Compactifications of quantum groups

Matthew Daws

Leeds

October 2012

Gelfand duality

Theorem

Any commutative C^*-algebra A has the form $C_0(X)$ where X is a locally compact Hausdorff space, isomorphic to the character space of A.

If X, Y are compact, then there is a bijection between continuous maps $X \to Y$ and unital $*$-homomorphisms $C(Y) \to C(X)$.

$$f : X \to Y \quad \leftrightarrow \quad \theta : C(Y) \to C(X); a \mapsto a \circ f.$$

What if X, Y are only locally compact? That $a \mapsto a \circ f$ maps $C_0(Y)$ to $C_0(X)$ corresponds to f being "proper".

Locally compact case

Let $C^b(X)$ be the bounded continuous functions on X. Then $f : X \to Y$ induces a $*$-homomorphism $\theta : C_0(Y) \to C^b(X); a \mapsto a \circ f$.
Not every $*$-homomorphism arises in this way: an arbitrary $\theta : C_0(Y) \to C^b(X)$ gives a continuous map $f : X \to Y_\infty$ to the one-point compactification of Y.
To single out those maps which "never take the value ∞" you need to look at "non-degenerate $*$-homomorphisms":

$$\overline{\text{lin}}\{\theta(a)b : a \in C_0(Y), b \in C_0(X)\} = C_0(X).$$

Then we get:

The category of locally compact spaces with continuous maps	$\xleftrightarrow[\textit{isomorphic}]{\textit{anti}}$	The category of commutative C*-algebras and non-degenerate $*$-homomorphisms

Multiplier algebras

The *multiplier algebra* of a C*-algebra A is the largest C*-algebra B which contains A as a two-sided ideal, in an "essential" way:

$$\text{For } b \in B, \quad ab = ba = 0 \quad (a \in A) \implies b = 0.$$

Write $M(A)$ for the multiplier algebra (there are various constructions).

- If $A = C_0(X)$ then $M(A) = C^b(X)$.
- If $A = \mathcal{K}(H)$, compact operators on a Hilbert space, then $M(A) = \mathcal{B}(H)$, all operators on a Hilbert space.

A $*$-homomorphism $\theta : A \to M(B)$ is non-degenerate when

$$\overline{\text{lin}}\{\theta(a)b : a \in A, b \in B\} = B.$$

Then θ extends to a $*$-homomorphism $M(A) \to M(B)$ and in this way we can compose two non-degenerate $*$-homomorphisms, and get another non-degenerate $*$-homomorphism.

Intuition

- We say that a "morphism" (a la Woronowicz) $A \to B$ is a non-degenerate $*$-homomorphism $\theta : A \to M(B)$.
- Intuition: "This corresponds to a continuous function from the non-commutative space of B to the non-commutative space of A."

Application: Quantum semigroups

Let S be a locally compact semigroup: so we have a continuous multiplication $S \times S \to S$ which is associative.

$$\rightsquigarrow \quad \Delta : C_0(S) \to C^b(S \times S); \; \Delta(a)(s,t) = a(st).$$

That multiplication is associative corresponds to Δ being "coassociative": $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$,

$$(\Delta \otimes \iota)\Delta(a)(s,t,r) = \Delta(a)(st,r) = a((st)r),$$
$$(\iota \otimes \Delta)\Delta(a)(s,t,r) = \Delta(a)(s,tr) = a(s(tr)).$$

A "quantum semigroup" is simply a C*-algebra A together with a non-degenerate $*$-homomorphism $\Delta : A \to M(A \otimes A)$ which is coassociative.

Application: Compact case

If S is compact, don't need to worry about multiplier algebras: $A = C(S)$ and $\Delta : A \to A \otimes A$.
Then introduce the "quantum cancellation conditions":

$$\overline{\text{lin}}\{\Delta(a)(b \otimes 1) : a, b \in A\} = \overline{\text{lin}}\{\Delta(a)(1 \otimes b) : a, b \in A\} = A \otimes A.$$

As the objects on the left are $*$-subalgebras of $C(S \times S)$, Stone–Weierstrass says that the first condition holds if, given $(s, t) \neq (s', t')$

$$\begin{aligned} &\exists\, a, b \in C(S),\ \Delta(a)(b \otimes 1)(s, t) \neq \Delta(a)(b \otimes 1)(s', t') \\ \Leftrightarrow &\exists\, a, b \in C(S),\ a(st)b(s) \neq a(s't')b(s'). \end{aligned}$$

Clearly this holds if $s \neq s'$, or if $s = s'$, $st \neq st'$. So the conditions are equivalent to

$$st = st' \implies t = t', \qquad ts = t's \implies t = t'.$$

Application: Compact groups

Folklore

Let S be a compact semigroup with cancellation ($st = st'$ or $ts = t's$ implies $t = t'$.) Then S is a compact group.

Fun exercise: Do this when S is finite. Then "topologize" your proof.

Definition (Woronowicz)

A compact quantum group is a unital C*-algebra A with a coassociative $*$-homomorphism $\Delta : A \to A \otimes A$, with

$$\overline{\text{lin}}\{\Delta(a)(b \otimes 1) : a, b \in A\} = \overline{\text{lin}}\{\Delta(a)(1 \otimes b) : a, b \in A\} = A \otimes A.$$

Group C*-algebras

For example, let Γ be a discrete group, and let Γ act on $\ell^2(\Gamma)$ by left translation:

$$\lambda(s)f : t \mapsto f(s^{-1}t) \qquad (s, t \in \Gamma, f \in \ell^2(\Gamma)).$$

Let $C_r^*(\Gamma)$ be the (reduced) group C*-algebra: that is, the norm closed algebra, acting on $\ell^2(\Gamma)$, generated by $\lambda(\Gamma)$. So $C_r^*(\Gamma)$ is commutative if and only if Γ is.

There is a $*$-homomorphism

$$\Delta : C_r^*(\Gamma) \to C_r^*(\Gamma) \otimes_{\min} C_r^*(\Gamma),$$
$$\Delta : \lambda(s) \mapsto \lambda(s) \otimes \lambda(s) \qquad (s \in \Gamma).$$

Cancellation is clear:

$$\begin{aligned} \operatorname{lin}\{\Delta(a)(b \otimes 1)\} &= \operatorname{lin}\{(\lambda(s) \otimes \lambda(s))(\lambda(t) \otimes 1)\} \\ &= \operatorname{lin}\{\lambda(st) \otimes \lambda(s)\} = \operatorname{lin}\{\lambda(r) \otimes \lambda(s)\}. \end{aligned}$$

Corepresentation theory

A (finite-dimensional) corepresentation of (A, Δ) is a matrix $u \in \mathbb{M}_n(A)$ with

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj} \qquad (1 \leq i, j \leq n).$$

Let $A = C(G)$, identify $M_n(A)$ with $A \otimes M_n = C(G, M_n)$.

- So u corresponds to some continuous function $\pi : G \to M_n$.
- So $\pi(s)_{ij} = u_{ij}(s)$ for $s \in G$.
- Then

$$\big(\pi(s)\pi(t)\big)_{ij} = \sum_k \pi(s)_{ik}\pi(t)_{kj} = \Delta(u_{ij})(s, t) = \pi(st)_{ij}.$$

- Can reverse this; so corepresentations of $C(G)$ correspond to representations of G.

Corepresentation theory cont.

Let (A, Δ) be any compact quantum group.

- All irreducible corepresentations of (A, Δ) are finite-dimensional.
- It's possible to show that any finite-dimensional corepresentation u is *equivalent* to a unitary corepresentation: $u^*u = uu^* = I_n$.
- There is a notion of infinite-dimensional corepresentation: but these split up into direct sums of irreducibles.
- There is a character theory for corepresentations.
- All of this completely generalises the theory for compact groups.

Contragradient (co)representations

Definition

Let $u = (u_{ij}) \in \mathbb{M}_n(A)$ be a corepresentation of (A, Δ). The contragradient to u is $\overline{u} = (u_{ij}^*)$.

That Δ is a $*$-homomorphism shows that

$$\Delta(\overline{u}_{ij}) = \Delta(u_{ij})^* = \sum_k u_{ik}^* \otimes u_{kj}^* = \sum_k \overline{u}_{ik} \otimes \overline{u}_{kj}.$$

- This is *not* the adjoint of the matrix u; instead we take the entry-by-entry adjoint.
- If $A = C(G)$ then u corresponds to $\pi : G \to \mathbb{M}_n$. Then this corresponds to the usual contragradient representation, assuming π is unitary.

Is the contragradient unitary?

- If $A = C(G)$ then everything is commutative, and $\overline{u}$ is unitary if u is.
- But in general, it's not even clear that $\overline{u}$ is an invertible element of the algebra $\mathbb{M}_n(A)$, even if u is unitary.

The general theory of compact quantum groups tells us that if u is unitary and irreducible, then $\overline{u}$ is similar to an irreducible unitary corepresentation.

Corollary

Let $\mathcal{A}$ be the linear span of elements $u_{ij} \in A$ where u is a unitary corepresentation. The $\mathcal{A}$ is a dense $*$-subalgebra of A.

So there is $T \in \mathbb{M}_n$ such that $T^{-1}\overline{u}T$ is unitary. If we take the polar decomposition $T = F^{1/2}U$, then if we are only interested in the *unitary equivalence class* of $T^{-1}\overline{u}T$, then only $F = T^*T$ is of interest.

Automorphisms

So u unitary corepresentation implies there is positive invertible F with $F^{-1/2}\overline{u}F^{1/2}$ unitary.

Theorem

For each $z \in \mathbb{C}$ there is a character $f_z : \mathcal{A} \to \mathbb{C}$ given by $f_z(u_{ij}) = \mathrm{tr}(F)^{-z/2}\big(F^{-z}\big)_{ij}$.

Here F^{-z} is formed by a functional calculus argument.

Theorem

For each $z, w \in \mathbb{C}$ there is an automorphism $\rho_{z,w}$ of $\mathcal{A}$ given by

$$\rho_{z,w}(u_{ij}) = \sum_{k,l} f_w(u_{ik})u_{kl}f_z(u_{lj}).$$

Example application...

There always exists a "Haar state", a state φ on A such that

$$(\varphi \otimes \iota)\Delta(a) = (\iota \otimes \varphi)\Delta(a) = \varphi(a)1.$$

If $A = C(G)$ then φ is "integrate against the Haar measure".
In general φ is not a trace, but if we set $\sigma_z = \rho_{iz,iz}$ on $\mathcal{A}$ then:

- for each $t \in \mathbb{R}$, σ_t is $*$-automorphism and so extends to A; it leaves φ invariant.
- for $a, b \in \mathcal{A}$ we have that

$$\varphi\big(ab\big) = \varphi\big(b\sigma_{-i}(a)\big) \qquad (a, b \in \mathcal{A}).$$

- This means that φ is a KMS state.

Moral: we can see an analytic property from von Neumann algebra theory in the corepresentation theory of (A, Δ).

Counter-example (Brown; Wang–Woronowicz)

Let $n \in \mathbb{N}$ (e.g. $n = 2$). Let A be the universal C*-algebra generated by elements $(u_{ij})_{i,j=1}^n$ subject to the relations which turn $u = (u_{ij}) \in \mathbb{M}_n(A)$ into a unitary.
Define $\Delta : A \to A \otimes A$ by

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$

(This exists, by universality, because right hand side is a unitary element of $\mathbb{M}_n(A \otimes A)$).
If (A, Δ) were a compact quantum group, then $\overline{u}$ would be, in particular, invertible. This is not the case...

Continued...

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then set

$$u' = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & I_{2n-4} \end{pmatrix} \in \mathbb{M}_{2n}.$$

Then u' is unitary, so by universality, there is a $*$-homomorphism $\pi : A \to \mathbb{M}_2$ such that

$$\pi \otimes I : A \otimes \mathbb{M}_n \to \mathbb{M}_2 \otimes \mathbb{M}_n \cong \mathbb{M}_{2n}; \quad (\pi \otimes I)(u) = u'.$$

Then calculation shows that if $\overline{u}$ is invertible, then $\overline{u'}$ is as well, because $(\pi \otimes I)(\overline{u}^{-1})$ would be the inverse. But $\overline{u'}$ is not invertible.

Interpretation

If S is a semigroup, and $\pi : S \to \mathbb{M}_n$ is a unitary representation, then π induces a (semi)group homomorphism $S \to U_n$.

- Get a $*$-homomorphism $\theta : C(U_n) \to C^b(S)$ with $(\theta \otimes \theta)\Delta_{U_n} = \Delta_S \theta$.
- 'Think about it' to see that $B = \theta(C(U_n))$ is the unital C*-subalgebra of $C^b(S)$ generated by the elements u_{ij}, where u is the corepresentation associated to π.

So this doesn't work for Quantum Semigroups: we just constructed (A, Δ) and a unitary corepresentation u such that the C*-algebra generated by the elements u_{ij}, in this case all of A, was not a (Compact Quantum) Group.

Theorem (Sołtan, Woronowicz)

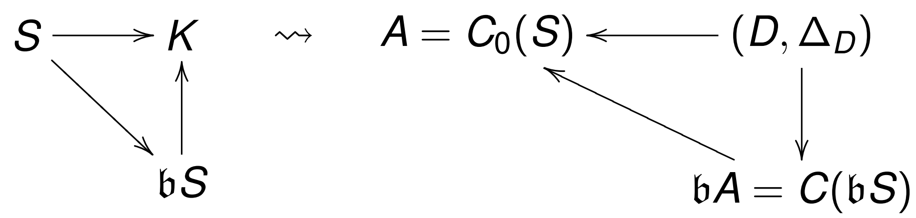
Let (A, Δ) be a quantum semigroup, let u be a corepresentation, and suppose that also $\overline{u}$ is invertible. If B is the C*-algebra generated by the u_{ij} in $M(A)$, then $(B, \Delta|_B)$ is a compact quantum group.

Sołtan's Quantum Bohr Compactification

Theorem

Let $\mathfrak{b}A$ be the union of all such B. Then $(\mathfrak{b}A, \Delta|_{\mathfrak{b}A})$ is a compact quantum group.

This compact quantum group is maximal:



So this gives a "quantum Bohr compactification".

Problem

How do you actually test if $\overline{u}$ invertible?

Locally compact quantum groups

If A is a non-unital C*-algebra, and $\Delta : A \rightarrow M(A \otimes A)$ a coassociative non-degenerate $*$-homomorphism, then seemingly it is not enough to ask just for "cancellation", but also to *assume* the existence of suitable generalisations of the left/right Haar measure.

- However, once this is done, one gets a very satisfactory theory (Kustermans–Vaes).
- In particular, given (A, Δ) we can form the "dual" quantum group $(\hat{A}, \hat{\Delta})$ which generalises Pontryagin duality.

$$A = C_0(G) \implies \hat{A} = C_r^*(G).$$

- We have $\hat{\hat{A}} = A$.
- A *discrete* quantum group is the dual of a compact quantum group. So $c_0(\Gamma)$ for discrete Γ, or $C^*(G)$ for compact G.

Compactifications of discrete quantum groups

Theorem (D., following Sołtan)

Let (A, Δ) *be a discrete quantum group, and let* u *be a finite-dimensional unitary corepresentation of* (A, Δ). *Then* $\overline{u}$ *is automatically invertible.*

Sketch proof.

Idea of Vaes, as used by Sołtan shows that it's enough to consider a "quotient" quantum group of (A, Δ) which is of "Kac type". This means that the antipode, the map which represents the group inverse, is bounded. □

Theorem (D.)

For a Kac algebra (A, Δ), *we have that* $\mathfrak{b}A$ *is the closure of the set of elements* $x \in M(A)$ *with* $\Delta(x)$ *a finite-rank tensor.*

Links with Banach algebras

Given a Banach algebra $\mathfrak{A}$, we turn $\mathfrak{A}^*$ into an $\mathfrak{A}$ bimodule via:

$$\langle a \cdot \mu, b \rangle = \langle \mu, ba \rangle, \quad \langle \mu \cdot a, b \rangle = \langle \mu, ab \rangle \qquad (a, b \in \mathfrak{A}, \mu \in \mathfrak{A}^*).$$

For $\mu \in \mathfrak{A}$ let

$$L_\mu : \mathfrak{A} \to \mathfrak{A}^*, \ a \mapsto \mu \cdot a.$$

Definition

We say that μ is *almost periodic* if L_μ is a compact operator.

If $\mathfrak{A} = L^1(G)$ for a locally compact group G, then $\mathfrak{A}^* = L^\infty(G)$, and the collection of almost periodic elements coincides with (the image of) $\mathfrak{b}C_0(G)$ inside $C^b(G) \subseteq L^\infty(G)$.

Stronger form of "compact"

Definition

For a Banach algebra $\mathfrak{A}$, say that $\mu \in \mathfrak{A}^*$ is "strongly almost periodic" if there is a sequence (T_n) of finite-rank right module maps $\mathfrak{A} \to \mathfrak{A}^*$ such that $\|L_\mu - T_n\| \to 0$.

So "compact" becomes "approximated by finite-ranks" (which for $L^1(G)$ is no change); and we also impose an "algebra" condition.
For a locally compact quantum group (A, Δ), there is a Banach algebra $L^1(A)$:

$$A = C_0(G) \implies L^1(A) = L^1(G), \quad A = C_r^*(G) \implies L^1(A) = A(G).$$

Then $L^1(A)^* = L^\infty(A)$ a von Neumann algebra which contains $M(A)$, and hence A.

Theorem

If (A, Δ) is a Kac algebra, then $\mathfrak{b}A$ is precisely the collection of strongly almost periodic elements of $L^1(A)^*$.

Future work

These techniques rely strongly on the fact that for a Kac algebra, the antipode S is bounded.

Claim

Let $u \in \mathbb{M}_n(A)$ be a corepresentation. If we know that $u_{ij}^* \in D(S)$, then $\overline{u}$ is invertible.

Claim

Let $x \in L^1(A)^*$ be strongly almost periodic. If we know that $x \in D(S)$, then $x \in \mathfrak{b}A$.

When $D(S) = L^1(A)^*$, as in the Kac case, we're done.
General problem: $D(S)$ is a bit mysterious.