(za) = |z|^{-1} C_{pi}(a) \qquad (a \neq 0, z \in \mathbb{C}).$$

For ultrapowers

Theorem

For a unital Banach algebra, the following are equivalent:

1. $(A)_{\mathcal{U}}$ *is purely infinite;*
2. $\sup\{C_{pi}(a) : \|a\| = 1\} < \infty$.

Sketch.

(1)⇒(2). If not, then there is sequence (a_n) in the unit sphere of A with $C_{pi}(a_n) \to \infty$. With $a = (a_n) \in (A)_{\mathcal{U}}$ there are $b, c \in (A)_{\mathcal{U}}$ with $bac = 1$. Picking representatives $b = (b_n)$, $c = (c_n)$ we find

$$\lim_{n \to \mathcal{U}} \|b_n a_n c_n - 1\| = 0.$$

So eventually $b_n a_n c_n$ is invertible, with norm control, from which it follows that $C_{pi}(a_n)$ can be controlled by $\|b\|\|c\|$, contradiction. □

For ultrapowers, cont.

Theorem

For a unital Banach algebra, the following are equivalent:

1. $(A)_{\mathcal{U}}$ *is purely infinite;*
2. $\sup\{C_{pi}(a) : \|a\| = 1\} < \infty$.

Corollary

If $(A)_{\mathcal{U}}$ *is purely infinite, then so is* A.

What about the converse?

Examples

Result

If A is a simple unital purely infinite C^*-algebra, then $C_{pi}(a) = 1$ for each $\|a\| = 1$.

For a Banach space E, let $\mathcal{B}(E)$ and $\mathcal{K}(E)$ be the algebras of bounded, respectively, compact operators. Sometimes, $\mathcal{K}(E)$ is the unique closed, two-sided ideal in $\mathcal{B}(E)$, so that $\mathcal{B}(E)/\mathcal{K}(E)$ is simple.

Theorem

For $E = c_0$ or ℓ^p, the algebra $\mathcal{B}(E)/\mathcal{K}(E)$ has purely infinite ultrapowers.

Proof.

A result of Ware gives exactly that $C_{pi}(T + \mathcal{K}(E)) = 1/\|T + \mathcal{K}(E)\|$ for each non-compact $T \in \mathcal{B}(E)$. □

Towards a counter-example

We seek a Banach algebra which *is* purely infinite, but with no good control of $C_{pi}(\cdot)$. This is hard, because being purely infinite is a "global" property.

Proposition

Let A, B be unital Banach algebras. Let A have purely infinite ultrapowers. When $\theta : A \to B$ is a homomorphism, θ is automatically bounded below.

Proof.

If $\|a\| = 1$ and $\|\theta(a)\| < \delta$ then there are $b, c \in A$ with $\|b\|\|c\| < 2C_{pi}(a)$ and $bac = 1$ so $\theta(b)\theta(a)\theta(c) = 1$ so

$$1 \leq \|\theta(b)\|\|\theta(c)\|\|\theta(a)\| < \|\theta\|^2 2C_{pi}(a)\delta,$$

which puts a lower-bound on δ. $\square$

The Cuntz monoid

(Or "Cuntz semigroup", but that has multiple meanings.)

$$Cu_2 = \langle s_1, s_2, t_1, t_2 : t_1 s_1 = t_2 s_2 = 1, t_1 s_2 = t_2 s_1 = \lozenge \rangle$$

where $\lozenge$ is a "semigroup zero", meaning $s\lozenge = \lozenge s = \lozenge$ for all s.
So Cu_2 is all words in these generators, subject to the relations. For example:

$$s_1 s_2 t_2 s_1 t_2 = s_1 s_2 \lozenge t_2 = \lozenge, \quad s_1 s_2 t_2 s_2 t_2 = s_1 s_2 t_2.$$

In fact, any word reduces to either $\lozenge$ or a word starting in s_1, s_2 and ending in t_1, t_2.

ℓ^1 algebras

We form the usual ℓ^1 algebra of this monoid:

- $\ell^1(Cu_2)$ is all sequences indexed by Cu_2 with finite ℓ^1-norm:

$$\|(a_s)_{s\in Cu_2}\| = \sum_{s\in Cu_2} |a_s|.$$

- Write elements as sums of "point-mass measures" δ_s:

$$(a_s) = \sum_{s\in Cu_2} a_s\delta_s.$$

- Use the convolution product: $\delta_s\delta_t = \delta_{st}$.

Notice that $\mathbb{C}\delta_\Diamond$ is a two-sided ideal. So we can quotient by it:

$$\mathcal{A} := \ell^1(Cu_2)/\mathbb{C}\delta_\Diamond.$$

This is equivalent to identify $\delta_\Diamond$ with the algebra 0, so e.g.

$$\delta_{t_1}\delta_{s_1} = 1, \qquad \delta_{t_1}\delta_{s_2} = 0.$$

Comparison with the Cuntz algebra $\mathcal{O}_2$

$\mathcal{O}_2$ is generated by isometries s_1, s_2 (so $s_1^* s_1 = s_2^* s_2 = 1$) with relation

$$s_1 s_1^* + s_2 s_2^* = 1.$$

This implies that s_1 and s_2 have orthogonal ranges, so $s_1^* s_2 = s_2^* s_1 = 0$.

Let $\mathcal{J} \subseteq \mathcal{A}$ be the closed ideal generated by

$$1 - \delta_{s_1 t_1} - \delta_{s_2 t_2}.$$

- So in the quotient algebra $\mathcal{A}/\mathcal{J}$ we do have that $\delta_{s_1 t_1} + \delta_{s_2 t_2} = 1$.

Theorem

The algebra $\mathcal{A}/\mathcal{J}$ is simple.

Towards a proof

Consider the Banach space ℓ^1, with standard unit vector basis $(e_n)_{n\geq 1}$. Define isometries

$$S_1 : e_n \mapsto e_{2n}, \qquad S_2 : e_n \mapsto e_{2n-1}.$$

and define surjections

$$T_1 : e_n \mapsto \begin{cases} e_{n/2} & : n \text{ even}, \\ 0 & : n \text{ odd}, \end{cases} \qquad T_2 : e_n \mapsto \begin{cases} 0 & : n \text{ even}, \\ e_{(n+1)/2} & : n \text{ odd}. \end{cases}$$

Then

$$T_1 S_1 = 1, \qquad T_2 S_2 = 1, \qquad T_1 S_2 = 0, \qquad T_2 S_1 = 0,$$

and

$$S_1 T_1 + S_2 T_2 = 1.$$

We have a representation

So we obtain a representation $\mathcal{A} \to \mathcal{B}(\ell^1)$ which annihilates $\mathcal{J}$, and so drops to a representation of $\mathcal{A}/\mathcal{J}$.

Proposition

The representation $\Theta : \mathcal{A}/\mathcal{J} \to \mathcal{B}(\ell^1)$ is not bounded below.

Proof.

Let $T = T_1 + T_2$ so for $(\xi_n) \in \ell^1$,

$$T(\xi_n) = (\xi_1 + \xi_2, \xi_3 + \xi_4, \xi_5 + \xi_6, \cdots).$$

Hence $\|T\| = 1$. Consider

$$a = (\delta_{t_1} + \delta_{t_2})^N = \sum \{\delta_s : s \text{ is a word in } t_1, t_2 \text{ of length } N\}$$

So $\|a\| = 2^N$ and one can show that $\|a + \mathcal{J}\| = 2^N$ as well. Notice that $\Theta(a + \mathcal{J}) = T^N$, so $\|\Theta(a + \mathcal{J})\| \leq 1$. □

Purely infinite

Theorem

$\mathcal{A}/\mathcal{J}$ is purely infinite.

The proof is a careful but direct construction: given $a \in \mathcal{A}$ with $a \notin \mathcal{J}$, we find $b, c \in \mathcal{A}$ with $bac = 1$.

- Of use is identifying $\mathcal{J}^\perp$ in $\mathcal{A}^* \cong \ell^\infty(Cu_2 \setminus \{\Diamond\})$ and playing Hahn-Banach games.
- Consider $a = 1 - \delta_{s_1 t_1} - \delta_{s_2 t_2} \in \mathcal{J}$. Then

  $$\delta_{t_1} a = \delta_{t_1} - \delta_{t_1 s_1 t_1} - \delta_{t_1 s_2 t_2} = 0,$$

  similarly $\delta_{t_2} a = 0$ and $a\delta_{s_1} = a\delta_{s_2} = 0$.
- So we can only left-multiply by s_1, s_2 and right multiply by t_1, t_2, but then no cancellation can occur. So we can never get $bac = 1$.

Corollaries

Corollary

$\mathcal{A}/\mathcal{J}$ is simple.

Corollary

$\mathcal{A}/\mathcal{J}$ does not have purely infinite ultrapowers.

Proof.

It is purely infinite, but we found a non-bounded below homomorphism. □

Interesting (to me) that the example is rather "natural". We didn't "build in" to the algebra some "bad norm control".

References

- Our papers: arXiv:1912.07108 [math.FA] and arXiv:2104.14989 [math.FA]
- Ilijas Farah, Bradd Hart, David Sherman, a series of papers "Model theory of Operator algebras".

Furthermore, Phillips studied (amongst other things) the *closure* of $\Theta(\mathcal{A}/\mathcal{J})$ in $\mathcal{B}(\ell^1)$, showing that this is also purely infinite, see arXiv:1201.4196 [math.FA] and arXiv:1309.0115 [math.FA]

- If $A \to B$ is a homomorphism with dense range, there seems to be no relationship between A being purely infinite, and B being purely infinite.