Kaplansky Density for automorphism groups

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UCLan

Banach Algebras 2019

Outline

Operator algebras

A C^*-*algebra* is either:

- A norm closed, self-adjoint, subalgebra A of $\mathcal{B}(H)$ (algebra of bounded operators on a Hilbert space).
- A Banach algebra A with an involution $*$ with $\|a^*a\| = \|a\|^2$ for $a \in A$.

A *von Neumann algebra* is either:

- A SOT closed, self-adjoint, subalgebra M of $\mathcal{B}(H)$.
  So if (x_i) a net in M, and $x \in \mathcal{B}(H)$, with $\|x_i(\xi) - x(\xi)\| \to 0$ for $\xi \in H$, then $x \in M$.
- A C^*-algebra M which is isometrically isomorphic to the dual of some Banach space M_*.

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Trace class operators and preduals

Let $\mathcal{T}(H)$ be the space of trace-class operators on H: those $x \in \mathcal{B}(H)$ for which $|x|$ has finite trace, $\mathrm{tr}(|x|) < \infty$.

There is a dual pairing between $\mathcal{T}(H)$ and $\mathcal{B}(H)$:

$$\langle x, y \rangle = \mathrm{tr}(xy) \qquad (x \in \mathcal{B}(H), y \in \mathcal{T}(H)).$$

- Under this, $\mathcal{B}(H)$ is the dual space of $\mathcal{T}(H)$.
- We often write $\mathcal{B}(H)_*$ for $\mathcal{T}(H)$ as $\mathcal{T}(H)$ is the *predual* of $\mathcal{B}(H)$.

Given a von Neumann algebra $M \subseteq \mathcal{B}(H)$, that M is SOT closed means that M is closed in $\mathcal{B}(H)$ for the weak*-topology induced by $\mathcal{B}(H)_*$.

- Equivalently (Hahn-Banach) the quotient $M_* = \mathcal{B}(H)_* / {}^\perp M$ is the predual of M:

$$\big(\mathcal{B}(H)_* / {}^\perp M\big)^* = ({}^\perp M)^\perp = M.$$

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Kaplansky Density

Theorem (Kaplansky)

Let M be a von Neumann algebra, and $A \subseteq M$ be a C^*-algebra which is weak*-dense in M. Then the unit ball of A is weak*-dense in the unit ball of M.

There exist weak*-closed subalgebra $M \subseteq \mathcal{B}(H)$ and a norm-closed subalgebra $A \subseteq M$ such that:

- A is weak*-dense in M;
- the unit ball of A is not weak*-dense in the unit ball of M.
- Dowson found an example with A and M commutative, with M self-adjoint, and such that $\{a \in A : \|a\| \leq r\}$ is not weak*-dense in the unit ball of M for any r.

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Automorphism groups

Definition

Let E be a Banach space. A one-parameter group of isometries of E is a family $(\alpha_t)_{t\in\mathbb{R}}$ with:

- Each α_t is a contraction in $\mathcal{B}(E)$;
- $\alpha_0 = 1$;
- $\alpha_{t+s} = \alpha_t \circ \alpha_s$ for $s, t \in \mathbb{R}$.

Then $\alpha_{-t} \circ \alpha_t = \alpha_t \circ \alpha_{-t} = \alpha_0 = 1$ so each α_t is a bijective isometry.

Say that (α_t) is *strongly-continuous* or a *C_0-group* if

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Examples

Let $E = H$ a Hilbert space, so that each α_t is a unitary on H.

Theorem (Stone)

There is an (unbounded) self-adjoint operator T with $\alpha_t = \exp(iTt)$ for $t \in \mathbb{R}$.

Let $T \in \mathbb{M}_n$ be self-adjoint, so $u_t = \exp(iTt)$ forms a 1-parameter unitary group on $\mathbb{C}^n$. For $x \in \mathbb{M}_n$ define

$$\alpha_t(x) = u_t x u_{-t} = e^{iTt} x e^{-iTt} \qquad (x \in \mathbb{M}_n).$$

- Each α_t is an isometry for the operator norm.
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Examples cont.

Consider $C_0(\mathbb{R})$, the C^*-algebra of continuous functions $f : \mathbb{R} \to \mathbb{C}$ with $\lim_{|t|\to\infty} f(t) = 0$.

- Define $\alpha_t(f)$ to be the function $s \mapsto f(s - t)$.
- Then (α_t) is a 1-parameter group of $*$-automorphisms of $C_0(\mathbb{R})$.

Let $L^\infty(\mathbb{R})$ be the von Neumann algebra of (equivalence classes) of (essentially) bounded measurable functions $f : \mathbb{R} \to \mathbb{C}$.

- Define $\alpha_t(f)$ to be the function $s \mapsto f(s - t)$.
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Notice that $C_0(\mathbb{R})$ is weak*-dense in $L^\infty(\mathbb{R})$, and that the automorphism groups are compatible with this inclusion.

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Holomorphic functions

Let E be a Banach space, $D \subseteq \mathbb{C}$ a domain, and $f : D \to E$ a function. The following are equivalent:

- f is *analytic* in the sense that for each $\alpha \in D$ there is an absolutely convergence power series for f, near α:

$$f(z) = \sum_{n \geq 0} a_n (z - \alpha)^n \qquad |z - \alpha| < r.$$

- f is holomorphic, in the sense that there is $F \subseteq E^*$ norming, with $D \to \mathbb{C}; z \mapsto \phi(f(z))$ is differentiable, for each $\phi \in F$.

Here *norming* means that

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Regular functions

Given $\alpha \in \mathbb{C}$ let

$$S(\alpha) = \left\{ z \in \mathbb{C} : \begin{matrix} 0 \leq \operatorname{Im}(z) \leq \operatorname{Im}(\alpha) & \text{if } \operatorname{Im}(\alpha) \geq 0 \\ 0 \geq \operatorname{Im}(z) \geq \operatorname{Im}(\alpha) & \text{if } \operatorname{Im}(\alpha) \leq 0 \end{matrix} \right\}.$$

That is, the closed horizontal strip bounded by $\mathbb{R}$ and $\mathbb{R} + \alpha$.

A function $f : S(\alpha) \to E$ is *regular* if f is continuous, analytic in the interior of $S(\alpha)$, and bounded on $\mathbb{R}$ and $\mathbb{R} + \alpha$:

$$M := \sup_{t \in \mathbb{R}} \max \left(\|f(t)\|, \|f(\alpha + t)\| \right) < \infty.$$

The 3-Lines Theorem shows that then $\|f(z)\| \leq M$ for all $z \in S(\alpha)$.
Some link with complex interpolation?

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Analytic generators

Given (α_t), a 1-parameter group on E, and $z \in \mathbb{C}$, define an operator $D(\alpha_z) \to E$ by

$$x \in D(\alpha_z) \text{ when there is } f : S(z) \to E \text{ regular with } f(t) = \alpha_t(x) \ (t \in \mathbb{R}).$$

Then we set $\alpha_z(x) = f(z)$.

- Morera's Theorem and the Reflection Principle imply that such an f is unique. So α_z is well-defined.
- Think of α_z as an "analytic extension" of the mapping $t \mapsto \alpha_t(x)$.
- Can show that $D(\alpha_z)$ is dense in E and that α_z is *closed*.
- Then α_{-i} is the *analytic generator* of (α_t).

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Examples

When (α_t) is a continuous unitary group on a Hilbert space H, with $\alpha_t = \exp(iTt)$, then

$$\alpha_{-i} = \exp(T).$$

Define $\exp(T)$ by functional calculus. The equality means with equality of domains. (Of course formally obvious; but the LHS and RHS have different definitions.)

If (α_t) on $\mathbb{M}_n$ is

$$\alpha_t(x) = u_t x u_{-t} = e^{iTt} x e^{-iTt},$$

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Some properties

α_z is *closed* in the sense that the *graph*

$$\mathcal{G}(\alpha_z) = \{(x, \alpha_z(x)) : x \in D(\alpha_z)\} \subseteq E \oplus E$$

is closed.

Recall how to compose two unbounded operators $T : D(T) \to E, S : D(S) \to E$:

$$D(ST) = \{x \in D(T) : T(x) \in D(S)\}; \quad ST : D(ST) \ni x \mapsto S(T(x)).$$

Then $S = T$ means $\mathcal{G}(S) = \mathcal{G}(T)$; and $S \subseteq T$ means $\mathcal{G}(S) \subseteq \mathcal{G}(T)$. As closed operators, we have that

- $\alpha_t \circ \alpha_z = \alpha_z \circ \alpha_t = \alpha_{t+z}$
- If z, w lie on the same side of the real axis, then $\alpha_z \alpha_w = \alpha_{z+w}$
- In general, $\alpha_z \alpha_w \subseteq \alpha_{z+w}$.

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Then $S = T$ means $\mathcal{G}(S) = \mathcal{G}(T)$; and $S \subseteq T$ means $\mathcal{G}(S) \subseteq \mathcal{G}(T)$.

As closed operators, we have that

- $\alpha_t \circ \alpha_z = \alpha_z \circ \alpha_t = \alpha_{t+z}$
- If z, w lie on the same side of the real axis, then $\alpha_z \alpha_w = \alpha_{z+w}$
- In general, $\alpha_z \alpha_w \subseteq \alpha_{z+w}$.

Some properties

α_z is *closed* in the sense that the *graph*

$$\mathcal{G}(\alpha_z) = \{(x, \alpha_z(x)) : x \in D(\alpha_z)\} \subseteq E \oplus E$$

is closed.

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Examples, cont.

$$\alpha_t(f)(s) = f(s-t) \qquad (s,t \in \mathbb{R}, f \in C_0(\mathbb{R})).$$

- Let $f \in D(\alpha_{-i})$;
- Let $F : S(-i) \to C_0(\mathbb{R})$ be the associated regular function.
- Define $g : S(i) \to \mathbb{C}$ by $g(z) = F(-z)(0)$.
- Then $g(t) = F(-t)(0) = \alpha_{-t}(f)(0) = f(t)$.
- Also g is regular.
- Can reverse this: given regular $g : S(i) \to \mathbb{C}$ then define $F : S(-i) \to C_0(\mathbb{R})$ by $F(z)(t) = g(t-z)$, so that F becomes a $C_0(\mathbb{R})$-valued regular function.

So f itself analytically extends to $S(i)$, and $F(-i)$ is this extension of f, evaluated on $\mathbb{R} + i$.

(Somehow like a Hardy space. . .)

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Some properties: C^*-algebra case

Now suppose $E = A$ is a C*-algebra and each α_t is a $*$-automorphism.
Given $a, b \in D(\alpha_z)$ with associated regular functions

$$F_a, F_b : S(z) \to A$$

we can pointwise multiply to obtain

$$F : S(z) \to A; \quad w \mapsto F_a(w)F_b(w).$$

- F is regular (local power series expansion).
- $F(t) = F_a(t)F_b(t) = \alpha_t(a)\alpha_t(b) = \alpha_t(ab)$ for $t \in \mathbb{R}$.
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Given $a \in D(\alpha_{-i})$ with regular $F : S(-i) \to A$ define

$$F^\star : S(-i) \to A; \quad w \mapsto F(\overline{w} - i)^*.$$

That is, use the involution on A.

- $F^\star$ is regular (local power series expansion).
- $F^\star(t) = F(t - i)^* = (\alpha_t \alpha_{-i}(a))^* = \alpha_t(\alpha_{-i}(a)^*)$ for $t \in \mathbb{R}$.
- $F^\star(-i) = F(0)^* = a^*$

So $\alpha_{-i}(a)^* \in D(\alpha_{-i})$ and $\alpha_{-i}(\alpha_{-i}(a)^*) = a^*$.

Equivalently $(a, b) \in \mathcal{G}(\alpha_{-i}) \implies (b^*, a^*) \in \mathcal{G}(\alpha_{-i})$. So $\mathcal{G}(\alpha_{-i})$ is a $*$-algebra, but for this twisted involution.

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Outline

Locally compact quantum groups

The Operator algebraic approach to Quantum Groups uses C^* and von Neumann algebras to generalise the notion of a locally compact group, and Pontryagin duality.

- Write $\mathbb{G}$ for the "abstract quantum group" and $L^\infty(\mathbb{G})$ and $C_0(\mathbb{G})$ for the associated algebras.
- The correct notion of the "group inverse" here is the *antipode* S, which in interesting examples turns out to be unbounded.
- Can "polar decompose" $S = R\tau_{-i/2}$ where R is the *unitary antipode* (and anti-$*$-automorphism), and…
- (τ_t) is the *scaling group*, a 1-parameter group of $*$-automorphisms of $L^\infty(\mathbb{G})$.
- $S^2 = \tau_{-i}$.

Furthermore, S, R and (τ_t) all drop to $C_0(\mathbb{G})$ which is weak*-dense in $L^\infty(\mathbb{G})$.

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Von Neumann setting

Each α_t is normal, and for $x \in M$, the orbit map $R \to M; \quad t \mapsto \alpha_t(x)$ is weak*-continuous.

- Form α_z in the same way, but we only require a weak*-regular extension.
- (But weak*-holomorphic implies norm analytic. The extension to the boundary is only weak*-continuous).
- Then $\mathcal{G}(\alpha_z)$ is weak*-closed.
- Still $\mathcal{G}(\alpha_z)$ is an algebra, and $\mathcal{G}(\alpha_{-i})$ is a $*$-algebra. (Harder to prove, as the product is only *separately* continuous now.)

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Related ideas: spectral subspaces

Arveson introduced and studied the notion of a *spectral subspace*.

- For a strongly continuous unitary group (u_t) on a Hilbert space, we have $u_t = e^{-itH}$ for some self-adjoint H.
- We can understand H using its spectral decomposition.
- Arveson's ideas generalise this away from Hilbert spaces.

An example: $H^\infty(\alpha)$ is those $a \in A$ such that $a \in D(\alpha_z)$ for any z in the upper-half plane, and $\limsup_n \|\alpha_{in}(a)\|^{1/n} \leq 1$.

- Equivalently, for each $\mu \in A^*$, the scalar-valued function $z \mapsto \langle \mu, \alpha_z(a) \rangle$ is in H^∞ of the upper-half-plane.
- This is a *closed* subspace of A.

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Related ideas: subdiagonal algebras

If you apply this to a von Neumann algebra M, then $H^\infty(\alpha)$ is (often) an example of a *maximal subdiagonal algebra*.

- Consider the "shift semigroup" on $L^\infty(\mathbb{R})$. Then you exactly obtain the classical Hardy space H^∞ of the upper-half-plane.
- Also generalises familiar non-self-adjoint operator algebras.
- For example, with $\alpha_t(x) = P^{it}xP^{-it}$ for $x \in \mathbb{M}_n$, if P is diagonal with increasing real entries, then $H^\infty(\alpha)$ is the upper-triangular matrices.

There is a rich theory of e.g. L^p-spaces for maximal subdiagonal algebras: e.g. $H^1(\alpha)$ can be factored in $H^2(\alpha)$ pairs.

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$$\alpha_{-i}^A \text{ a norm-closed, norm-densely defined operator on } A,$$
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Almost by definition, we have that α^M_{-i} *extends* α^A_{-i}, which means that

$$\mathcal{G}(\alpha^A_{-i}) \subseteq \mathcal{G}(\alpha^M_{-i}),$$

under the obvious inclusions $A \oplus A \subseteq M \oplus M$.

- In fact, $\mathcal{G}(\alpha^A_{-i}) = \mathcal{G}(\alpha^M_{-i}) \cap (A \oplus A)$.

One can show that actually

$$\mathcal{G}(\alpha^A_{-i}) \text{ is weak}^* \text{ dense in } \mathcal{G}(\alpha^M_{-i}).$$

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Kaplansky

Theorem

The unit ball of $\mathcal{G}(\alpha^A_{-i})$ is weak-dense in the unit ball of $\mathcal{G}(\alpha^M_{-i})$.*

To be concrete, this means that given $x \in D(\alpha^M_{-i})$ with

$$\|x\| \leq 1 \text{ and } \|\alpha^M_{-i}(x)\| \leq 1,$$

there is a net (a_j) in $D(\alpha^A_{-i})$ with $a_j \to x$ and $\alpha^A_{-i}(a_j) \to \alpha^M_{-i}(x)$ weak*, and with

$$\|a_j\| \leq 1 \text{ and } \|\alpha^A_{-i}(a_j)\| \leq 1.$$

Sketch of proof

The key idea is von Neumann algebraic:

- Using Kaplansky density for $A \subseteq M$ we see that A norms the predual M_*.
- Equivalently, the induced map $M_* \to A^*$ (given by restricting functions in M_* to $A \subseteq M$) is an isometry.
- The resulting subspace of A^* is an A-bimodule, and so there is a central projection $z \in A^{**}$ with $A^* z = M_*$.
- Thus $A^{**} z \cong M$.

We now consider $\mathcal{G}(\alpha^A_{-i})^{**} \subseteq A^{**} \oplus A^{**}$. One can carefully show that

$$\mathcal{G}(\alpha^M_{-i}) \cong \mathcal{G}(\alpha^A_{-i})^{**}(z \oplus z) \text{ and } \mathcal{G}(\alpha^M_{-i}) \subseteq \mathcal{G}(\alpha^A_{-i})^{**}.$$

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- So there are $(a^{**}, b^{**}) \in \mathcal{G}(\alpha^A_{-i})^{**}$ with $a^{**}z = a^{**}$, $b^{**}z = b^{**}$ and (a^{**}, b^{**}) corresponds to (x, y).
- By Hahn-Banach ("Goldstine theorem") there is a net (a_j, b_j) in $\mathcal{G}(\alpha^A_{-i})$ converging to (a^{**}, b^{**}), with norm control: $\|a_j\| \leq 1$ and $\|b_j\| \leq 1$.
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Dualise everything

Swap things about:

- The adjoints of (α_t^A) give rise to a weak*-continuous 1-parameter isometry group on A^*.
- The pre-adjoints of (α_t^M) give rise to a norm-continuous 1-parameter isometry group on M_*.

We have the isometric inclusion $M_* \to A^*$ which leads to

$$\mathcal{G}(\alpha_{-i}^{M_*}) \subseteq \mathcal{G}(\alpha_{-i}^{A^*}),$$

which is weak*-dense.

Theorem ("Automatic normality")

Let $\omega \in M_*$ be such that $\omega \in D(\alpha_{-i}^{A^*})$. Then $\omega \in D(\alpha_{-i}^{M_*})$, that is, $\alpha_{-i}^{A^*}(\omega) \in M_*$.

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Open questions

- Does an analogue of Kaplansky Density hold for $\mathcal{G}(\alpha_{-i}^{M_*}) \subseteq \mathcal{G}(\alpha_{-i}^{A^*})$?
- Under some "weakly complemented" conditions, this is true.
- This clarifies (slightly) a proof of Daws & Salmi that if $\mathbb{G}$ is *coamenable* then $L^1_\sharp(\mathbb{G}) \to M^\sharp(\mathbb{G})$ satisfies Kaplansky Density. (This is equivalent to working with the scaling group (τ_t)).
- Broad question: Study $\mathcal{G}(\alpha_{-i})$ as a Banach $*$-algebra.

Proposition (After Verding; Kustermans; Van Daele)

Let (α_t) be an automorphism group of a Banach algebra A. If A has a bounded (contractive) approximate identity then so does $\mathcal{G}(\alpha_{-i})$.