Types of L^1 Banach and operator algebras

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Asymptotic sequence algebras

Question

How might we convert "approximate" relations into exact relations?

Given a Banach algebra A, we can consider two sequence algebras:

$$\ell^\infty(A) = \{\text{all bounded sequences in } A \text{ with pointwise operations}\}$$

and the ideal $c_0(A)$.

The quotient $A_\infty = \ell^\infty(A)/c_0(A)$ is the "asymptotic sequence algebra".

The quotient norm is the same as

$$\|a\| = \limsup_{n\to\infty} \|a_n\| \qquad (a = (a_n) \in \ell^\infty(A)).$$

In particular, the "constant sequence map" $\iota\colon A \to A_\infty$ is an isometry.

Simple application

Proposition

Let A be separable. The following are equivalet:

1. *there is a non-zero $x \in A_\infty$ with $xa = ax$ for each $a \in A$;*
2. *for each finite $F \subseteq A$ and $\epsilon > 0$, there is $x \in A$ with $\|x\| = 1$ and $\|xa - ax\| < \epsilon$ for $a \in F$.*

Notice the "metric condition" in (2).

We can also form E_∞ for a Banach space E. If H is a Hilbert space, then H_∞ has no obvious inner-product; indeed, E_∞ always contains a copy of ℓ^∞ / c_0.

Ultrafilters

Recall that a *filter* $\mathcal{F}$ on a set I is a collection of subsets of I such that: $\emptyset \notin \mathcal{F}$; if $U \in \mathcal{F}$ and $U \subseteq V$, then $V \in \mathcal{F}$; if $U, V \in \mathcal{F}$ then $U \cap V \in \mathcal{F}$. Think: "subsets of $\mathcal{F}$ are large".

Filters are partially ordered by inclusion, and a maximal filter is an *ultrafilter*. Examples are the *principal ultrafilters*,

$$\mathcal{U}_i = \{U \subseteq I : i \in U\} \qquad (i \in I).$$

Zorn's lemma shows that non-principal ultrafilters exists: take a maximal refinement of the *Fréchet filter*

$$\mathfrak{F}_0 = \{U \subseteq I : |I \setminus U| < \infty\}.$$

Result

A filter $\mathcal{F}$ is an ultrafilter if and only if, for any $U \subseteq I$, either $U \in \mathcal{F}$ or $I \setminus U \in \mathcal{F}$.

Ultrafilters converge

Proposition

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, and (x_n) a bounded sequence in $\mathbb{C}$ (or $\mathbb{R}$). There is a unique $x \in \mathbb{C}$ with, for each $\epsilon > 0$,

$$\big\{ n : |x - x_n| < \epsilon \big\} \in \mathcal{U}.$$

Write $\lim_{n\to\mathcal{U}} x_n = x$.

We have the usual rules for limits:

$$\lim_{n\to\mathcal{U}} \lambda x_n = \lambda \lim_{n\to\mathcal{U}} x_n, \quad \lim_{n\to\mathcal{U}} x_n + y_n = \lim_{n\to\mathcal{U}} x_n + \lim_{n\to\mathcal{U}} y_n.$$

Think: an ultrafilter is a selection method, which chooses which subsequence we converge down.

Ultraproducts and powers

Given a family of Banach spaces $(E_i)_{i\in I}$ and a (non-principal) ultrafilter $\mathcal{U}$ on I, the collection

$$N_{\mathcal{U}} = \{(x_i) \in \ell^\infty(E_i) : \lim_{i\to\mathcal{U}} \|x_i\| = 0\}$$

is a closed subspace of $\ell^\infty(E_i)$.

- When (A_i) is a family of Banach algebras, this is a closed ideal.
- Denote $(E_i)_{\mathcal{U}}$ the quotient $\ell^\infty(E_i)/N_{\mathcal{U}}$, the *ultraproduct*. The norm satisfies

$$\|(x_i)\| = \|(x_i) + N_{\mathcal{U}}\| = \lim_{i\to\mathcal{U}} \|x_i\|.$$

- So $(A_i)_{\mathcal{U}}$ is a Banach algebra, and a C^*-algebra, if each A_i is.

When $E_i = E$ for each i, we write $(E)_{\mathcal{U}}$, the *ultrapower*.

Ultraproducts and powers cont.

This construction preserves lots of structure; for example, given a family of Hilbert spaces (H_i), we can define

$$\big((x_i)\big|(y_i)\big) = \lim_{i\to\mathcal{U}} (x_i|y_i),$$

and obtain an inner-product on $(H_i)_{\mathcal{U}}$ compatible with the norm.

- Ultraproducts of L^p spaces are again L^p spaces;
- Ultraproducts of $C(K)$ spaces are commutative C^*-algebras, so again $C(K)$ spaces.

Unital algebras

This is joint work with Bence Horváth. Fix a Banach algebra A.

Question

When is $(A)_{\mathcal{U}}$, or $\mathsf{Asy}(A)$, unital?

- If A is unital, under the diagonal embedding $A \to \mathsf{Asy}(A)$, the unit becomes a unit for $\mathsf{Asy}(A)$.
- Conversely, let $e \in \mathsf{Asy}(A)$ be a unit for $\mathsf{Asy}(A)$. This has a representative $(e_n) \in \ell^\infty(A)$, which satisfies

$$\lim_n \|e_n a_n - a_n\| = 0, \quad \lim_n \|a_n e_n - a_n\| = 0 \qquad ((a_n) \in \ell^\infty(A)).$$

- By picking (a_n) suitably, this shows that, for example,

$$\lim_n \sup\{\|e_n a - a\| : a \in A, \|a\| \leqslant 1\} = 0.$$

Unital algebras cont.

$$\lim_n \sup\{\|e_n a - a\|, \|a e_n - a\| : a \in A, \|a\| \leqslant 1\} = 0.$$

- Extract a subsequence (e_n) with $\|e_n a - a\|, \|a e_n - a\| \leqslant \frac{1}{n}\|a\|$ for $a \in A$.
- We can also arrange that e.g. $\|e_n\| \leqslant 2\|(e_n)\|_{\mathsf{Asy}} = K$ say.
- Thus $\|e_n - e_m\| \leqslant \|e_n - e_n e_m\| + \|e_n e_m - e_m\| \leqslant K(\frac{1}{m} + \frac{1}{n})$.
- So (e_n) is Cauchy in A, so converges in A, say to e. Clearly e is a unit.

The argument for an ultrapower is similar, just with more bookkeeping.

Ring-theoretic infiniteness

Definition

$p \in A$ is an *idempotent* if $p^2 = p$.
Two idempotents p, q are *equivalent*, written $p \sim q$, if there are $a, b \in A$ with $p = ab$ and $q = ba$.

[If $q \sim r$, say $q = cd, r = dc$, then $p = p^2 = abab = aqb = (ac)(db)$ and $(db)(ac) = dqc = dcdc = r^2 = r$ so $p \sim r$.]
In e.g. $\mathbb{M}_n$, idempotents are equivalent if and only if they have the same rank.

Definition

Let A be a unital algebra. A is *Dedekind finite* if $p \sim 1$ implies $p = 1$.

That is, if $a, b \in A$ with $ab = 1$ then also $ba = 1$ (as then $p = ba$ has $p^2 = baba = b1a = p$ so $p \sim 1$).
[Might mention C^*-algebras?]

For asymptotic sequence algebras

Theorem

Let A be a unital Banach algebra. If A is Dedekind-finite then so is $\mathsf{Asy}(A)$*.*

Proof.

Let $p^2 = p \sim 1$ in $\mathsf{Asy}(A)$. We need to show that $p = 1$.

- Let $(x_n) \in \ell^\infty(A)$ be a representative of p. Of course, (x_n) will not be an idempotent in general. A Functional Calculus argument shows that we can find a different representative (p_n) with $p_n^2 = p_n$ for each n.
- That $p \sim 1$ means there are $a = (a_n)$ and $b = (b_n)$ with $(a_n b_n - p_n) \in c_0(A)$ and $(b_n a_n - 1) \in c_0(A)$.
- So eventually $u_n = b_n a_n$ is invertible. With $q_n = a_n u_n^{-1} b_n$ we find $q_n^2 = q_n$ and $q_n \sim b_n a_n u_n^{-1} = 1$ so hypothesis implies $q_n = 1$.

For asymptotic sequence algebras cont.

Proof continued.

$$(a_n b_n - p_n) \in c_0(A) \qquad (b_n a_n - 1) \in c_0(A).$$

We established that with $u_n = b_n a_n$ we have $q_n = a_n u_n^{-1} b_n = 1$ eventually.
For large n, now compute:

$$\begin{aligned} \|1 - p_n\| = \|q_n - p_n\| &\leqslant \|a_n u_n^{-1} b_n - a_n b_n\| + \|a_n b_n - p_n\| \\ &\leqslant \|a_n\| \|u_n^{-1} - 1\| \|b_n\| + \|a_n b_n - p_n\|, \end{aligned}$$

which is small for large n.
Thus $(1 - p_n) \in c_0(A)$ so $p = 1$ in $\mathrm{Asy}(A)$. □

This also holds for $\mathrm{Asy}(A_n)$ for a sequence (A_n) of Dedekind-finite algebras.

The converse?

Question

If $\mathrm{Asy}(A)$ is Dedekind-finite, is A Dedekind-finite?

Motivated by either:

- Some intuition from working with examples, or
- metric model theory considerations,

it is natural to consider a *metric* reformulation.

Definition

Say that A is *Dedekind-infinite* if it is not Dedekind-finite. Define

$$C_{\mathrm{DI}}(A) = \inf \left\{ \|a\|\|b\| : a, b \in A, ab = 1, ba \neq 1 \right\}.$$

Set $C_{\mathrm{DI}}(A) = \infty$ if A is Dedekind-finite.

With metric control

Proposition

Let (A_n) be a sequence of unital Banach algebras with $C_{DI}(A_n) \leqslant K$ for all n. Then $\mathsf{Asy}((A_n))$ *is Dedekind-infinite.*

Proposition

Let (A_n) be such that $C_{DI}(A_n) \to \infty$. Then $\mathsf{Asy}(A_n)$ *is Dedekind-finite.*

Proof.

Much the same idea as before, starting with $a, b \in \mathsf{Asy}(A_n)$ with $ab = 1$, we lift to representatives in $\ell^\infty(A_n)$, say $a = (a_n)$ and $b = (b_n)$, and adjust so that $a_n b_n = 1$ for n large enough.

As $\|a_n\|\|b_n\|$ bounded, eventually $\|a_n\|\|b_n\| < C_{\mathrm{DI}}(A_n)$, and so $a_n b_n = 1 \implies b_n a_n = 1$.

Conclude $ba = 1$. □

Counter-example for Banach algebras

Question

If A is a Dedekind-infinite Banach algebra, is it true that $C_{\mathrm{DI}}(A) \leqslant K$ for some absolute constant K?

True for C^*-algebras.

For Banach Algebras: Of course not!

Our counter-example will be a *weighted-semigroup* algebra.

Let C be the bicyclic semigroup, so C has generators α, β with $\alpha\beta = 1$ and no other relations.

So C is all *reduced words* which are of the form $\beta^n \alpha^m$ with $n, m \in \mathbb{Z}_{\geqslant 0}$. Exercise to the reader to work out the multiplication.

Weights

Definition

A *weight* on a semigroup is $\omega : S \to (0, \infty)$ with $\omega(st) \leqslant \omega(s)\omega(t)$.

The *weighted semigroup algebra* is $\ell^1(S, \omega)$, which is those $a \in \ell^1(S)$ with $\|a\|_\omega = \sum_s |a_s|\omega(s) < \infty$. The condition on the weight ensures that $\ell^1(S, \omega)$ is an algebra under convolution:

$$\delta_s \star \delta_t = \delta_{st}, \qquad a \star b = \Big(\sum_{\{t,r : tr = s\}} a_t b_r \Big)_{s \in S}.$$

Proposition

Let C be the bicyclic semigroup Suppose $\omega(1) = 1, \omega(s) \geqslant n$ otherwise. Then $C_{DI}(\ell^1(C, \omega)) \geqslant (n/86)^{1/3}$.

So, no, there is no absolute constant!

Renormings

The definitions of $\mathrm{Asy}(A_n)$ and $(A_n)_{\mathcal{U}}$ are very "metric".

Example

If we allow a "unital" Banach algebra to have a unit which is not a unit vector, then it is easily possible to have a sequence of unital Banach algebras (A_n) such that $\mathrm{Asy}(A_n)$ is not unital.

$$C_{\mathrm{DI}}(A) = \inf\left\{\|a\|\|b\| : a, b \in A, ab = 1, ba \neq 1\right\}.$$

Question

If A is Dedekind infinite, can we renorm A so that $C_{\mathrm{DI}}(A) = 1$, or perhaps less than some absolute constant?

Proper infiniteness

Definition

A (Banach) algebra is *properly infinite* when there are idempotents $p \sim 1$ and $q \sim 1$ which are orthogonal, $pq = qp = 0$.

- Using ideas of Friis and Rørdam, one can show that *commuting* idempotents $p, q \in \mathsf{Asy}(A_n)$ can be lifted to commuting idempotents $P, Q \in \ell^\infty(A_n)$.
- Using this, one can show that if $\mathsf{Asy}(A_n)$ is properly infinite, then A_n is properly infinite, for n sufficiently large.

For the converse, we proceed in a rather similar way to before; define

$$C_{\mathrm{PI}}(A) = \inf \big\{ \|a\|\|b\|\|c\|\|d\| : ab = 1 = cd, ad = 0 = cb \big\}.$$

Theorem

$\mathsf{Asy}(A_n)$ *is properly infinite if and only if* $\limsup_{n\to\infty} C_{PI}(A_n) < \infty$.

(Counter-)example

Again, $C_{\mathrm{PI}}(A)$ can be arbitrarily large for a properly infinite A.

- We now have two idempotents, so we want two copies of the bicylic semigroup...
- The Cuntz semigroup is

$$Cu_2 = \big\langle s_1, s_2, t_1, t_2 \,\big|\, t_1 s_1 = 1 = t_2 s_2, t_1 s_2 = t_2 s_1 = \diamond \big\rangle,$$

  where $\diamond$ is a "semigroup zero".

Directly forming the semigroup algebra $\ell^1(Cu_2)$ seems wrong, as it is more natural to identify the semigroup zero and the algebra zero.

- Note that $\mathbb{C}\delta_\diamond$ is a (closed, two-sided) ideal in $\ell^1(Cu_2)$, so...
- We can form $\ell^1(Cu_2)/\mathbb{C}\delta_\diamond$ as an algebra,
- Which as a space is isometric with $\ell^1(Cu_2 \setminus \{\diamond\})$.
- So really we have $\ell^1(Cu_2 \setminus \{\diamond\})$ with the obvious product.

(Counter-)example continued

Proposition

Let ω be a weight on Cu_2 with (for example) $\omega(s) \geqslant n$ for each $s \neq 1$. Then $C_{PI}(A) \geqslant (n/86)^{1/3}$.

Again, by choosing ω_n suitably, with $A_n = \ell^1(Cu_2 \setminus \{\diamond\}, \omega_n)$, we produce a sequence of properly infinite algebras with $C_{\mathrm{PI}}(A_n) \to \infty$, so $\mathrm{Asy}(A_n)$ is not properly-infinite.

Question

Same renorming question for C_{PI}.

All the same results hold for ultraproducts.

Pure infiniteness

For C^*-algebras, being "purely infinite" is related again to projections, and hereditary subalgebras. For general algebras, there is a different definition (equivalent for C^*-algebras).

Definition (Ara, Goodearl, Pardo)

A unital algebra is *purely infinite* if it is not a division algebra, and for each $a \neq 0$ there are b, c with $bac = 1$.

Clearly, a purely infinite algebra is simple, so we consider now only ultraproducts.

A metric condition

For $a \neq 0$ define

$$C_{\mathrm{pi}}^{A}(a) = \inf \left\{ \|b\|\|c\| : bac = 1 \right\}.$$

This is homogeneous, and so we can consider only $\|a\| = 1$.

Proposition

Let $A \neq \mathbb{C}$ and $\mathcal{U}$ be countably incomplete. Then $(A)_{\mathcal{U}}$ is purely infinite if and only if $\sup\{C_{pi}^{A}(a) : \|a\| = 1\} < \infty$.

Proposition

Suppose that $\mathcal{U}$ is countably incomplete, and $(A)_{\mathcal{U}}$ is simple. When $\psi\colon A \to B$ is a non-zero bounded homomorphism to a Banach algebra B, we have that ψ is bounded below.

E.g. $A = \mathcal{B}(\ell^p)/\mathcal{K}(\ell^p)$ has $(A)_{\mathcal{U}}$ purely infinite.

Towards a counter-example

We seek an example A which is purely infinite, but with $(A)_{\mathcal{U}}$ not purely infinite. Some analogue of the Cuntz algebra is perhaps a good guess!

$$Cu_2 = \left\langle s_1, s_2, t_1, t_2 \,\middle|\, t_1 s_1 = 1 = t_2 s_2, t_1 s_2 = t_2 s_1 = \diamond \right\rangle,$$

- We start with $\ell^1(Cu_2 \setminus \{\diamond\}) = \ell^1(Cu_2)/\mathbb{C}\delta_\diamond$.
- We now impose the condition

$$\delta_{s_1 t_1} + \delta_{s_2 t_2} = 1.$$

  Do this by defining I to be the closed ideal generated by the element $\delta_{s_1 t_1} + \delta_{s_2 t_2} - 1$ and considering the quotient algebra $A = \ell^1(Cu_2 \setminus \{\diamond\})/I$.
- The purely algebra version of this construction is the *Leavitt algebra* L_2, so you can consider A as an "ℓ^1-completion of L_2".

Combinatorics

As ℓ^1 algebras are rather "combinatorial", we can show:

Proposition

Let $a \in \ell^1(Cu_2 \setminus \{\diamond\})$. TFAE:

1. $a \in I$;
2. *some combinatorial condition on the coefficients of* a;
3. *there are not* $b, c \in \ell^1(Cu_2 \setminus \{\diamond\})$ *with* $bac = 1$.

As such, I is the unique maximal ideal in $\ell^1(Cu_2 \setminus \{\diamond\})$.

Corollary

The quotient algebra A is purely infinite.

Representations

The following construction was studied by C. Phillips. For $1 \leqslant p < \infty$, consider $\ell^p(\mathbb{N})$ and define operators

$$S_1(\xi) = (\xi_1, 0, \xi_2, 0, \xi_3, 0, \cdots), \quad S_2(\xi) = (0, \xi_1, 0, \xi_2, 0, \xi_3, \cdots),$$
$$T_1(\xi) = (\xi_1, \xi_3, \xi_5, \cdots), \quad T_2(\xi) = (\xi_2, \xi_4, \xi_6, \cdots).$$

These are contractions, with the relations

$$T_1 S_1 = T_2 S_2 = I, \quad T_2 S_1 = T_1 S_2 = 0, \quad S_1 T_1 + S_2 T_2 = I.$$

By universality, there is a contractive homomorphism

$$A \to \mathcal{B}(\ell^p); \quad \delta_{s_i} + I \mapsto S_i, \ \delta_{t_i} + I \mapsto T_i.$$

In particular, A is non-trivial!

Putting it all together

Theorem

The representation $A \to \mathcal{B}(\ell^p)$ is injective, but not bounded below. As such, A does not have purely infinite ultrapowers.

Proof.

We estimate the norms of certain elements of A, compare these to their images in $\mathcal{B}(\ell^p)$, and conclude that the homomorphism cannot be bounded below. As A is purely infinite, it is simple, so the homomorphism must be injective. If $(A)_{\mathcal{U}}$ were purely infinite, it would be simple, and so any homomorphism would be bounded below, contraction.

To compute norms in A, we study the dual space

$$A^* = I^\perp = \{f \in \ell^\infty(Cu_2 \setminus \{\diamond\}) : f(x) = 0 \ (x \in I)\}.$$

Again, this can be done in a combinatorial manner. □

Closing remarks

- It seems surprising to me that $A = \ell^1(Cu_2 \setminus \{\diamond\})/I$ hasn't been studied before.
- $\ell^1(Cu_2 \setminus \{\diamond\})$ was studied by Dales, Laustsen, Read.
- "Quotients of ℓ^1-semigroup algebras" seem like a natural class of algebra to look at.
- Cu_2 is an inverse semigroup, and many of our combinatorial techniques have clear analogues for any inverse semigroup. (We could indeed have been more explicit about this!)

Phillips' algebras $\mathcal{O}_2^p$ are the closure of the image of A in $\mathcal{B}(\ell^p)$. The natural map $A \to \mathcal{O}_2^1$ is not an isomorphism; looking at bounded traces shows that there can be no isomorphism between A and $\mathcal{O}_2^p$ for any p.

Question

If A amenable?

Closing remarks 2

Recently, [Bardadyn, Kwaśniewski and McKee] have looked at Banach space analogues of this notion of "tight" representation (from Exel's theory for Groupoid C^*-algebras) of Cu_2 on a Banach space.
Given any $i = (i_1, \cdots, i_n) \in \{1, 2\}^n$, we have an idempotent

$$e_i = s_{i_1} s_{i_2} \cdots s_{i_n} t_{i_n} \cdots t_{i_2} t_{i_1}.$$

Their algebra B is "maximal", subject to requiring that the span of the e_i is isomorphic to $\ell^\infty_{2^n}$, for each n.

Proposition

The natural map $A \to B$ is not bounded below.

What is the algebra B?