Compactifications of quantum groups

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Gelfand duality

Theorem

Any commutative C*-algebra A has the form C₀(X) where X is a locally compact Hausdorff space, isomorphic to the character space of A.

If X, Y are compact, then there is a bijection between continuous maps \( f : X \to Y \) and unital \(*\)-homomorphisms \( C(Y) \to C(X) \).

\[
f : X \to Y \quad \leftrightarrow \quad \theta : C(Y) \to C(X); \quad a \mapsto a \circ f.
\]

What if X, Y are only locally compact? That \( a \mapsto a \circ f \) maps \( C₀(Y) \) to \( C₀(X) \) corresponds to \( f \) being “proper”.
Locally compact case

Let $C^b(X)$ be the bounded continuous functions on $X$. Then $f : X \to Y$ induces a $\ast$-homomorphism $\theta : C_0(Y) \to C^b(X); a \mapsto a \circ f$.

Not every $\ast$-homomorphism arises in this way: an arbitrary $\theta : C_0(Y) \to C^b(X)$ gives a continuous map $f : X \to Y_\infty$ to the one-point compactification of $Y$.

To single out those maps which “never take the value $\infty$” you need to look at “non-degenerate $\ast$-homomorphisms”:

$$\overline{\text{lin}} \{ \theta(a)b : a \in C_0(Y), b \in C_0(X) \} = C_0(X).$$

Then we get:

The category of locally compact spaces with continuous maps \(\xrightarrow{\text{anti}}\) The category of commutative $C^*$-algebras and non-degenerate $\ast$-homomorphisms

Multiplier algebras

The multiplier algebra of a $C^*$-algebra $A$ is the largest $C^*$-algebra $B$ which contains $A$ as a two-sided ideal, in an “essential” way:

For $b \in B$, $ab = ba = 0$ ($a \in A$) \(\implies\) $b = 0$.

Write $M(A)$ for the multiplier algebra (there are various constructions).

- If $A = C_0(X)$ then $M(A) = C^b(X)$.
- If $A = \mathcal{K}(H)$, compact operators on a Hilbert space, then $M(A) = B(H)$, all operators on a Hilbert space.

A $\ast$-homomorphism $\theta : A \to M(B)$ is non-degenerate when

$$\overline{\text{lin}} \{ \theta(a)b : a \in A, b \in B \} = B.$$

Then $\theta$ extends to a $\ast$-homomorphism $M(A) \to M(B)$ and in this way we can compose two non-degenerate $\ast$-homomorphisms, and get another non-degenerate $\ast$-homomorphism.
Intuition

- We say that a “morphism” (a la Woronowicz) \( A \to B \) is a non-degenerate \(*\)-homomorphism \( \theta : A \to M(B) \).
- Intuition: “This corresponds to a continuous function from the non-commutative space of \( B \) to the non-commutative space of \( A \).”

Application: Quantum semigroups

Let \( S \) be a locally compact semigroup: so we have a continuous multiplication \( S \times S \to S \) which is associative.

\[
\Delta : C_0(S) \to C^b(S \times S); \quad \Delta(a)(s, t) = a(st).
\]

That multiplication is associative corresponds to \( \Delta \) being “coassociative”: \((\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta,\)

\[
(\Delta \otimes \iota)\Delta(a)(s, t, r) = \Delta(a)(st, r) = a((st)r),
\]

\[
(\iota \otimes \Delta)\Delta(a)(s, t, r) = \Delta(a)(s, tr) = a(s(tr)).
\]

A “quantum semigroup” is simply a \( C^* \)-algebra \( A \) together with a non-degenerate \(*\)-homomorphism \( \Delta : A \to M(A \otimes A) \) which is coassociative.
Application: Compact case

If $S$ is compact, don’t need to worry about multiplier algebras:

$$A = C(S) \text{ and } \Delta : A \rightarrow A \otimes A.$$ 

Then introduce the “quantum cancellation conditions”:

$$\text{lin}\{\Delta(a)(b \otimes 1) : a, b \in A\} = \text{lin}\{\Delta(a)(1 \otimes b) : a, b \in A\} = A \otimes A.$$ 

As the objects on the left are $*$-subalgebras of $C(S \times S)$, Stone–Weierstrass says that the first condition holds if, given $(s, t) \neq (s', t')$

$$\exists \ a, b \in C(S), \ \Delta(a)(b \otimes 1)(s, t) \neq \Delta(a)(b \otimes 1)(s', t')$$

$$\iff \exists \ a, b \in C(S), \ a(st)b(s) \neq a(s't')b(s').$$

Clearly this holds if $s \neq s'$, or if $s = s', st \neq st'$. So the conditions are equivalent to

$$st = st' \implies t = t', \quad ts = t's \implies t = t'.$$

Application: Compact groups

Folklore

Let $S$ be a compact semigroup with cancellation ($st = st'$ or $ts = t's$ implies $t = t'$.) Then $S$ is a compact group.

Fun exercise: Do this when $S$ is finite. Then “topologize” your proof.

Definition (Woronowicz)

A compact quantum group is a unital $C^*$-algebra $A$ with a coassociative $*$-homomorphism $\Delta : A \rightarrow A \otimes A$, with

$$\text{lin}\{\Delta(a)(b \otimes 1) : a, b \in A\} = \text{lin}\{\Delta(a)(1 \otimes b) : a, b \in A\} = A \otimes A.$$
Group C*-algebras
For example, let \( \Gamma \) be a discrete group, and let \( \Gamma \) act on \( \ell^2(\Gamma) \) by left translation:
\[
\lambda(s)f : t \mapsto f(s^{-1}t) \quad (s, t \in \Gamma, f \in \ell^2(\Gamma)).
\]
Let \( C^*_r(\Gamma) \) be the (reduced) group C*-algebra: that is, the norm closed algebra, acting on \( \ell^2(\Gamma) \), generated by \( \lambda(\Gamma) \). So \( C^*_r(\Gamma) \) is commutative if and only if \( \Gamma \) is.

There is a ∗-homomorphism
\[
\Delta : C^*_r(\Gamma) \to C^*_r(\Gamma) \otimes_{\text{min}} C^*_r(\Gamma),
\]
\[
\Delta : \lambda(s) \mapsto \lambda(s) \otimes \lambda(s) \quad (s \in \Gamma).
\]
Cancellation is clear:
\[
\text{lin}\{\Delta(a)(b \otimes 1)\} = \text{lin}\{(\lambda(s) \otimes \lambda(s))(\lambda(t) \otimes 1)\} = \text{lin}\{\lambda(st) \otimes \lambda(s)\} = \text{lin}\{\lambda(r) \otimes \lambda(s)\}.
\]

Corepresentation theory
A (finite-dimensional) corepresentation of \( (A, \Delta) \) is a matrix \( u \in \mathbb{M}_n(A) \) with
\[
\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj} \quad (1 \leq i, j \leq n).
\]
Let \( A = C(G) \), identify \( M_n(A) \) with \( A \otimes M_n = C(G, M_n) \).
- So \( u \) corresponds to some continuous function \( \pi : G \to M_n \).
- So \( \pi(s)_{ij} = u_{ij}(s) \) for \( s \in G \).
- Then
\[
(\pi(s)\pi(t))_{ij} = \sum_k \pi(s)_{ik}\pi(t)_{kj} = \Delta(u_{ij})(s, t) = \pi(st)_{ij}.
\]
- Can reverse this; so corepresentations of \( C(G) \) correspond to representations of \( G \).
Let \((A, \Delta)\) be any compact quantum group.

- All irreducible corepresentations of \((A, \Delta)\) are finite-dimensional.
- It’s possible to show that any finite-dimensional corepresentation \(u\) is equivalent to a unitary corepresentation: \(u^*u = uu^* = I_n\).
- There is a notion of infinite-dimensional corepresentation: but these split up into direct sums of irreducibles.
- There is a character theory for corepresentations.
- All of this completely generalises the theory for compact groups.

### Contragradient (co)representations

**Definition**

Let \(u = (u_{ij}) \in M_n(A)\) be a corepresentation of \((A, \Delta)\). The contragradient to \(u\) is \(\bar{u} = (u_{ij}^*)\).

That \(\Delta\) is a \(*\)-homomorphism shows that

\[
\Delta(\bar{u}_{ij}) = \Delta(u_{ij})^* = \sum_k u_{ik}^* \otimes u_{kj}^* = \sum_k u_{ik} \otimes \bar{u}_{kj}.
\]

- This is not the adjoint of the matrix \(u\); instead we take the entry-by-entry adjoint.
- If \(A = C(G)\) then \(u\) corresponds to \(\pi : G \to M_n\). Then this corresponds to the usual contragradient representation, assuming \(\pi\) is unitary.
Is the contragradient unitary?

- If $A = C(G)$ then everything is commutative, and $\overline{u}$ is unitary if $u$ is.
- But in general, it's not even clear that $\overline{u}$ is an invertible element of the algebra $\mathbb{M}_n(A)$, even if $u$ is unitary.

The general theory of compact quantum groups tells us that if $u$ is unitary and irreducible, then $\overline{u}$ is similar to an irreducible unitary corepresentation.

**Corollary**

Let $\mathcal{A}$ be the linear span of elements $u_{ij} \in A$ where $u$ is a unitary corepresentation. The $\mathcal{A}$ is a dense *-subalgebra of $A$.

So there is $T \in \mathbb{M}_n$ such that $T^{-1} \overline{u}T$ is unitary. If we take the polar decomposition $T = F^{1/2}U$, then if we are only interested in the unitary equivalence class of $T^{-1} \overline{u}T$, then only $F = T^*T$ is of interest.

**Automorphisms**

So $u$ unitary corepresentation implies there is positive invertible $F$ with $F^{-1/2}\overline{u}F^{1/2}$ unitary.

**Theorem**

For each $z \in \mathbb{C}$ there is a character $f_z : \mathcal{A} \rightarrow \mathbb{C}$ given by

$$f_z(u_{ij}) = \text{tr}(F)^{-z/2}(F^{-z})_{ij}.$$  

Here $F^{-z}$ is formed by a functional calculus argument.

**Theorem**

For each $z, w \in \mathbb{C}$ there is an automorphism $\rho_{z,w}$ of $\mathcal{A}$ given by

$$\rho_{z,w}(u_{ij}) = \sum_{k,l} f_w(u_{ik})u_{kl}f_z(u_{lj}).$$
Example application...

There always exists a “Haar state”, a state $\varphi$ on $A$ such that

$$(\varphi \otimes \iota)\Delta(a) = (\iota \otimes \varphi)\Delta(a) = \varphi(a)1.$$ 

If $A = C(G)$ then $\varphi$ is “integrate against the Haar measure”.

In general $\varphi$ is not a trace, but if we set $\sigma_z = \rho_{iz,iz}$ on $A$ then:

- for each $t \in \mathbb{R}$, $\sigma_t$ is $*$-automorphism and so extends to $A$; it leaves $\varphi$ invariant.
- for $a, b \in A$ we have that

$$\varphi(ab) = \varphi(b\sigma_{-i}(a)) \quad (a, b \in A).$$

This means that $\varphi$ is a KMS state.

Moral: we can see an analytic property from von Neumann algebra theory in the corepresentation theory of $(A, \Delta)$.

Counter-example (Brown; Wang–Woronowicz)

Let $n \in \mathbb{N}$ (e.g. $n = 2$). Let $A$ be the universal C*-algebra generated by elements $(u_{ij})_{i,j=1}^n$ subject to the relations which turn $u = (u_{ij}) \in M_n(A)$ into a unitary.

Define $\Delta : A \to A \otimes A$ by

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$ 

(This exists, by universality, because right hand side is a unitary element of $M_n(A \otimes A)$).

If $(A, \Delta)$ were a compact quantum group, then $\bar{u}$ would be, in particular, invertible. This is not the case...
Continued...

\[
a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Then set

\[
u' = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & I \end{pmatrix} \in \mathbb{M}_{2n}.
\]

Then \(u'\) is unitary, so by universality, there is a \(*\)-homomorphism \(\pi: A \to \mathbb{M}_2\) such that

\[
\pi \otimes I: A \otimes \mathbb{M}_n \to \mathbb{M}_2 \otimes \mathbb{M}_n \cong \mathbb{M}_{2n}; \quad (\pi \otimes I)(u) = u'.
\]

Then calculation shows that if \(\bar{u}\) is invertible, then \(\bar{u}'\) is as well, because \((\pi \otimes I)(\bar{u}^{-1})\) would be the inverse. But \(\bar{u}'\) is not invertible.

**Interpretation**

If \(S\) is a semigroup, and \(\pi: S \to \mathbb{M}_n\) is a unitary representation, then \(\pi\) induces a (semi)group homomorphism \(S \to U_n\).

- Get a \(*\)-homomorphism \(\theta: C(U_n) \to C^b(S)\) with
  \[(\theta \otimes \theta)\Delta_{U_n} = \Delta_S \theta.\]
- ‘Think about it’ to see that \(B = \theta(C(U_n))\) is the unital \(C^*\)-subalgebra of \(C^b(S)\) generated by the elements \(u_{ij}\), where \(u\) is the corepresentation associated to \(\pi\).

So this doesn’t work for Quantum Semigroups: we just constructed \((A, \Delta)\) and a unitary corepresentation \(u\) such that the \(C^*\)-algebra generated by the elements \(u_{ij}\), in this case all of \(A\), was not a (Compact Quantum) Group.

**Theorem (Sołtan, Woronowicz)**

*Let \((A, \Delta)\) be a quantum semigroup, let \(u\) be a corepresentation, and suppose that also \(\bar{u}\) is invertible. If \(B\) is the \(C^*\)-algebra generated by the \(u_{ij}\) in \(M(A)\), then \((B, \Delta|_B)\) is a compact quantum group.*
Sołtan’s Quantum Bohr Compactification

Theorem

Let $bA$ be the union of all such $B$. Then $(bA, \Delta_{|bA})$ is a compact quantum group.

This compact quantum group is maximal:

\[
S \rightarrow K \quad \sim \quad A = C_0(S) \quad \leftarrow \quad (D, \Delta_D)
\]

\[
bS \quad \downarrow \quad bA = C(bS)
\]

So this gives a “quantum Bohr compactification”.

Problem

How do you actually test if $\bar{u}$ invertible?

Locally compact quantum groups

If $A$ is a non-unital $C^*$-algebra, and $\Delta : A \rightarrow M(A \otimes A)$ a coassociative non-degenerate $\ast$-homomorphism, then seemingly it is not enough to ask just for “cancellation”, but also to assume the existence of suitable generalisations of the left/right Haar measure.

- However, once this is done, one gets a very satisfactory theory (Kustermans–Vaes).
- In particular, given $(A, \Delta)$ we can form the “dual” quantum group $(\hat{A}, \hat{\Delta})$ which generalises Pontryagin duality.

\[
A = C_0(G) \quad \Rightarrow \quad \hat{A} = C^*_r(G).
\]

- We have $\hat{\hat{A}} = A$.
- A discrete quantum group is the dual of a compact quantum group. So $c_0(\Gamma)$ for discrete $\Gamma$, or $C^*(G)$ for compact $G$. 
Compactifications of discrete quantum groups

Theorem (D., following Sołtan)

Let \((A, \Delta)\) be a discrete quantum group, and let \(u\) be a finite-dimensional unitary corepresentation of \((A, \Delta)\). Then \(\bar{u}\) is automatically invertible.

Sketch proof.

Idea of Vaes, as used by Sołtan shows that it’s enough to consider a “quotient” quantum group of \((A, \Delta)\) which is of “Kac type”. This means that the antipode, the map which represents the group inverse, is bounded.

Theorem (D.)

For a Kac algebra \((A, \Delta)\), we have that \(\mathcal{b}A\) is the closure of the set of elements \(x \in M(A)\) with \(\Delta(x)\) a finite-rank tensor.

Links with Banach algebras

Given a Banach algebra \(\mathcal{A}\), we turn \(\mathcal{A}^*\) into an \(\mathcal{A}\) bimodule via:

\[
\langle a \cdot \mu, b \rangle = \langle \mu, ba \rangle, \quad \langle \mu \cdot a, b \rangle = \langle \mu, ab \rangle \quad (a, b \in \mathcal{A}, \mu \in \mathcal{A}^*).
\]

For \(\mu \in \mathcal{A}\) let

\[L_\mu : \mathcal{A} \to \mathcal{A}^*, \quad a \mapsto \mu \cdot a.\]

Definition

We say that \(\mu\) is almost periodic if \(L_\mu\) is a compact operator.

If \(\mathcal{A} = L^1(G)\) for a locally compact group \(G\), then \(\mathcal{A}^* = L^\infty(G)\), and the collection of almost periodic elements coincides with (the image of) \(\mathcal{b}C_0(G)\) inside \(C^b(G) \subseteq L^\infty(G)\).
Stronger form of “compact”

**Definition**

For a Banach algebra \( \mathcal{A} \), say that \( \mu \in \mathcal{A}^* \) is “strongly almost periodic” if there is a sequence \((T_n)\) of finite-rank right module maps \( \mathcal{A} \to \mathcal{A}^* \) such that \( \|L_\mu - T_n\| \to 0 \).

So “compact” becomes “approximated by finite-ranks” (which for \( L^1(G) \) is no change); and we also impose an “algebra” condition.

For a locally compact quantum group \((\mathcal{A}, \Delta)\), there is a Banach algebra \( L^1(\mathcal{A}) \):

\[
\mathcal{A} = C_0(G) \implies L^1(\mathcal{A}) = L^1(G), \quad \mathcal{A} = C^*_r(G) \implies L^1(\mathcal{A}) = \mathcal{A}(G).
\]

Then \( L^1(\mathcal{A})^* = L^\infty(\mathcal{A}) \) a von Neumann algebra which contains \( M(\mathcal{A}) \), and hence \( \mathcal{A} \).

**Theorem**

*If \((\mathcal{A}, \Delta)\) is a Kac algebra, then \( \mathfrak{b} \mathcal{A} \) is precisely the collection of strongly almost periodic elements of \( L^1(\mathcal{A})^* \).*

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**Future work**

These techniques rely strongly on the fact that for a Kac algebra, the antipode \( S \) is bounded.

**Claim**

Let \( u \in M_n(\mathcal{A}) \) be a corepresentation. If we know that \( u_{ij}^* \in D(S) \), then \( \overline{u} \) is invertible.

**Claim**

Let \( x \in L^1(\mathcal{A})^* \) be strongly almost periodic. If we know that \( x \in D(S) \), then \( x \in \mathfrak{b} \mathcal{A} \).

When \( D(S) = L^1(\mathcal{A})^* \), as in the Kac case, we’re done.

General problem: \( D(S) \) is a bit mysterious.