Quantum groups by way of Operator algebras

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Leeds

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What is a compact group?

Well, it's a compact topological space G with the structure of a group such that the group action is jointly continuous, and the inverse is continuous.

It's a unital commutative C*-algebra A with a unital $*$-homomorphism $\Delta : A \to A \otimes_{\text{min}} A$ which is:

- Co-associative, $(\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta$
- "Cancellative", that is, the sets

$$\{(a \otimes 1)\Delta(b) : a, b \in A\}, \quad \{(1 \otimes a)\Delta(b) : a, b \in A\},$$

  have dense linear span in $A \otimes_{\text{min}} A$.

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have dense linear span in $A \otimes_{\min} A$.

Equivalence, easy direction

If G is a compact group, set

$$A = C(G) = \{\text{continuous functions } G \to \mathbb{C}\},$$

identify $A \otimes_{\min} A = C(G \times G)$, define

$$\Delta(f) \in C(G \times G), \quad \Delta(f) : (s,t) \mapsto f(st) \qquad (f \in C(G), s,t \in G).$$

Finally observe that

$$(a \otimes 1)\Delta(b) : (s,t) \mapsto a(s)b(st),$$

will separate the points of $G \times G$ (by varying a and b) so by Stone-Weierstrass,

$$\text{lin}\{(a \otimes 1)\Delta(b) : a,b \in A\}$$

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Equivalence, hard direction

Gelfand-Naimark tells us that a unital commutative C*-algebra A has the form $C(X)$ for some compact space X. So again $A \otimes_{\text{min}} A = C(X \times X)$. Then $\Delta : C(X) \to C(X \times X)$ a unital $*$-homomorphism induces a continuous map $\theta : X \times X \to X$ such that

$$f(\theta(s,t)) = \Delta(f)(s,t) \qquad (s,t \in X, f \in C(X)).$$

Δ co-associative implies that θ is associative, so X is a compact semigroup.
The cancellation rules for Δ imply that X is cancellative, that is

$$st = rt \implies s = r, \quad ts = tr \implies s = r.$$

Exercise: A compact semigroup with cancellation is a compact group.

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Compact quantum groups

Simply remove the word “commutative”!

For example, let Γ be a discrete group, and let Γ act on $\ell^2(\Gamma)$ by left translation:

$$\lambda(s)f : t \mapsto f(s^{-1}t) \qquad (s, t \in \Gamma, f \in \ell^2(\Gamma)).$$

Let $C_r^*(\Gamma)$ be the (reduced) group C*-algebra: that is, the norm closed algebra, acting on $\ell^2(\Gamma)$, generated by $\lambda(\Gamma)$. So $C_r^*(\Gamma)$ is commutative if and only if Γ is.

There is a $*$-homomorphism

$$\Delta : C_r^*(\Gamma) \to C_r^*(\Gamma) \otimes_{\min} C_r^*(\Gamma) = C_r^*(\Gamma \times \Gamma),$$
$$\Delta : \lambda(s) \mapsto \lambda(s) \otimes \lambda(s) = \lambda(s, s) \qquad (s \in \Gamma).$$

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Compact or Discrete?

Hang on: we're saying that for *discrete* Γ, we have that $C_r^*(\Gamma)$ is a *compact* quantum group?

If Γ were abelian, then the fourier transform tells us that

$$C_r^*(\Gamma) \cong C(\hat{\Gamma}),$$

where $\hat{\Gamma}$ is the Pontryagin dual of Γ. As Γ is discrete, $\hat{\Gamma}$ is compact.

As $C(G)$ is our "commutative" base algebra, this terminology is forced upon us.

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Twisted $SU(2)$

{From Woronowicz in the C*-setting, but independently discovered by Soibelman and Vaksman}

$C(SU(2))$ is the (commutative) C*-algebra generated by a, b with

$$a^*a + b^*b = 1.$$

$$aa^* + bb^* = 1, \quad b^*b = bb^*, \quad ab = ba, \quad ab^* = b^*a.$$

We introduce a real parameter $\mu \in [-1, 1] \setminus \{0\}$, and let $C(SU_\mu(2))$ be the (non-commutative) C*-algebra generated by a, b with

$$a^*a + b^*b = 1, \quad aa^* + \mu^2 bb^* = 1,$$
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There exists a coproduct Δ with

$$\Delta(a) = a \otimes a - \mu b^* \otimes b, \quad \Delta(b) = b \otimes a + a^* \otimes b.$$

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Corepresentation theory

A (finite-dimensional) corepresentation of (A, Δ) is a matrix $u \in \mathbb{M}_n(A)$ with

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj} \qquad (1 \leq i, j \leq n).$$

Let $A = C(G)$, identify $M_n(A)$ with $A \otimes M_n = C(G, M_n)$.

- So u corresponds to some continuous function $\pi : G \to M_n$;
- So $\pi(s)_{ij} = u_{ij}(s)$ for $s \in G$.
- Then

$$\big(\pi(s)\pi(t)\big)_{ij} = \sum_k \pi(s)_{ik}\pi(t)_{kj} = \Delta(u_{ij})(s, t) = \pi(st)_{ij}.$$

- Can reverse this; so corepresentations of $C(G)$ correspond to representations of G.

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Corepresentation theory cont.

Let (A, Δ) be any compact quantum group.

- All irreducible corepresentations of (A, Δ) are finite-dimensional.
- It's possible to show that any finite-dimensional corepresentation u is *equivalent* to a unitary corepresentation: $u^* u = uu^* = I_n$.
- There is a notion of infinite-dimensional corepresentation: but these split up into direct sums of irreducibles.
- There is a character theory for corepresentations.
- All of this completely generalises the theory for compact groups.

For $SU(2)$, for each integer $n \geq 1$, there is precisely one (up to equivalence) irreducible representation on M_n, say u_{n-1}; also

$$u_n \otimes u_m \cong u_{|m-n|} \oplus u_{|m-n|+2} \oplus \cdots \oplus u_{m+n}.$$

Exactly the same is true for $SU_\mu(2)$.

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- All of this completely generalises the theory for compact groups.

For $SU(2)$, for each integer $n \geq 1$, there is precisely one (up to equivalence) irreducible representation on M_n, say u_{n-1}; also

$$u_n \otimes u_m \cong u_{|m-n|} \oplus u_{|m-n|+2} \oplus \cdots \oplus u_{m+n}.$$

Exactly the same is true for $SU_\mu(2)$.

Corepresentation theory cont.

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Back to $C(G)$

If G is a compact group, then $C(G)$ is a sufficiently nice algebra that ϵ and S extend to bounded maps on $C(G)$:

- $\epsilon(f) = f(e)$ where $e \in G$ is the group identity;
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- So the counit ϵ represents the group identity, and the antipode S represents the group inverse.
- Notice that for general (A, Δ), the multiplication map is unbounded, so it's not even clear what axioms the antipode should satisfy…
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So, a C*-algebraic compact quantum group (A, Δ) gives rise to a Hopf $*$-algebra $(\mathcal{A}, \Delta)$. Can we go the other way?

Theorem (Dijkhuizen and Koornwinder)

Given a Hopf $*$-algebra $(\mathcal{A}, \Delta)$, the following are equivalent:

1. *$\mathcal{A}$ is given by a compact quantum group (A, Δ);*
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3. *there is a functional $h : \mathcal{A} \rightarrow \mathbb{C}$ which is positive ($h(a^*a) \geq 0$) and invariant ($(h \otimes \mathrm{id})\Delta(a) = h(a)1$).*

However, the (A, Δ) occurring in (1) might not be unique. Also, the h occuring in (3) extends to A: for $A = C(G)$, this is just the functional given by integrating against the Haar measure.

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Multiplier algebras

To handle the non-compact case (either algebraic, or in the C*-algebra language) we need to consider non-unital algebras.

- Let $\mathcal{A}$ be an algebra;
- The *multiplier algebra*, $M(\mathcal{A})$, is the largest unital algebra which contains $\mathcal{A}$ as an ideal, such that if $x \in M(\mathcal{A})$, and $axb = 0$ for $a, b \in \mathcal{A}$, then $x = 0$.
- For example, if $\mathcal{A}$ the algebra of finitely supported complex functions on G, then $M(\mathcal{A})$ is the algebra of all complex functions on G.
- A homomorphism $\theta : \mathcal{A} \rightarrow M(\mathcal{B})$ is *non-degenerate* if $\operatorname{lin}\{\theta(a)b : a \in \mathcal{A}, b \in \mathcal{B}\}$ and $\operatorname{lin}\{b\theta(a) : a \in \mathcal{A}, b \in \mathcal{B}\}$ are equal to $\mathcal{B}$
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- For example, if $\mathcal{A}$ the algebra of finitely supported complex functions on G, then $M(\mathcal{A})$ is the algebra of all complex functions on G.
- A homomorphism $\theta : \mathcal{A} \to M(\mathcal{B})$ is *non-degenerate* if $\mathrm{lin}\{\theta(a)b : a \in \mathcal{A}, b \in \mathcal{B}\}$ and $\mathrm{lin}\{b\theta(a) : a \in \mathcal{A}, b \in \mathcal{B}\}$ are equal to $\mathcal{B}$
- Then θ has a unique extension to $M(\mathcal{A})$: we define $\theta(x)\theta(a)b = \theta(xa)b$ and $b\theta(a)\theta(x) = b\theta(ax)$ for $x \in M(\mathcal{A}), a \in \mathcal{A}, b \in \mathcal{B}$.

Multiplier Hopf algebras

Let G be any group (without topology, but maybe infinite).

- Let $\mathcal{A}$ be the algebra of complex-valued, finitely supported functions on G, with the pointwise product;
- The coproduct Δ should be as before: $\Delta(f)(s,t) = f(st)$; But this might not be finitely supported.
- However, $\Delta : \mathcal{A} \to M(\mathcal{A} \otimes \mathcal{A})$ makes sense;
- Furthermore, $\Delta(a)(1 \otimes b)$ and $\Delta(a)(b \otimes 1)$ are members of $\mathcal{A} \otimes \mathcal{A}$.

A *multiplier Hopf $*$-algebra* is an $*$-algebra $\mathcal{A}$, a non-degenerate, coassociative, $*$-homomorphism $\Delta : \mathcal{A} \to M(\mathcal{A} \otimes \mathcal{A})$ such that the maps

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Multiplier Hopf algebras (cont)

Then we can *construct* a counit and an antipode with the same axioms as before.

- So a Hopf $*$-algebra is just a unital multiplier Hopf $*$-algebra.
- If a multiplier Hopf $*$-algebra has an invariant positive functional, then we have the notion of an *algebraic quantum group*.
- As for compact quantum groups, such algebras admit C*-algebraic completions, and the counit and anitpode, in some sense, extend to this C*-algebra. These C*-algebraic quantum groups fit into the framework of Locally Compact Quantum Groups in the sense of Kustermans and Vaes, and Masuda, Nakagami and Woronowicz; these have a vast amount of structure;
- Algebraic quantum groups also admit a duality theory: we can turn a subset of the dual $\mathcal{A}'$ into an algebra, with a coproduct, which becomes an algebraic quantum group in its own right, say $\hat{\mathcal{A}}$. Then $\hat{\hat{\mathcal{A}}} \cong \mathcal{A}$.

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Compactifications

Suppose we start with a multiplier bialgebra $(\mathcal{A}, \Delta)$ (so $\Delta : \mathcal{A} \rightarrow M(\mathcal{A} \otimes \mathcal{A})$ is a non-degenerate coassociative homomorphism). Can we find a maximal (algebraic) compact quantum group in $M(\mathcal{A})$?

- This is equivalent to the classical question of starting with a semigroup S and finding the maximal compact group G with a dense-range homomorphism $S \rightarrow G$.
- Soltan showed how to do this: one looks at the algebra generated by the matrix coefficients of certain unitary corepresentations of S;
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From semigroups to groups

However, if we start with a non-compact *group* H, then the construction is easier.

- We first form the maximal compact *semigroup* G which contains a dense-range homomorphic image of H;
- This is related to *almost periodic functions* on H;
- Then we lift the inverse from H to G, showing that G is a group.

Theorem

Let $(\mathcal{A}, \Delta)$ be a multiplier bialgebra. Then $M(\mathcal{A})$ contains a maximal unital algebra $\mathcal{B}$ such that $\Delta(\mathcal{B}) \subseteq \mathcal{B} \otimes \mathcal{B}$.
In fact, we have that $\mathcal{B} = \{x \in M(\mathcal{A}) : \Delta(x) \in M(\mathcal{A}) \otimes M(\mathcal{A})\}$.
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Future work

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