How to make the bilateral shift weak*-continuous.

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Consider the bilateral shift S on $\ell^1(\mathbb{Z})$,

$$S(x)(n) = x(n-1) \qquad (x \in \ell^1(\mathbb{Z}), n \in \mathbb{Z}).$$

This is obviously weak*-continuous with respect to the predual $c_0(\mathbb{Z})$. Can it ever be weak*-continuous for another predual?

One class of *isometric* preduals for $\ell^1(\mathbb{Z})$ arise in the following way. Let K be a countable, locally compact Hausdorff space, and consider the Banach space $C_0(K)$. Then the dual is $C_0(K)'$, which may be identified with the space of regular measures on K, $M(K)$. As K is countable and every measure is countably additive, we see that $M(K) = \ell^1(K)$. By choosing some bijection between K and $\mathbb{Z}$, we have get an isometric isomorphism between the dual of $C_0(K)$ and $\ell^1(\mathbb{Z})$. However, a few moment's thought reveals that if S is weak*-continuous, then something odd is happening to the accumulation points in K. It hence seems unlikely that S will be weak*-continuous: indeed, we shall show later that S is only weak*-continuous when K is discrete

If S is weak*-continuous, then so is every operator in the algebra generated by S; it follows that the convolution product on the Banach algebra $\ell^1(\mathbb{Z})$ becomes separately weak*-continuous.

Runde defined such algebras to be *dual Banach algebras*: they are somehow a generalisation of von Neumann algebras.

The author has shown that dual Banach algebras are precisely the algebras which arise as weak*-closed subalgebras of $\mathcal{B}(E)$, for reflexive Banach spaces E. Here $\mathcal{B}(E)$ is the algebra of operators on E, which is a dual Banach algebra with respect to the predual $E\widehat{\otimes}E'$, the projective tensor product of E with E'. See "Dual Banach algebras: representations and injectivity", Studia Math. 178 (2007).

Big question: Does varying the predual alter the properties of the algebra? Indeed, *can* we vary the predual?

Of course, my interest is in the question: what are the algebraic preduals of $\ell^1(\mathbb{Z})$? However, it is nice to introduce the problem is a very concrete way, as it shows how basic the question really is.

Of course, it is well-known that von Neumann algebras have unique preduals. This is best stated as follows: if $\mathcal{M}$ is a von Neumann algebra, E is a Banach space and $\phi : \mathcal{M} \to E'$ is an *isometry*, then ϕ is weak*-continuous.

Pełczyński showed that $\ell^\infty(\mathbb{N})$ and $L^\infty[0,1]$ are isomorphic, so we really do need the condition "isometric".

Theorem (D.). Let $\mathcal{M}$ be a commutative von Neumann algebra, let $\mathcal{A}$ be a dual Banach algebra, and let $\phi : M \rightarrow \mathcal{A}$ be an algebra isomorphism. Then ϕ is weak*-continuous.

See the previously mentioned Studia paper. Here, of course, an *algebra isomorphism* is a Banach space isomorphism which is also an algebra homomorphism. So the "uniqueness of predual" property still holds if we ignore the involution, and consider merely bounded maps instead of isometric maps.

Theorem (D. & White). This holds for general von Neumann algebras $\mathcal{M}$.

This follows easily, as, in a sense, the weak*-topology on a von Neumann algebra is determined by its abelian subalgebras. Then observe that a maximal abelian subalgebra must be weak*-closed regardless of the predual.

We have a few other general statements about algebras with unique preduals. Of interest here is that for some *semigroups*, the semigroup algebra $\ell^1(S)$ can have a unique predual. This holds when $S = \mathbb{N}$ with the operation max, for example.

If E is a predual for a dual Banach algebra $\mathcal{A}$, then we have the canonical map

$$E \hookrightarrow E'' = \mathcal{A}',$$

and Runde showed that E must be a *submodule* of $\mathcal{A}'$.

By fixing E as a subspace of $\mathcal{A}'$, we have the useful property that if $F \subseteq \mathcal{A}'$ is another predual, then E and F give the same weak*-topology if and only if $E = F$ as subsets of $\mathcal{A}'$.

Furthermore, if E is any closed submodule of $\mathcal{A}'$, then E is a predual if and only if E separates the points of $\mathcal{A}$, and every functional on E is implemented by a member of $\mathcal{A}$. This is useful in calculations.

Back to $\ell^1(\mathbb{Z})$: notice that the dual $\ell^\infty(\mathbb{Z})$ is a C*-algebra, and the usual predual $c_0(\mathbb{Z})$ is a C*-algebra. So let us ask an easier question: if a predual $E \subseteq \ell^\infty(\mathbb{Z})$ is a C*-algebra, is $E = c_0(\mathbb{Z})$?

Theorem (D. & White). If $E \subseteq \ell^\infty(\mathbb{Z})$ is a predual for $\ell^1(\mathbb{Z})$ such that E is a C*-algebra, then $E = c_0(\mathbb{Z})$.

Proof: We have $E = C_0(\Omega)$, where $\Omega \subseteq E' = \ell^1(\mathbb{Z})$ is the *character space* of E.

We can check that each member of Ω must be a point mass in $\mathbb{Z}$, and so we get a map $\Omega \to \mathbb{Z}$.

As E is a predual, we can show that this map is a bijection.

So Ω becomes a group, but maybe not a discrete group. As E is a submodule, we can show that the group product on Ω is at least *separately* continuous.

Choi's trick: As Ω is locally compact, Hausdorff and countable, we can write it as the countable union of closed singletons. So Baire Category implies that for some $n_0 \in \mathbb{Z}$, $\{n_0\}$ is open. As the product is continuous, this shows that actually $\{n\}$ is open for all n; that is, Ω is discrete.

This is due to Yemon Choi. I mean "trick" to mean a nice, but unexpected, argument. It's nice to use the Baire Category theorem is such a setting!

Actually, we show that the above holds for any discrete group G in the place of $\mathbb{Z}$. Everything works up to showing that Ω becomes a group such that the product is separately continuous. Then Ellis's Theorem tells us that actually Ω is a topological group, as Ω is locally compact. An argument using the Haar measure then yields that Ω is discrete.

The Haar measure is not in general in $M(\Omega)$, but we can restrict the measure to a compact set. Then as $M(\Omega) = C_0(\Omega)' = E' = \ell^1(G)$, we have that the Haar measure, restricted to a compact set, is discrete. By left invariance, it follows that the Haar measure is a multiple of the counting measure, and so Ω is discrete.

It is well known that $\ell^1(G)$, as an algebra, does not determine G: for example, $\ell^1(C_4)$ and $\ell^1(C_2 \times C_2)$ are isomorphic.

Not, of course, isometrically. Infact, the *isometric* isomorphism class of $L^1(G)$ is an invariant of the group G. This is Wendel's Theorem (or a corollary thereof).

To better encode the group structure, we consider the following *coassociative* product:

$$\Gamma : \ell^1(G) \to \ell^1(G \times G), \quad \delta_g \mapsto \delta_{(g,g)}.$$

This of course is one of the central ideas behind quantum group theory, in all its variations. It is more common to have the coassociative product on $C_0(G)$ or $L^\infty(G)$, but in some sense both are dual to each other. See "Operator space tensor products and Hopf convolution algebras" by Effros and Ruan, J. Operator Theory, for a much more comprehensive study of such ideas.

It thus seems natural to insist further that this map is weak*-continuous.

If E is a predual for $\ell^1(G)$, then there is some completion of the tensor product $E \otimes E$ which is a predual for $\ell^1(G \times G)$. It hence makes sense to ask for a map

$$\Gamma_* : E \check{\otimes} E \to E, \text{ with } \Gamma_*' = \Gamma.$$

The tensor product to use is infact the *injective tensor product*, a fact hinted at by the notation used here.

A simple calculation shows that this exists if and only if E is a subalgebra of $\ell^\infty(G)$.

Theorem (D. & White)**.** Let E be a predual for $\ell^1(G)$ which is an algebra. Then $E = c_0(G)$.

Proof: We proceed as before, using the Gelfand Transform to get a contractive homomorphism $E \to C_0(\Omega)$. It is possible to easily extract that this must be an isometry onto its range.

We then have to use the *Principle of Local Reflexivity*. This tells us that $E'' = \ell^\infty(G)$ is "locally" like E, in some technical sense. After a fair amount of work, one can again show that every character on E is actually a point mass in $\ell^1(G)$. The rest of the proof follows as in the C*-case.

This gives some evidence that the "correct" way to think of $\ell^1(G)$ as a dual Banach algebra is with this coassociative product.

The situation for semigroups is more complicated. $\ell^1(\mathbb{N})$ has a unique predual which is an algebra, but we have not been able to perform the calculation for $\ell^1(S_2)$, for example. S_2 is the free semigroup on 2 generators. However, $\ell^1(\mathbb{N}, \max)$ has a unique predual in full generality.

This seems a little odd, as we generally tend to think of semigroups algebras has having a lot less structure than group algebras, and so we would expect it would be easier to find new preduals.

The "dual" objects to algebras $\ell^1(G)$, for a discrete group G, are algebras $A(H)$, for a compact groups H. Here $A(H)$ is the *Fourier algebra* as defined by Eymard. When H is abelian, $A(H) = \ell^1(\hat{H})$, and so the above applies.

For general compact H, we have that $A(H)$ is the dual of group C*-algebra $C^*(H)$, and the dual of $A(H)$ is the group von Neumann algebra $VN(G)$. When $E \subseteq VN(G)$ is an algebra, we can attempt to analyse E in terms of its *spectrum*, which is the space of primitive ideals, with the Hull-Kernel topology. This is a rather weak topology, which for example fails in general to be Hausdorff.

That said, $C^*(H)$ is a very simple C*-algebra; the irreducible representations of H are all finite-dimensional; the tensor product of representations (which corresponds to the product in $A(H)$) are well understood in terms of the *character theory* of H.

Theorem (D. & White). Let $E \subseteq VN(H)$ be a predual for $A(H)$ such that E is a C*-algebra, and the spectrum of E is Hausdorff. Then $E = C^*(H)$.

We hope to be able to prove that the spectrum of E is automatically Hausdorff. A key tool here is the Duans-Hoffman theorem, relating to continuous functions on the spectrum of E to the centre of the multiplier algebra of E.

In general, $\ell^1(\mathbb{Z})$ does not have a unique predual which makes the bilateral shift weak*-continuous. We have the following, for example.

Theorem (D., Haydon, Schlumprecht & White)**.** Let $J \subseteq \mathbb{Z}$ be a sufficiently lacunary set, and let $a \in \ell^1(\mathbb{Z})$ be a vector such that $\|a\| < 1$. Then there is a predual E for $\ell^1(\mathbb{Z})$ such that $\delta_n \to a$ weak*, as n tends through the set J.

Of course, $\delta_n \to 0$ weak* for the usual predual $c_0(\mathbb{Z})$.

For example, $J = \{2^n : n \in \mathbb{N}\}$ or $J = \{\pm n! : n \in \mathbb{N}\}$ will both suffice.

For the later choice, the involution on $\ell^1(\mathbb{Z})$ is also weak*-continuous, while it is not weak*-continuous for the first choice of J.

It should not surprise us that the involution on a general Banach $*$-algebra need have little connection to the algebra product.

Curiously, it appears that all the preduals we can currently construct are, purely as Banach spaces, isomorphic to c_0.

It would, however, probably be a foolhardy conjecture that every predual E of $\ell^1(\mathbb{Z})$ which makes the product weak*-continuous is such that $E \cong c_0$.