Multipliers and Abstract Harmonic Analysis

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Outline

1. Multiplier algebras; Dual Banach algebras
2. The Fourier algebra; Extending homomorphisms
3. Hopf convolution algebras

Multiplier algebras

Let A be an algebra. A multiplier of A is a pair (L, R) of maps $A \to A$ such that $aL(b) = R(a)b$ for $a, b \in A$. Let $M(A)$ be the collection of such maps, made into an algebra for the product $(L, R)(L', R') = (LL', R'R)$.

Henceforth assume that A is *faithful*: if $a \in A$ and $bac = 0$ for all $b, c \in A$, then $a = 0$. Then we can show that

$$L(ab) = L(a)b, \quad R(ab) = aR(b) \qquad (a, b \in A),$$

and furthermore, the map $A \to M(A)$,

$$a \mapsto (L_a, R_a), \quad L_a(b) = ab, R_a(b) = ba \qquad (a, b \in A),$$

is an injective algebra homomorphism.

Then A becomes an ideal in $M(A)$. If B is an algebra containing A as an ideal, we say that A is *essential* if $x \in B$ is such that $axb = 0$ for $a, b \in A$, then $x = 0$. Then B embeds into $M(A)$. In this sense, $M(A)$ is the "largest" algebra containing A as an essential ideal.

Banach algebras; Examples

If A is a Banach algebra, then a little closed graph argument shows that if $(L, R) \in M(A)$, then L and R are bounded. We norm $M(A)$ by regarding it as a subspace of $\mathcal{B}(A) \oplus_\infty \mathcal{B}(A)$.

If A is unital, then $A = M(A)$.

If A is a C*-algebra then so is $M(A)$. For a commutative C*-algebra $A = C_0(X)$, the multiplier algebra can be identified with $C^b(X)$, which in turn is $C(\beta X)$. So multiplier algebras are Stone-Cech compactifications.
Notice that $M(A)$ is rarely a von Neumann algebra.

Let E be a Banach algebra, and $A = \mathcal{K}(E)$ the compact operators on E. Then $M(A) = \mathcal{B}(E)$.
Notice that $\mathcal{B}(E)$ may or may not be a dual space.

For a locally compact group G, consider the algebra $L^1(G)$. Then $M(L^1(G)) = M(G)$ [Wendel's Theorem]. A bit of measure theory shows that $L^1(G)$ is an ideal in $M(G)$, and so we have an embedding $M(G) \to M(L^1(G))$. A bounded approximate identity argument gives that this surjects.
Notice that $M(G)$ is always a dual space (and indeed a dual Banach algebra).

Dual Banach algebras

Let A be a Banach algebra which is the dual Banach space of A_* say. We say that A is a dual Banach algebra (for A_*) if the product is separately weak*-continuous.

Let's assemble some ingredients. Let A be a Banach algebra such that $\{ab : a, b \in A\}$ is linearly dense in A. Let (B, B_*) be a dual Banach algebra such that:

- we have an isometric homomorphism $\iota : A \to B$;
- $\iota(A)$ is an (essential) ideal in B;
- the resulting map $B \to M(A)$ injects.

We'll construct a predual for $M(A)$.

If you are interested in the one-sided case, compare with [Selivanov], Monatsh. Math. (1999).

The construction

Consider

$$X = (A \widehat{\otimes} B_*) \oplus_1 (A \widehat{\otimes} B_*) \quad \text{so that} \quad X^* = \mathcal{B}(A, B) \oplus_\infty \mathcal{B}(A, B).$$

Let $Y \subseteq X$ be the linear span of

$$\big(b \otimes \mu \cdot \iota(a)\big) \oplus \big(-a \oplus \iota(b) \cdot \mu\big) \qquad (a, b \in A, \mu \in B_*).$$

Then $Y^\perp \subseteq X^*$ is a weak*-closed subspace with predual X/Y. A calculation shows that

$$(T, S) \in Y^\perp \ \Leftrightarrow \ \iota(a)T(b) = S(a)\iota(b) \qquad (a, b \in A).$$

Now argue that as products are dense in A, actually $T(A), S(A) \subseteq \iota(A)$, and so we really have maps $L, R : A \to A$ with $T = \iota L, S = \iota R$. But then $(L, R) \in M(A)$; so we've shown that $M(A) \cong Y^\perp$.

A final, slightly technical, check shows that $M(A)$ is indeed a dual Banach algebra.

Uniqueness

Following the construction through the weak*-topology on $M(A)$ satisfies: a bounded net (L_α, R_α) in $M(A)$ is weak*-null if and only if

$$\lim_\alpha \langle \iota L_\alpha(a), \mu \rangle + \langle \iota R_\alpha(b), \lambda \rangle = 0 \qquad (a, b \in A, \mu, \lambda \in B_*).$$

Let $\theta : B \to M(A)$ be the map induced by $\iota : A \to B$. Then there is one and only one weak*-topology on $M(A)$ such that:

- $M(A)$ is a dual Banach algebra;
- for a bounded net (b_α) in B, we have that (b_α) is weak* null in B if and only if $(\theta(b_\alpha))$ is weak* null in $M(A)$.

So what we've done is taken a dual Banach algebra B which isn't quite large enough to be all of $M(A)$, and boot-strapped the weak*-topology from B to $M(A)$.

The Fourier algebra

Let G be a locally compact group, and let λ be the left-regular representation of G on $L^2(G)$:

$$\lambda(s)\xi : t \mapsto \xi(s^{-1}t) \qquad (s, t \in G, \xi \in L^2(G)).$$

Let $VN(G)$ be the group von Neumann algebra, which is generated by $\{\lambda(s) : s \in G\}$.

There exists a normal $*$-homomorphism $\Delta : VN(G) \to VN(G \times G)$ which satisfies $\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s)$. This exists, as we can define a unitary $W \in \mathcal{B}(L^2(G \times G))$ by $W\xi(s, t) = \xi(ts, t)$, and then

$$\Delta(x) = W^*(1 \otimes x)W \qquad (x \in VN(G)),$$

does the job.

Let $A(G)$ be the predual of $VN(G)$. As Δ is normal, for $\omega, \sigma \in A(G)$, there exists $\omega\sigma \in A(G)$ such that

$$\langle \Delta(x), \omega \otimes \sigma \rangle = \langle x, \omega\sigma \rangle \qquad (x \in VN(G)).$$

Thus we've turned $A(G)$ into a Banach algebra.

As a function algebra

As $\{\lambda(s) : s \in G\}$ generates $VN(G)$, an element $\omega \in A(G)$ is uniquely determined by $\{\langle\lambda(s), \omega\rangle : s \in G\}$ so we can think of ω as a function $G \to \mathbb{C}; s \mapsto \omega(s) = \langle\lambda(s), \omega\rangle$.

Then the product on $A(G)$ is just the pointwise product, as

$$(\omega\sigma)(s) = \langle\Delta(\lambda(s)), \omega \otimes \sigma\rangle = \langle\lambda(s) \otimes \lambda(s), \omega \otimes \sigma\rangle = \omega(s)\sigma(s).$$

Alternatively, starting with λ, we could integrate this to get a $*$-homomorphism $\lambda : L^1(G) \to \mathcal{B}(L^2(G))$. Then the norm closure of the image is $C_r^*(G)$, the (reduced) group C*-algebra. Then Δ restricts to give a map

$$\Delta : C_r^*(G) \to M\big(C_r^*(G \times G)\big).$$

Using this, we turn $C_r^*(G)^*$ into a commutative, dual Banach algebra, $B_r(G)$ the (reduced) Fourier-Stieltjes algebra.

If G is abelian with dual group $\hat{G}$, then

$$VN(G) \cong L^\infty(\hat{G}), \quad A(G) \cong L^1(\hat{G}), \quad C_r^*(G) \cong C_0(\hat{G}), \quad B_r(G) \cong M(\hat{G}).$$

Multipliers of $A(G)$

By analogy with the abelian case, we might hope that $M(A(G)) = B_r(G)$ always. However, $B_r(G)$ is unital if and only if G is amenable [Leptin, Cowling?].

As $A(G)$ is commutative, it follows that for $(L, R) \in M(A(G))$ we have $L = R$, and that actually

$$M(A(G)) = \{T : A(G) \to A(G) : T(\omega\sigma) = T(\omega)\sigma \ (\omega, \sigma \in A(G))\}.$$

Indeed, we can actually identify $M(A(G))$ with a space of functions:

$$M(A(G)) = \{f \in C^b(G) : f\omega \in A(G) \ (\omega \in A(G))\}.$$

We do always have that $A(G) \to B_r(G) = C_r^*(G)^*$ isometrically (Kaplansky density), and that $A(G)$ is an ideal in $B_r(G)$ (the Fell absorption principle). A little check shows that $A(G)$ is essential in $B_r(G)$. Thus we can run our programme, and $M(A(G))$ is a dual Banach algebra.

This was first shown by [De Canniere, Haagerup]. I think it's nice that we don't really need to know very much about the structure of $A(G)$ at all. Also, this construction happily extends to locally compact quantum groups.

Operator spaces; Completely bounded maps

Given a map $T : A \to B$ between C*-algebras, we can dilate T to a map

$$T \otimes \iota_n : A \otimes M_n = M_n(A) \to B \otimes M_n = M_n(B)$$

between the matrix algebras over A and B. If $\sup_n \|T \otimes \iota_n\| < \infty$, then T is *completely bounded*.

Given a multiplier $T \in M(A(G))$, if $T^* : VN(G) \to VN(G)$ is completely bounded, then T is completely bounded, $T \in M_{cb}A(G)$.

Of course, the definition of completely bounded makes sense for operators between subspaces of C*-algebras, and this leads to the notion of an *operator space*. The category of operators spaces and completely bounded maps is nicely behaved, and we can run our construction again, showing that $M_{cb}A(G)$ is a dual Banach algebra.

Self-induced algebras and modules

Let $A\widehat{\otimes}_A A$ be the quotient of $A\widehat{\otimes}A$ by the linear closure of

$$\{ab \otimes c - a \otimes bc : a, b, c \in A\}.$$

The product map $\pi : A\widehat{\otimes}A \to A; a \otimes b \mapsto ab$ respects this quotient, so we get a map $A\widehat{\otimes}_A A \to A$. If this is an isomorphism, then A is *self-induced*.

Similarly, for a left A-module E, we can form $A\widehat{\otimes}_A E$, and we say that E is *induced* if the product map implements an isomorphism $A\widehat{\otimes}_A E \cong E$.

Let $\theta : A \to M(B)$ be a homomorphism: we say that this is *non-degenerate* if $\{\theta(a_1)b\theta(a_2) : a_1, a_2 \in A, b \in B\}$ is linearly dense in B. We can turn B into an A-bimodule by setting

$$a \cdot b = \theta(a)b, \quad b \cdot a = b\theta(a) \qquad (a \in A, b \in B).$$

Then θ is non-degenerate if B is an essential A-module.

[Johnson]$\Rightarrow$ that if A has a bounded approximate identity, then any non-degenerate homomorphism extends to $\theta : M(A) \to M(B)$.

Extending homomorphisms in the induced case

Let $\theta : A \to M(B)$ be a homomorphism which turns B into an induced A-bimodule. We can extend θ to $M(A)$ as follows. Given $(L, R) \in M(A)$, we define $L' : B \to B$ by

$$\begin{array}{ccccc} B & \xleftarrow{\cong} & A\widehat{\otimes}_A B & \twoheadleftarrow & A\widehat{\otimes} B \\ \downarrow L' & & \vdots & & \downarrow L\otimes\iota \\ B & \xleftarrow{\cong} & A\widehat{\otimes}_A B & \twoheadleftarrow & A\widehat{\otimes} B \end{array}$$

This commutes, as $L \otimes \iota$ maps $N = \overline{\text{lin}}\{a_1 a_2 \otimes b - a_1 \otimes \theta(a_2)b : a_1, a_2 \in A, b \in B\}$ into itself. This follows, as

$$L(a_1 a_2) \otimes b - L(a_1) \otimes \theta(a_2)b = L(a_1)a_2 \otimes b - L(a_1) \otimes \theta(a_2)b \in N.$$

Similarly we form $R' : B \to B$, check that $(L', R') \in M(A)$, and that $(L, R) \mapsto (L', R')$ is homomorphism.

Example of an induced algebra

[Rieffel; Forrest, Lee, Samei]$\Rightarrow$ If A has a bounded approximate identity, then A is self-induced and any essential module is induced.

We now work in the category of Operator Spaces. We'll show that $A(G)$ is always self-induced: we want that $A(G)\widehat{\otimes}A(G)/N = A(G \times G)/N$ is isomorphic to $A(G)$ under the product map. Dualising, we want that

$$VN(G) \xrightarrow{\Delta} N^\perp \subseteq VN(G)\overline{\otimes}VN(G) = VN(G \times G)$$

is an isomorphism. All we need to prove is that it's onto. However,

$$\begin{aligned} N^\perp &= \{x \in VN(G)\overline{\otimes}VN(G) : \langle x, ab \otimes c - a \otimes bc\rangle = 0\} \\ &= \{x \in VN(G)\overline{\otimes}VN(G) : \langle(\Delta \otimes \iota)x - (\iota \otimes \Delta)x, a \otimes b \otimes c\rangle = 0\} \\ &= \{x \in VN(G)\overline{\otimes}VN(G) : (\Delta \otimes \iota)x = (\iota \otimes \Delta)x\}. \end{aligned}$$

We can then check that the *support* of $x \in N^\perp$ must be contained in the diagonal $\{(s, s) : s \in G\} \subseteq G \times G$. As this is a set of synthesis, by [Herz; Takesaki, Tatsumma] we have that $x \in \Delta(VN(G))$, so we're done.

Other extensions

For a homomorphism $\theta : A \to M(B)$, we can extend to $M(A)$ when:

- A has a bounded approximate identity, and θ is non-degenerate;
- B becomes an induced A-bimodule;
- [Ilie, Stokke] A has a bounded approximate identity, and $M(B)$ is a dual Banach algebra.

For more on induced algebras, see work of [Gronbaek].

Haagerup tensor products

For a C*-algebra A, we define a norm $\|\cdot\|_h$ on $A \otimes A$ by

$$\|\tau\|_h = \inf\Big\{\Big\|\sum_k a_k a_k^*\Big\|\Big\|\sum_k b_k^* b_k\Big\| : \tau = \sum_k a_k \otimes b_k\Big\}.$$

Write $A \otimes^h A$ for the completed tensor product.

If $A \subseteq \mathcal{B}(H)$ then the (maybe infinite) column vector $b = (b_1, b_2, \cdots)^T$ can be regarded as a map $H \to H^{(\infty)}$, and then $b^*b = \sum_k b_k^* b_k$. Similarly the row vector $a = (a_1, a_2, \cdots)$ is such that $aa^* = \sum_k a_k a_k^*$. Then notice that

$$ab = \sum_k a_k b_k, \quad \|ab\| \leq \|a\|\|b\| = \|aa^*\|^{1/2}\|b^*b\|^{1/2} = \Big\|\sum_k a_k a_k^*\Big\|\Big\|\sum_k b_k^* b_k\Big\|.$$

So multiplication $A \otimes^h A \to A; a \otimes b \mapsto ab$ is (completely) contractive. (Multiplication from $A \otimes_{\min} A$ is rarely even bounded).

Haagerup tensor products (cont.)

The same definition works to form $A \otimes^h B$. The Haagerup tensor product is injective (and projective). So it makes perfect sense on Operator Spaces as well. The Haagerup tensor product is then *self-dual* in the sense that

$$E^* \otimes^h F^* \subseteq (E \otimes^h F)^* \text{ (completely) isometrically,}$$

for Operator Spaces E and F.

For a von Neumann algebra M, we define the *extended Haagerup tensor product* by

$$M \otimes^{eh} M = (M_* \otimes^h M_*)^*.$$

(This can also be defined as before, but with the sums $\sum_k a_k a_k^*$, and so forth, being interpreted in the σ-weak topology, not the norm topology).

For an operator space E, we define

$$E \otimes^{eh} E = (E^* \otimes^h E^*)^*_\sigma,$$

the separately-weak*-continuous functionals $E^* \otimes^h E^* \to \mathbb{C}$.

Hopf convolution algebras

I introduced the Fourier algebra by:

- defining a von Neumann algebra $VN(G)$;
- defining a "coproduct", a normal $*$-homomorphism $\Delta : VN(G) \to VN(G)\overline{\otimes}VN(G)$ with $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.
- then Δ induces the algebra structure on $A(G) = VN(G)_*$.

Can we do everything at the level of $A(G)$? We'd need a map

$$m : A(G) \to A(G) \otimes A(G) \quad \text{for some tensor product}$$

whose adjoint $m : VN(G) \otimes VN(G) \to VN(G)$ was the product on $VN(G)$.

[Quigg] tried to do with with the projective tensor product, but that only works (in any sense) if G is abelian by finite.

We could try the Haagerup tensor product $A(G) \otimes^h A(G)$, as then the adjoint is the product map $VN(G) \otimes^{eh} VN(G) \to VN(G)$, which is (completely) contractive.

The problem

However, this is too small: even if G is abelian but non-compact, then $A(G) \otimes^h A(G)$ won't work.

[Effros+Ruan] solve this by working with $A(G) \otimes^{eh} A(G)$. So we get a map $m : A(G) \to A(G) \otimes^{eh} A(G)$. For a good analogy, this should be a homomorphism.

To turn $A(G) \otimes^{eh} A(G)$ into an algebra, we use the *shuffle map*

$$S : (A(G) \otimes^{eh} A(G))\widehat{\otimes}(A(G) \otimes^{eh} A(G)) \to (A(G)\widehat{\otimes}A(G)) \otimes^{eh} (A(G)\widehat{\otimes}A(G))$$

which sends $(a \otimes b) \otimes (c \otimes d)$ to $(a \otimes c) \otimes (b \otimes d)$. Then the product on $A(G) \otimes^{eh} A(G)$ is given by:

$$\begin{array}{ccc} (A(G) \otimes^{eh} A(G))\widehat{\otimes}(A(G) \otimes^{eh} A(G)) & \xrightarrow{S} & (A(G)\widehat{\otimes}A(G)) \otimes^{eh} (A(G)\widehat{\otimes}A(G)) \\ & & \Big\downarrow {\scriptstyle \Delta_* \otimes \Delta_*} \\ & & A(G) \otimes^{eh} A(G) \end{array}$$

Using multipliers

Remember that, if I don't like von Neumann algebras, I can work with $C_r^*(G)$. However, here Δ now maps from $C_r^*(G)$ to the multiplier algebra $M(C_r^*(G) \otimes_{\min} C_r^*(G))$.

So can we analogously replace $A(G) \otimes^{eh} A(G)$ by, for example, $M_{cb}(A(G) \otimes^h A(G))$? (So, if G is compact, then actually we should be able to work with $A(G) \otimes^h A(G)$).

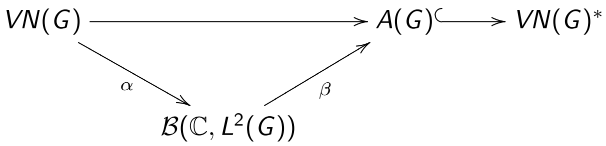
We can indeed do so. Firstly note that $A(G) \otimes^h A(G) \subseteq A(G) \otimes^{eh} A(G)$ (completely) isometrically, and so it follows that $A(G) \otimes^h A(G)$ is a subalgebra of $A(G) \otimes^{eh} A(G)$. The strategy will be to show that $m : A(G) \to A(G) \otimes^{eh} A(G)$ maps into the *idealiser*

$$\{\tau \in A(G) \otimes^{eh} A(G) : \tau\sigma \in A(G) \otimes^h A(G) \ (\sigma \in A(G) \otimes^h A(G))\}.$$

If we can do this, then we obviously have a map $A(G) \to M_{cb}(A(G) \otimes^h A(G))$.

The proof

An alternative description of $(E \otimes^h F)^*$ is those maps $F \to E^*$ which cb-factor through a *column* Hilbert space. For $a, b, c \in A(G)$, consider $m(a)(b \otimes c) \in A(G) \otimes^{eh} A(G) = \big(VN(G) \otimes^h VN(G)\big)^*_\sigma$; let's see how to view this as a map which factors through the column $L^2(G) = \mathcal{B}(\mathbb{C}, L^2(G))$:



Let $a \in A(G)$ be the normal functional $\langle x, a\rangle = (x\xi_0|\eta_0)$ for $x \in VN(G)$. Then we have

$$\alpha(x) = (c \cdot x)(\xi_0) \qquad (x \in VN(G)),$$
$$\beta(\xi) = \omega_{\xi,\eta_0} b \qquad (\xi \in L^2(G)),$$

where $\omega_{\xi,\eta_0} : VN(G) \to \mathbb{C}; x \mapsto (x\xi|\eta_0)$.

The proof continued

Let's think more about $\alpha : VN(G) \to L^2(G); x \mapsto (c \cdot x)(\xi_0)$. Let $c = \omega_{\xi_1,\eta_1}$, so that

$$\big(\alpha(x)\big|\eta\big) = \langle c \cdot x, \omega_{\xi_0,\eta}\rangle = \big(\Delta(x)\xi_0 \otimes \xi_2 \big| \eta \otimes \eta_2\big) = \big((1 \otimes x)W(\xi_0 \otimes \xi_2)\big| W(\eta \otimes \eta_2)\big).$$

Let (e_i) be an orthonormal basis for $L^2(G)$, so we can write

$$W(\xi_0 \otimes \xi_2) = \sum_i e_i \otimes \phi_i,$$

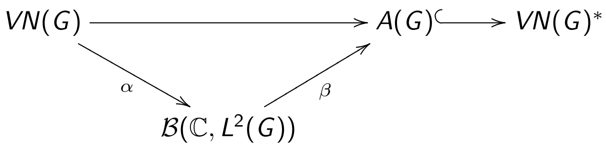
for some (ϕ_i). Then, letting $\sigma : L^2(G) \otimes L^2(G) \to L^2(G) \otimes L^2(G)$ be the swap map,

$$\big(\alpha(x)\big|\eta\big) = \sum_i \big(\sigma W^*(e_i \otimes x(\phi_i))\big|\eta_2 \otimes \eta\big) = \sum_i \big((\omega_{e_i,\eta_2} \otimes \iota)(\sigma W^*)x\phi_i\big|\eta\big).$$

Now, the key idea is that $(\omega_{e_i,\eta_2} \otimes \iota)(\sigma W^*)$ is a compact operator, which we can approximate by finite-ranks. So, being careful, we can approximate α, in the cb-norm, by a finite-rank map.

Finishing up

Recall we had $a, b, c \in A(G)$, and we viewed $m(a)(b \otimes c)$ as:



We can cb-norm approximate α by a finite-rank map, so we can approximate $m(a)(b \otimes c)$, in the extended Haagerup norm, by a finite-rank tensor in $A(G) \otimes A(G)$. As $A(G) \otimes^h A(G)$ is closed in $A(G) \otimes^{eh} A(G)$, it follows that $m(a)(b \otimes c) \in A(G) \otimes^h A(G)$, as required.

Various open problems

We have a complete contraction $A(G) \to C_0(G)$ and so also complete contractions

$$A(G \times G) = A(G)\widehat{\otimes}A(G) \to A(G) \otimes^h A(G) \to C_0(G) \otimes^h C_0(G)$$
$$\to C_0(G) \otimes_{\min} C_0(G) = C_0(G \times G).$$

So we can view $\mathfrak{A} = A(G) \otimes^h A(G)$ as an algebra of functions on $G \times G$. A result of [Gelbaum, Robbins] easily generalises to show that the spectrum of $\mathfrak{A}$ is $G \times G$.

- What sort of spectral synthesis properties does $\mathfrak{A}$ have?
- When is $m : A(G) \to M_{cb}(A(G) \otimes^h A(G))$ induced?
- Can we otherwise form an extension $M_{cb}A(G) \to M_{cb}(A(G) \otimes^h A(G))$ when G is non-amenable?

Is there any use of the fact that $A(G)$ is a (completely contractive) self-induced algebra? For a general locally compact quantum group, do we have $\{x \in L^\infty(\mathbb{G}) : (\Delta \otimes \iota)x = (\iota \otimes \Delta)x\} = \Delta(L^\infty(\mathbb{G}))$?