Positive definite functions on locally compact quantum groups

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Bochner's Theorem

Theorem (Herglotz, Bochner)

$f \in C^b(\mathbb{R})$ is positive definite if and only if f is the Fourier transform of a positive, finite Borel measure on $\mathbb{R}$.

Recall that f is positive definite if and only if, for $s_1, \cdots, s_n \in \mathbb{R}$ the matrix $(f(s_i^{-1} s_j))$ is positive (semi-definite). That is,

$$\sum_{i,j=1}^{n} a_j \overline{a_i} f(s_i^{-1} s_j) \geq 0 \qquad ((a_j)_{j=1}^n \subseteq \mathbb{C}).$$

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Bochner's Theorem, general case

Recall that for a locally compact abelian group G, we have the Pontryagin dual $\widehat{G}$, the collection of continuous characters $\phi : G \to \mathbb{T}$, with pointwise operations, and the compact-open topology.

Theorem (Bochner, 1932)

$f \in C^b(G)$ is positive definite if and only if f is the Fourier transform of a positive, finite Borel measure on $\widehat{G}$.

Group C*-algebras

- Recall that we turn $L^1(G)$ into a Banach $*$-algebra for the convolution product. The group C^*-algebra $C^*(G)$ is the universal C^*-completion of $L^1(G)$.
- We have bijections between:
  - unitary representations of G;
  - $*$-representations of $L^1(G)$;
  - $*$-representations of $C^*(G)$.
- Then the adjoint of $L^1(G) \to C^*(G)$ allows us to identify $C^*(G)^*$ with a (non-closed) subspace of $L^\infty(G)$.
- A bit of calculation shows that we actually get a subspace of $C^b(G)$ (or even uniformly continuous functions on G).
- Write $B(G)$ for this space— it is a subalgebra of $C^b(G)$. The multiplication follows by tensoring respresentations. Get the "Fourier-Stieltjes algebra".

Non-commutative generalisations

- If G is abelian, then the Fourier transform gives a unitary

$$\mathcal{F} : L^2(G) \to L^2(\widehat{G}).$$

- Then conjugating by $\mathcal{F}$ gives a $*$-isomorphism

$$C^*(G) \cong C_0(\widehat{G}) \implies B(G) \cong M(\widehat{G}).$$

- Define the positive part of $B(G)$ to be the positive functionals on $C^*(G)$. This is not the same as being "positive" in $C^b(G)$.

Theorem (Abstract Bochner)

The continuous positive definite functions on G form precisely the positive part of $B(G)$.

For abelian G this is just a re-statement of Bochner's Theorem. But it's true for arbitrary G.

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Sketch proof

Theorem (Abstract Bochner)

The continuous positive definite functions on G form precisely the positive part of $B(G)$.

- $f \in C^b(G)$ is positive definite if and only if $K(g,h) = f(g^{-1}h)$ defines a positive kernel of G.
- By GNS or Kolmogorov decomposition, there is a Hilbert space H and a map $\theta : G \to H$ with

$$\big(\theta(g)\big|\theta(h)\big) = K(g,h) = f(g^{-1}h) \qquad (g,h \in G).$$

- Then $\pi(g)\theta(h) = \theta(gh)$ extends by linearity to a unitary $\pi(g)$; the map $g \to \pi(g)$ is a unitary representation.
- Set $\xi = \theta(e) \in H$, so that

$$(\pi(g)\theta(h_1)|\theta(h_2)) = f(h_2^{-1}gh_1),$$

which shows both that π is weakly (and hence strongly) continuous, and that $f = \omega_\xi \circ \pi$ is a positive functional on $C^*(G)$.

The Fourier algebra

- The Fourier-Stieltjes algebra $B(G)$ is the space of coefficient functionals of all unitary representations of G.
- The *Fourier algebra* $A(G)$ is the the space of coefficient functionals of the (left) regular representation of G.
- Fell absorption shows that $A(G)$ is an ideal in $B(G)$.
- We identify $A(G)$ with a dense, non-closed subalgebra of $C_0(G)$.
- An alternative picture is that $A(G)$ forms a space of functionals on the *reduced* group C*-algebra $C_r^*(G)$.
- In fact, we get exactly the ultra-weakly continuous functionals, and so $A(G)$ is the predual of $VN(G) = C_r^*(G)''$.
- If G is abelian then $A(G) \cong L^1(\widehat{G})$.

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The Fourier algebra and multipliers

- A "multiplier" of $A(G)$ is a (continuous) function $F : G \to \mathbb{C}$ such that $Fa \in A(G)$ for each $a \in A(G)$.
- A Closed Graph argument shows that we get a bounded map $m_F : A(G) \to A(G); a \mapsto Fa$.
- We say that F is a *completely bounded multiplier* if $m_F^* : VN(G) \to VN(G)$ is completely bounded (matrix dilations are uniformly bounded).

Theorem (Gilbert, Herz, Jolissaint)

F is a completely bounded multiplier if and only if there is a Hilbert space H and continuous maps $\alpha, \beta : G \to H$ such that

$$F(g^{-1}h) = \big(\alpha(g)\big|\beta(h)\big).$$

Summary of Bochner's Theorem

The following are all equivalent notions:

1. Positive functionals on the Banach $*$-algebra $L^1(G)$;
2. Positive functionals on $C^*(G)$;
3. Completely positive multipliers of $A(G)$;
4. Positive definite functions on G.

The equivalence of (3) and (4) was first noted by de Canniere and Haagerup.

The von Neumann algebra of a group

Let G be a locally compact group $\implies$ has a Haar measure $\implies$ can form the von Neumann algebra $L^\infty(G)$ acting on $L^2(G)$.

Have lost the product of G. We recapture this by considering the injective, normal $*$-homomorphism

$$\Delta : L^\infty(G) \to L^\infty(G \times G); \quad \Delta(F)(s,t) = F(st).$$

The pre-adjoint of Δ gives the usual convolution product on $L^1(G)$

$$L^1(G) \otimes L^1(G) \to L^1(G); \quad \omega \otimes \tau \mapsto (\omega \otimes \tau) \circ \Delta = \omega \star \tau.$$

The map Δ is implemented by a unitary operator W on $L^2(G \times G)$,

$$W\xi(s,t) = \xi(s, s^{-1}t), \quad \Delta(F) = W^*(1 \otimes F)W,$$

where $F \in L^\infty(G)$ identified with the operator of multiplication by F.

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The other von Neumann algebra of a group

Let $VN(G)$ be the von Neumann algebra acting on $L^2(G)$ generated by the left translation operators $\lambda_s, s \in G$.

The predual of $VN(G)$ is the Fourier algebra $A(G)$.

Again, there is $\Delta : VN(G) \to VN(G)\overline{\otimes}VN(G)$ whose pre-adjoint induces the product on $A(G)$,

$$\Delta(\lambda_s) = \lambda_s \otimes \lambda_s.$$

That such a Δ exists follows as

$$\Delta(x) = \hat{W}^*(1 \otimes x)\hat{W} \quad \text{with} \quad \hat{W} = \sigma W^* \sigma,$$

where $\sigma : L^2(G \times G) \to L^2(G \times G)$ is the swap map.

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How W governs everything

The unitary W is multiplicative; $W_{12}W_{13}W_{23} = W_{23}W_{12}$; and lives in $L^\infty(G)\overline{\otimes}VN(G)$.

The map

$$L^1(G) \to VN(G); \quad \omega \mapsto (\omega \otimes \iota)(W),$$

is the usual representation of $L^1(G)$ on $L^2(G)$ by convolution. The image is σ-weakly dense in $VN(G)$, and norm dense in $C_r^*(G)$.

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The group inverse is represented by the antipode $S : L^\infty(G) \to L^\infty(G); S(F)(t) = F(t^{-1})$. We have that

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More non-commutative framework

- M a von Neumann algebra;
- Δ a normal injective $*$-homomorphism $M \to M\overline{\otimes}M$ with $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
- a "left-invariant" weight φ, with $(\iota \otimes \varphi)\Delta(\cdot) = \varphi(\cdot)1$ (in some loose sense).
- a "right-invariant" weight ψ, with $(\psi \otimes \iota)\Delta(\cdot) = \psi(\cdot)1$.

Let H be the GNS space for φ. There is a unitary W on $H \otimes H$ with

$$\Delta(x) = W^*(1 \otimes x)W, \quad M = \overline{\{(\iota \otimes \omega)(W) : \omega \in \mathcal{B}(H)_*\}}^{\sigma\text{-weak}}.$$

Again form $\widehat{W} = \sigma W^* \sigma$. Then

$$\widehat{M} = \overline{\text{lin}\{(\iota \otimes \omega)(\widehat{W})\}}^{\sigma\text{-weak}}$$

is a von Neumann algebra, we can define $\widehat{\Delta}(\cdot) = \widehat{W}^*(1 \otimes \cdot)\widehat{W}$ which is "coassociative". It is possible to define weights $\widehat{\varphi}$ and $\widehat{\psi}$.

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Antipode not bounded

Can again define

$$S\big((\iota \otimes \omega)(W)\big) = (\iota \otimes \omega)(W^*).$$

However, in general S will be an unbounded, σ-weakly-closed operator.

- We can factor S as $S = R \circ \tau_{-i/2}$.
- R is the "unitary antipode", a normal anti-$*$-homomorphism $M \to M$ which is an anti-homomorphism on M_*
- $S \circ * \circ S \circ * = \iota$.
- $\tau_{-i/2}$ is the analytic generator of a one-parameter automorphism group, the "scaling group", (τ_t) of M. Each τ_t also induces a homomorphism on M_*; equivalently, $(\tau_t \otimes \tau_t) \circ \Delta = \Delta \circ \tau_t$.

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Notation

- We tend to write $L^\infty(\mathbb{G})$ for M, write $L^2(\mathbb{G})$ for H, and write $L^1(\mathbb{G})$ for the predual of $L^\infty(\mathbb{G})$.
- Similarly, write $L^\infty(\widehat{\mathbb{G}})$ for $\widehat{M}$.
- If we take the norm closure of

$$\{(\iota \otimes \omega)(W) : \omega \in \mathcal{B}(H)_*\}$$

  then we obtain a C*-algebra, which we'll denote by $C_0(\mathbb{G})$.
- Δ restricts to a map $C_0(\mathbb{G}) \to M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ with, for example, $(1 \otimes a)\Delta(b) \in C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$ for $a, b \in C_0(\mathbb{G})$.
- Write $M(\mathbb{G})$ for $C_0(\mathbb{G})^*$; this becomes a Banach algebra for the adjoint of Δ.

Completely Bounded Multipliers

Definition

A "abstract left cb multiplier" (or "left cb centraliser") of $L^1(\mathbb{G})$ is a bounded linear map $L_* : L^1(\mathbb{G}) \to L^1(\mathbb{G})$ such that $L_*(\omega_1 \star \omega_2) = L_*(\omega_1) \star \omega_2$, and such that the adjoint $L = (L_*)^* : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ is completely bounded.

The left regular representation of $L^1(\mathbb{G})$ is the contractive map $\lambda : L^1(\mathbb{G}) \to C_0(\widehat{\mathbb{G}}); \omega \mapsto (\omega \otimes \iota)(W)$.

Definition

A "concrete left cb multiplier" (or a "represented left cb multplier") of $L^1(\mathbb{G})$ is an element $a \in L^\infty(\widehat{\mathbb{G}})$ such that there is L_* as above, with $\lambda(L_*(\omega)) = a\lambda(\omega)$.

Picture: An abstract multiplier of $A(G)$ is a right module homomorphism on $A(G)$, whereas a concrete multiplier is a continuous function on G multiplying $A(G)$ into itself. Notice that here these ideas coincide.

Completely Bounded Multipliers

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Concrete = Abstract

Theorem (Junge–Neufang–Ruan, D.)

For every (abstract) left cb multiplier L_* there is $a \in MC_0(\widehat{\mathbb{G}})$ with $\lambda(L_*(\omega)) = a\lambda(\omega)$.

Lemma (Aristov, Kustermans–Vaes, Meyer–Roy–Woronowicz)

Let N be a von Neumann algebra, and let $x \in L^\infty(\mathbb{G})\overline{\otimes}N$ with $(\Delta \otimes \iota)(x) \in L^\infty(\mathbb{G})\overline{\otimes}\mathbb{C}1\overline{\otimes}N$. Then $x \in \mathbb{C}1\overline{\otimes}N$.

Proof of Theorem, D..

Consider $X = (L \otimes \iota)(W)W^*$, and then calculate that $(\Delta \otimes \iota)(X) = X_{13}$. Key idea here is that L_* a right module homomorphism means that $\Delta \circ L = (L \otimes \iota) \circ \Delta$. By the lemma, there is $a \in L^\infty(\widehat{\mathbb{G}})$ with $X = 1 \otimes a$, that is, $(L \otimes \iota)(W) = (1 \otimes a)W$. Using the definition of λ, it follows that $\lambda(L_*(\omega)) = a\lambda(\omega)$. Furthermore, as $W \in M(C_0(\mathbb{G}) \otimes C_0(\widehat{\mathbb{G}}))$, it follows that

$$1 \otimes a = (L \otimes \iota)(W)W^* \in M(\mathcal{B}_0(L^2(\mathbb{G})) \otimes C_0(\widehat{\mathbb{G}})) \implies a \in MC_0(\widehat{\mathbb{G}}).$$ □

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Canonical extensions

Theorem (JNR)

Let L_* be a cb left multiplier. There is a normal cb extension of L to $\mathcal{B}(L^2(\mathbb{G}))$, say Φ, which is an $L^\infty(\widehat{\mathbb{G}})'$ module map. Indeed,

$$1 \otimes \Phi(\cdot) = W\big((L \otimes \iota)(W^*(1 \otimes \cdot)W)\big)W^*.$$

Sketch proof.

Define $T : \mathcal{B}(L^2(\mathbb{G})) \to L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(L^2(\mathbb{G}))$ using the formula on the right-hand-side. Show that $(\Delta \otimes \iota)T(x) = T(x)_{13}$, so the lemma again shows the existence of Φ. The rest is simple calculation. □

For an alternative proof working purely with $C_0(\mathbb{G})$ (and Hilbert C^*-modules) and only using the result of the previous slide, see [D., J. London Math. Soc.]

(Co)representations

Definition

A corepresentation of $\mathbb{G}$ on K is $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(K)$ with $(\Delta \otimes \iota)(U) = U_{13}U_{23}$. Usual to assume U is unitary.

If U has a right inverse, then an idea of Woronowicz (which follows an idea of Baaj–Skandalis) shows that $U \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(K))$.
As the antipode is unbounded, the usual way to define an involution on $L^1(\mathbb{G})$,

$$\langle x, \omega^\sharp \rangle = \overline{\langle S(x)^*, \omega \rangle} \qquad (x \in D(S) \subseteq L^\infty(\mathbb{G}), \omega \in L^1(\mathbb{G}))$$

is only densely defined, but we end up with a dense $*$-algebra, $L^1_\sharp(\mathbb{G})$.
Kustermans studied the universal C^*-enveloping algebra $C_0^u(\widehat{\mathbb{G}})$ which has all the behaviour of a quantum group, excepting that the invariant weights might fail to be faithful.

(Co)representations and the universal dual

Theorem (Kustermans)

There is a bijection between (unitary) corepresentations of $\mathbb{G}$ *and non-degenerate* $*$*-homomorphisms of* $C_0^u(\widehat{\mathbb{G}})$ *(or* $L^1_\sharp(\mathbb{G})$*).*

Indeed, there is a universal corepresentation $\widehat{\mathcal{V}} \in M(C_0(\mathbb{G}) \otimes C_0^u(\widehat{\mathbb{G}}))$, and then U bijects with $\phi : C_0^u(\widehat{\mathbb{G}}) \to \mathcal{B}(K)$ according to the relation

$$U = (\iota \otimes \phi)(\widehat{\mathcal{V}}).$$

(Advertisement: In [Brannan, D., Samei, Münster Journal Maths, to appear] we start a program of studying non-unitary corepresenations. It's very interesting to me what a corepresentation on a Banach (or Operator) space might be— the current theory is very "Hilbert space" heavy.)

Multipliers from (co)representations

Theorem

Let U be a unitary corepresentation on K, and let $\alpha, \beta \in K$. Then

$$L(\cdot) = (\iota \otimes \omega_{\alpha,\beta})(U(\cdot \otimes 1)U^*)$$

defines (the adjoint of) a left cb multiplier of $L^1(\widehat{\mathbb{G}})$. The element $a \in MC_0(\mathbb{G})$ "representing" this multiplier is $a = (\iota \otimes \omega_{\alpha,\beta})(U^*)$. If $\alpha = \beta$ we get a completely positive multiplier.

Notice the dual here— if $\mathbb{G} = G$ is commutative then this says that a unitary representation of G gives a cb multiplier of $A(G)$; if $\mathbb{G} = \widehat{G}$ is co-commutative this says that a unitary corpresentation of $\widehat{G}$, that is, a $*$-representation of $C_0(G)$, gives a cb multiplier of $L^1(G)$, that is, a measure on G.

Via consider universal quantum groups, we see that all these multipliers arise from functionals on $C_0^u(\widehat{\mathbb{G}})^*$. Not surprising, as $L^1(\widehat{\mathbb{G}})$ is an ideal in $C_0^u(\widehat{\mathbb{G}})^*$.

Completely positive multipliers are positive functionals on the dual

Theorem (D. 2012)

Let L_* be a completely positive multiplier of $L^1(\widehat{\mathbb{G}})$. Then L_* arises from a unitary corepresentation of $\mathbb{G}$, equivalently, from a positive functional in $C_0^u(\widehat{\mathbb{G}})^*$.

An unpublished result of Losert (see also Ruan) shows that the space of cb multipliers of $A(G)$ is equal to $B(G)$ (if and) only if G is amenable. So as a corollary, we see that the cb multipliers of $A(G)$ are equal to the span of the cp multipliers only if G is amenable.

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Some ideas of the proof

- First consider the adjoint $L : L^\infty(\widehat{\mathbb{G}}) \to L^\infty(\widehat{\mathbb{G}})$.
- Recall that we can extend this to a normal CP map Φ on $\mathcal{B}(L^2(\mathbb{G}))$ which is an $L^\infty(\mathbb{G})$-module map.
- Applying the Stinespring construction to $\Phi : \mathcal{K}(L^2(\mathbb{G})) \to \mathcal{B}(L^2(\mathbb{G}))$, and looking carefully at what you get, we find a family $(a_i)_{i\in I}$ in $L^\infty(\mathbb{G})'$ with

$$\sum_{i\in I} a_i^* a_i < \infty, \qquad \Phi(\cdot) = \sum_i a_i^*(\cdot)a_i.$$

- Let $H = \ell^2(I)$ with basis (e_i), and write $\nu_{\alpha,\beta} = \sum_i \langle a_i, \omega_{\alpha,\beta}\rangle e_i$.
- The family (a_i) is minimal in that such $\nu_{\alpha,\beta}$ span a dense subset of H.
- A similar result holds for cb module maps ([Smith], also [Blecher, Effros, Ruan]).

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Constructing the corepresentation

$$\Phi(\cdot) = \sum_{i \in I} a_i^*(\cdot) a_i, \qquad 1 \otimes \Phi(\cdot) = \widehat{W}\big((L \otimes \iota)(\widehat{W}^*(1 \otimes \cdot)\widehat{W})\big)\widehat{W}^*.$$

As $\widehat{W} = \sigma W^* \sigma$ and $\Delta(\cdot) = W^*(1 \otimes \cdot)W$, and using that Φ extends L, we can substitute the 1st equation into the 2nd, and find that

$$\sum_i a_i^*(\cdot) a_i \otimes 1 = \Phi(\cdot) \otimes 1 = \sum_i \Delta(a_i^*)(\cdot \otimes 1)\Delta(a_i).$$

Then putting this together, we find that there is an isometry U^* with

$$U^*(\xi \otimes \nu_{\alpha,\beta}) = \sum_i (\omega_{\alpha,\beta} \otimes \iota)\Delta(a_i)\xi \otimes e_i.$$

Recall that $\nu_{\alpha,\beta} = \sum_i \langle a_i, \omega_{\alpha,\beta} \rangle e_i$.

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- Formally $(\omega_{\xi,\eta} \otimes \iota)(U^*)\nu_{\alpha,\beta} = \sum_i \langle a_i, \omega_{\alpha,\beta} \star \omega_{\xi,\eta} \rangle e_i$.
- So if we think of H as some sort of Hilbert space completion of $L^1(\mathbb{G})$ (under the map $\Lambda : \omega_{\xi,\eta} \mapsto \nu_{\xi,\eta}$) then the (anti-)representation of $L^1(\mathbb{G})$ which U^* induces is $\omega_1 \cdot \Lambda(\omega_2) = \Lambda(\omega_2 \star \omega_1)$.
- Making this formal shows that $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ and $(\Delta \otimes \iota)(U) = U_{13}U_{23}$.
- Remains to show that U is unitary.

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Can recover our multiplier

There is $a_0 \in MC_0(\mathbb{G})$ "representing" the multiplier L_*. So $(1 \otimes a_0)\widehat{W} = (L \otimes \iota)(\widehat{W}) = (\Phi \otimes \iota)(\widehat{W}) = \sum_i (a_i^* \otimes 1)\widehat{W}(a_i \otimes 1)$. Re-arranging shows

$$\sum_i (1 \otimes a_i)\Delta(a_i) = a_0 \otimes 1.$$

Using this we can use Riesz to find $\alpha_0 \in H$ with

$$\Big(\sum_i \langle a_i, \omega\rangle e_i \Big| \alpha_0\Big) = \langle a_0, \omega\rangle \qquad (\omega \in L^1(\mathbb{G})).$$

It will then follows that

$$a_i = (\iota \otimes \omega_{\alpha_0, e_i})(U^*) \qquad (i \in I),$$

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$$L(\cdot) = \Phi(\cdot) = \sum_i a_i^*(\cdot)a_i = (\iota\otimes\omega_{\alpha_0})(U(\cdot\otimes 1)U^*).$$

Can recover our multiplier

There is $a_0 \in MC_0(\mathbb{G})$ "representing" the multiplier L_*. So $(1 \otimes a_0)\widehat{W} = (L \otimes \iota)(\widehat{W}) = (\Phi \otimes \iota)(\widehat{W}) = \sum_i (a_i^* \otimes 1)\widehat{W}(a_i \otimes 1)$. Re-arranging shows

$$\sum_i (1 \otimes a_i)\Delta(a_i) = a_0 \otimes 1.$$

Using this we can use Riesz to find $\alpha_0 \in H$ with

$$\Big(\sum_i \langle a_i, \omega \rangle e_i \Big| \alpha_0 \Big) = \langle a_0, \omega \rangle \qquad (\omega \in L^1(\mathbb{G})).$$

It will then follows that

$$a_i = (\iota \otimes \omega_{\alpha_0, e_i})(U^*) \qquad (i \in I),$$

and so

$$L(\cdot) = \Phi(\cdot) = \sum_i a_i^*(\cdot) a_i = (\iota \otimes \omega_{\alpha_0})(U(\cdot \otimes 1)U^*).$$

Linking the multipliers

We have two pictures of out multiplier: the map L (extended to Φ) and the representing element $a_0 \in MC_0(\mathbb{G})$.
Recall the scaling group (τ_t). There is a positive (unbounded) operator P such that $\tau_t(\cdot) = P^{it}(\cdot)P^{-it}$.

Theorem

Let $\xi, \eta \in D(P^{1/2})$ and $\alpha, \beta \in D(P^{-1/2})$. Consider the rank-one operator $\theta_{\xi,\eta}$. Then

$$\big(\Phi(\theta_{\xi,\eta})\alpha \big| \beta\big) = \langle \Delta(a_0), \omega_{\alpha,\eta} \otimes \overline{\omega^{\sharp}_{\xi,\beta}} \rangle.$$

So, at least in principle, we can compute Φ from a_0.

Extended Haagerup tensor product

Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and consider "tensor sums"

$$\sum_i x_i \otimes y_i \quad \text{with} \quad (x_i), (y_i) \subseteq M, \sum_i x_i x_i^* < \infty, \sum_i y_i^* y_i < \infty.$$

Write $M \overset{eh}{\otimes} M$ for the resulting linear space.
Let $\mathcal{CB}_{M'}(\mathcal{K}(H), \mathcal{B}(H))$ denote the space of completely bounded maps $\Phi : \mathcal{K}(H) \to \mathcal{B}(H)$ which are M' bimodule maps.
Then $M \overset{eh}{\otimes} M \cong \mathcal{CB}_{M'}(\mathcal{K}(H), \mathcal{B}(H))$ where

$$\sum_i x_i \otimes y_i \leftrightarrow \Phi \quad \Leftrightarrow \quad \Phi(\cdot) = \sum_i x_i(\cdot)y_i.$$

A little motivation

Have $a_0 \in L^\infty(\mathbb{G})$ and $\Phi : \mathcal{K}(L^2(G)) \to \mathcal{B}(L^2(\mathbb{G}))$ linked by

$$\big(\Phi(\theta_{\xi,\eta})\alpha\big|\beta\big) = \langle \Delta(a_0), \omega_{\alpha,\eta} \otimes \overline{\omega^\sharp_{\xi,\beta}} \rangle.$$

Then $\Phi(\theta) = \sum a_i \theta b_i$ say, so

$$\langle \Delta(a_0), \omega_{\alpha,\eta} \otimes \overline{\omega^\sharp_{\xi,\beta}} \rangle = \sum_i (b_i\alpha|\eta)(a_i\xi|\beta) = \sum_i \langle b_i \otimes S(a_i), \omega_{\alpha,\eta} \otimes \overline{\omega^\sharp_{\xi,\beta}} \rangle.$$

So, at least formally,

$$\Delta(a_0) = \sum_i b_i \otimes S(a_i),$$

or, very vaguely, $(\iota \otimes S^{-1})\Delta(a_0)$ is a completely bounded kernel.

Bochner for LCQGs

Consider $x \in L^\infty(\mathbb{G})$. Define:

1. x is a *positive definite function* if $\langle x^*, \omega \star \omega^\sharp \rangle \geq 0$ for $\omega \in L^1_\sharp(\mathbb{G})$;
2. x is the *Fourier-Stieltjes transform of a positive measure* if there is a unitary corepresentation $U \in M(C_0(\mathbb{G}) \otimes \mathcal{K}(H))$ and $\omega \in \mathcal{K}(H)^*_+$ with $x = (\iota \otimes \omega)(U)$;
3. x is a *completely positive multiplier* of $L^1(\widehat{\mathbb{G}})$, as already discussed;
4. x is a *completely positive definite function* if there is some CP $\Phi : \mathcal{K}(L^2(G)) \to \mathcal{B}(L^2(\mathbb{G}))$ with $(\Phi(\theta_{\xi,\eta})\alpha|\beta) = \langle x^*, \omega_{\xi,\beta} \star \omega^\sharp_{\eta,\alpha} \rangle$ for suitable α, β, ξ, η.

Notice we make no bimodule assumption in (4). That x or x^* appears is somehow related to S not being a $*$-map. Then (3) $\implies$ (4) and (3)$\Leftrightarrow$(2) we have seen; that (2) $\implies$ (1) and (4) $\implies$ (1) are easy.

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GNS constructions

1. x is a *positive definite function* if $\langle x^*, \omega \star \omega^\sharp \rangle \geq 0$ for $\omega \in L^1_\sharp(\mathbb{G})$;

The obvious thing to do is to try a GNS construction, but first we need:

Theorem (D., Salmi, 2013)

Give $L^1_\sharp(\mathbb{G})$ the norm $\|\omega\|_\sharp = \max(\|\omega\|, \|\omega^\sharp\|)$, under which $L^1_\sharp(\mathbb{G})$ is a Banac $*$-algebra. Then $\{\omega_1 \star \omega_2 : \omega_1, \omega_2 \in L^1_\sharp(\mathbb{G})\}$ is linearly dense in $L^1_\sharp(\mathbb{G})$.

Then we can produce a Hilbert space H, a map $\Lambda : L^1_\sharp(\mathbb{G}) \to H$ and a *non-degenerate* representation $\pi : L^1_\sharp(\mathbb{G}) \to \mathcal{B}(H)$ with

$$\pi(\omega_1)\Lambda(\omega_2) = \Lambda(\omega_1 \star \omega_2), \qquad (\Lambda(\omega_1)|\Lambda(\omega_2))_H = \langle x^*, \omega_2^\sharp \star \omega_1 \rangle.$$

Then [Kustermans] $\implies$ there is a unitary corepresentation U of $\mathbb{G}$ on H with $\pi(\cdot) = (\cdot \otimes \iota)(U)$.

CP Positive Definite is Fourier-Stieltjes transform

1. x is a *completely positive definite function* if there is some CP $\Phi : \mathcal{K}(L^2(G)) \to \mathcal{B}(L^2(\mathbb{G}))$ with $(\Phi(\theta_{\xi,\eta})\alpha|\beta) = \langle x^*, \omega_{\xi,\beta} \star \omega^\sharp_{\eta,\alpha}\rangle$ for suitable α, β, ξ, η.

Sketch proof.

- Show that the GNS space (H, π) constructed for x^* is isomorphic to the Stinespring space for Φ.
- This also shows that Φ is an $L^\infty(\mathbb{G})'$ bimodule map.
- Then use that the corepresentation U linked to π would agree with the corepresentation for Φ (if Φ actually came from a multiplier, which we don't know, yet).
- You reverse engineer from the corepresentation and Φ that, actually, Φ *was* given by a multiplier.
- Annoyingly seem to use complete *positivity* in an essential way!

□

Positive definite doesn't imply PD

1. x is a *positive definite function* if $\langle x^*, \omega \star \omega^\sharp \rangle \geq 0$ for $\omega \in L^1_\sharp(\mathbb{G})$;

If (1), then apply GNS to x and then applying Kustermans gives unitary corepresentation U. However, it's not clear how we find $\xi \in H$ with $x = (\iota \otimes \omega_\xi)(U)$.

Theorem

If $\mathbb{G} = \widehat{\mathbb{F}_2}$ (that is, $L^\infty(\mathbb{G}) = VN(\mathbb{F}_2)$) then there are positive definite x which do not come from positive functionals on $C_0^u(\widehat{\mathbb{G}})^* = \ell^1(\mathbb{F}_2)$.

Proof.

For a subset $E \subseteq \mathbb{F}_2$ define $A(E)$ to be the collection of functions in $A(\mathbb{F}_2)$ restricted to E, normed so that $A(\mathbb{F}_2) \to A(E)$ is a metric surjection.
Pick a Leinert set $E \subseteq \mathbb{F}_2$; so $A(E) \cong \ell^2(E)$. Then any positive $x \in \ell^2(E) \setminus \ell^1(E)$ gives the required counter-example. □

Co-amenability to the rescue

Recall that $\mathbb{G}$ is *coamenable* if $C_0(\mathbb{G}) = C_0^u(\mathbb{G})$, equivalently, if the counit is bounded on $C_0(\mathbb{G})$.
Then [Bedos, Tuset] shows this is equivalent to $L^1(\mathbb{G})$ having a bai.

Theorem

If $\mathbb{G}$ is coamenable, then $L^1_\sharp(\mathbb{G})$ has a bounded approximate identity in it's natural norm.

Proof.

To get something in $L^1_\sharp(\mathbb{G})$, we usually "smear" by the scaling group.
However, this would tend to destroy norm control of $\|\omega^\sharp\|$.
Instead, we adapt an idea of Kustermans (who attributes it to van Daele and Verding) and take the smeared limit in the "wrong direction".
This works essentially because we're trying to approximate the counit which is *invariant* for the scaling group. □

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Final Theorem

1. x is a *positive definite function* if $\langle x^*, \omega \star \omega^\sharp \rangle \geq 0$ for $\omega \in L^1_\sharp(\mathbb{G})$;
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Theorem (D.–Salmi, 2013)

Conditions (2)–(4) are equivalent, and imply (1). If $\mathbb{G}$ is coamenable, then they are all equivalent.