Ring-theoretical infiniteness and ultrapowers

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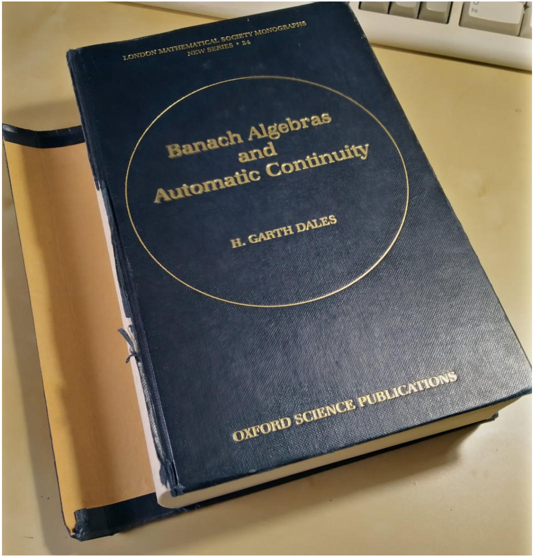
Garth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Banach Algebras
and
Automatic Continuity
H. GARTH DALES
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The plan

(In my humble opinion . . .) some links between my interests and Garth's work are the following:

- General theory of Banach Algebras;
- Compare and contrast to the theory of Operator Algebras;
- Interesting Examples of Banach Algebras.

Ultrafilters

A *filter* $\mathcal{F}$ on a set I is a non-empty collection of subsets of I with:

1. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;
2. If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.
3. $\emptyset \notin \mathcal{F}$ (this ensures $\mathcal{F} \neq 2^I$).

For example, the *Fréchet Filter* is the collection of A such that $I \setminus A$ is finite.

We order by inclusion, and define an *ultrafilter* to be a maximal filter.

Lemma

A filter $\mathcal{U}$ on I is an ultrafilter if and only if for each $A \subseteq I$ either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

- For example, for $i_0 \in I$ the *principle ultrafilter at i_0* is $\{A \subseteq I : i_0 \in A\}$.
- Use Zorn's Lemma to find a maximal filter which contains the Fréchet Filter. This ultrafilter is not principle.

Ultraproducts

In a metric space (X, d), a family (x_i) in X *converges along* a filter $\mathcal{F}$ to $x_0 \in X$ when

$$\forall\, \epsilon > 0, \quad \{i \in I : d(x_0, x_i) < \epsilon\} \in \mathcal{F}.$$

We write $x_0 = \lim_{i \to \mathcal{U}} x_i$. When (X, d) is compact, any family converges along an *ultra*filter.

Let $(E_i)_{i \in I}$ be a family of Banach spaces. Form $\ell^\infty(E_i)$. For an ultrafilter $\mathcal{U}$, define

$$N_{\mathcal{U}} = \{(x_i) \in \ell^\infty(E_i) : \lim_{i \to \mathcal{U}} \|x_i\| = 0\},$$

which is a closed subspace.

The *ultraproduct* is the quotient space $\ell^\infty(E_i)/N_{\mathcal{U}}$, denoted $(E_i)_{\mathcal{U}}$. When $E_i = E$ for all i, we have the *ultrapower* $(E)_{\mathcal{U}}$.

Applications

Studying ultraproducts and ultrapowers is an interesting way to convert *approximate* statements into *exact* statements.

- There is the notion of *finite-representation* of one Banach space in another: that finite-dimensional subspaces are "close to" finite-dimensional subspaces;
- A separable E is finitely-representable in F if and only if E is a subspace of $(F)_{\mathcal{U}}$.
- Also gives an interesting way to study biduals.

If (A_i) is a family of Banach algebras then the ultraproduct $(A_i)_{\mathcal{U}}$ is a Banach algebra, because $N_{\mathcal{U}}$ is an ideal.

Unital algebras

This is joint work with Bence Horváth. Fix a Banach algebra A.

Question

When is $(A)_{\mathcal{U}}$ unital?

- If A is unital, under the diagonal embedding $A \to (A)_{\mathcal{U}}$, the unit becomes a unit for $(A)_{\mathcal{U}}$.
- Conversely, let $e \in (A)_{\mathcal{U}}$ be a unit. This has a representative $(e_n) \in \ell^\infty(A)$, which satisfies

$$\lim_{n\to\mathcal{U}} \|e_n a_n - a_n\| = 0, \quad \lim_{n\to\mathcal{U}} \|a_n e_n - a_n\| = 0 \qquad ((a_n) \in \ell^\infty(A)).$$

- Can then extract a Cauchy sequence from the (e_n), which must converge in A, to the unit.

Also an ultra*product* $(A_n)_{\mathcal{U}}$ is unital if and only if "eventually" A_n is unital. (But this is only true because we assume a unit has *norm one*).

Idempotents and equivalence

Let A be a (Banach) algebra.

Definition

$p \in A$ is an *idempotent* if $p^2 = p$.
Two idempotents p, q are *equivalent*, written $p \sim q$, if there are $a, b \in A$ with $p = ab$ and $q = ba$.

[If $q \sim r$, say $q = cd, r = dc$, then $p = p^2 = abab = aqb = (ac)(db)$ and $(db)(ac) = dqc = dcdc = r^2 = r$ so $p \sim r$.]

For example, with $A = \mathbb{M}_n \cong \mathcal{B}(\mathbb{C}^n)$:

- idempotents correspond to direct sums $\mathbb{C}^n = V \oplus W = \mathrm{Im}(p) \oplus \ker(p)$;
- equivalence looks at the *dimension* of V.

For C^*-algebras, we usually look at *projections* and equivalence using partial-isometries. This gives the same notion of equivalence; and the same definitions in what follows.

Finiteness

Definition

Let A be a unital algebra. A is *Dedekind finite* if $p \sim 1$ implies $p = 1$.

- So $\mathbb{M}_n$ is Dedekind finite, via dimension.
- A Banach algebra like $\mathcal{B}(\ell^p)$ is not, as there are proper, complemented subspaces of ℓ^p isomorphic to ℓ^p.
    - Indeed, $ab = 1, ba = p$ can be achieved by letting:
    - b be the isometry of ℓ^p onto the subspace of elements with even support, and
    - a the projection onto this subspace composed with the inverse to b,
    - then p is the projection.

Purely infinite

Definition

A is *purely infinite* if $A \not\cong \mathbb{C}$ and for $a \neq 0$ there are $b, c \in A$ with $bac = 1$.

Theorem (Ara, Goodearl, Pardo)

Let A be a simple algebra. TFAE:

- *A is purely infinite;*
- *every non-zero right ideal of A contains an infinite idempotent.*

(An infinite idempotent is equivalent to a proper sub-idempotent of itself.)

To ultrapowers

Definition

For a unital Banach algebra A, for $a \neq 0$, define

$$C_{pi}(a) = \inf \{ \|b\| \|c\| : bac = 1 \}.$$

- Thus A is purely infinite if $C_{pi}(a) < \infty$ for each $a \neq 0$.

Theorem

For a unital Banach algebra, the following are equivalent:

1. $(A)_{\mathcal{U}}$ *is purely infinite;*
2. $\sup\{C_{pi}(a) : \|a\| = 1\} < \infty$.

Examples

Result

If A is a simple unital purely infinite C^*-algebra, then $C_{pi}(a) = 1$ for each $\|a\| = 1$.

For a Banach space E, let $\mathcal{B}(E)$ and $\mathcal{K}(E)$ be the algebras of bounded, respectively, compact operators. Sometimes, $\mathcal{K}(E)$ is the unique closed, two-sided ideal in $\mathcal{B}(E)$, so that $\mathcal{B}(E)/\mathcal{K}(E)$ is simple.

Theorem

For $E = c_0$ or ℓ^p, the algebra $\mathcal{B}(E)/\mathcal{K}(E)$ has purely infinite ultrapowers.

Proof.

A result of Ware gives exactly that $C_{pi}(T + \mathcal{K}(E)) = 1/\|T + \mathcal{K}(E)\|$ for each non-compact $T \in \mathcal{B}(E)$. □

Towards a counter-example

We seek a Banach algebra which *is* purely infinite, but with no good control of $C_{pi}(\cdot)$. This is hard, because being purely infinite is a "global" property.

Proposition

Let A, B be unital Banach algebras. Let A have purely infinite ultrapowers. When $\theta : A \to B$ is a homomorphism, θ is automatically bounded below.

Proof.

If $\|a\| = 1$ and $\|\theta(a)\| < \delta$ then there are $b, c \in A$ with $\|b\|\|c\| < 2C_{pi}(a)$ and $bac = 1$ so $\theta(b)\theta(a)\theta(c) = 1$ so

$$1 \leq \|\theta(b)\|\|\theta(c)\|\|\theta(a)\| < \|\theta\|^2 2C_{pi}(a)\delta,$$

which puts a lower-bound on δ. □

The Cuntz monoid

(Or "Cuntz semigroup", but that has multiple meanings.)

$$Cu_2 = \langle s_1, s_2, t_1, t_2 : t_1 s_1 = t_2 s_2 = 1, t_1 s_2 = t_2 s_1 = \Diamond \rangle$$

where $\Diamond$ is a "semigroup zero", meaning $s\Diamond = \Diamond s = \Diamond$ for all s.
So Cu_2 is all words in these generators, subject to the relations. For example:

$$s_1 s_2 t_2 s_1 t_2 = s_1 s_2 \Diamond t_2 = \Diamond, \quad s_1 s_2 t_2 s_2 t_2 = s_1 s_2 t_2.$$

In fact, any word reduces to either $\Diamond$ or a word starting in s_1, s_2 and ending in t_1, t_2.

ℓ^1 algebras

We form the usual ℓ^1 algebra of this monoid:

- $\ell^1(Cu_2)$ is all sequences indexed by Cu_2 with finite ℓ^1-norm:

$$\|(a_s)_{s\in Cu_2}\| = \sum_{s\in Cu_2} |a_s|.$$

- Write elements as sums of "point-mass measures" δ_s:

$$(a_s) = \sum_{s\in Cu_2} a_s \delta_s.$$

- Use the convolution product: $\delta_s \delta_t = \delta_{st}$.

Notice that $\mathbb{C}\delta_\Diamond$ is a two-sided ideal. So we can quotient by it:

$$\mathcal{A} := \ell^1(Cu_2)/\mathbb{C}\delta_\Diamond.$$

This is equivalent to identify $\delta_\Diamond$ with the algebra 0, so e.g.

$$\delta_{t_1}\delta_{s_1} = 1, \qquad \delta_{t_1}\delta_{s_2} = 0.$$

Comparison with the Cuntz algebra $\mathcal{O}_2$

$\mathcal{O}_2$ is generated by isometries s_1, s_2 (so $s_1^* s_1 = s_2^* s_2 = 1$) with relation

$$s_1 s_1^* + s_2 s_2^* = 1.$$

This implies that s_1 and s_2 have orthogonal ranges, so $s_1^* s_2 = s_2^* s_1 = 0$.

Let $\mathcal{J} \subseteq \mathcal{A}$ be the closed ideal generated by

$$1 - \delta_{s_1 t_1} - \delta_{s_2 t_2}.$$

- So in the quotient algebra $\mathcal{A}/\mathcal{J}$ we do have that $\delta_{s_1 t_1} + \delta_{s_2 t_2} = 1$.

Theorem

The algebra $\mathcal{A}/\mathcal{J}$ is simple.

Towards a proof

Consider the Banach space ℓ^1, with standard unit vector basis $(e_n)_{n\geq 1}$. Define isometries

$$S_1 : e_n \mapsto e_{2n}, \qquad S_2 : e_n \mapsto e_{2n-1}.$$

and define surjections

$$T_1 : e_n \mapsto \begin{cases} e_{n/2} & : n \text{ even}, \\ 0 & : n \text{ odd}, \end{cases} \qquad T_2 : e_n \mapsto \begin{cases} 0 & : n \text{ even}, \\ e_{(n+1)/2} & : n \text{ odd}. \end{cases}$$

Then

$$T_1 S_1 = 1, \qquad T_2 S_2 = 1, \qquad T_1 S_2 = 0, \qquad T_2 S_1 = 0,$$

and

$$S_1 T_1 + S_2 T_2 = 1.$$

We have a representation

So we obtain a representation $\mathcal{A} \to \mathcal{B}(\ell^1)$ which annihilates $\mathcal{J}$, and so drops to a representation of $\mathcal{A}/\mathcal{J}$.

Proposition

The representation $\Theta : \mathcal{A}/\mathcal{J} \to \mathcal{B}(\ell^1)$ is not bounded below.

Proof.

Let $T = T_1 + T_2$ so for $(\xi_n) \in \ell^1$,

$$T(\xi_n) = (\xi_1 + \xi_2, \xi_3 + \xi_4, \xi_5 + \xi_6, \cdots).$$

Hence $\|T\| = 1$. Consider

$$a = (\delta_{t_1} + \delta_{t_2})^N = \sum \{\delta_s : s \text{ is a word in } t_1, t_2 \text{ of length } N\}$$

So $\|a\| = 2^N$ and one can show that $\|a + \mathcal{J}\| = 2^N$ as well. Notice that $\Theta(a + \mathcal{J}) = T^N$, so $\|\Theta(a + \mathcal{J})\| \leq 1$. □

Purely infinite

Theorem

$\mathcal{A}/\mathcal{J}$ is purely infinite.

The proof is a careful but direct construction: given $a \in \mathcal{A}$ with $a \notin \mathcal{J}$, we find $b, c \in \mathcal{A}$ with $bac = 1$.

- Of use is identifying $\mathcal{J}^\perp$ in $\mathcal{A}^* \cong \ell^\infty(\mathcal{C}u_2 \setminus \{\Diamond\})$ and playing Hahn-Banach games.
- Consider $a = 1 - \delta_{s_1 t_1} - \delta_{s_2 t_2} \in \mathcal{J}$. Then

$$\delta_{t_1} a = \delta_{t_1} - \delta_{t_1 s_1 t_1} - \delta_{t_1 s_2 t_2} = 0,$$

  similarly $\delta_{t_2} a = 0$ and $a\delta_{s_1} = a\delta_{s_2} = 0$.
- So we can only left-multiply by s_1, s_2 and right multiply by t_1, t_2, but then no cancellation can occur. So we can never get $bac = 1$.

Corollaries

Corollary

$\mathcal{A}/\mathcal{J}$ is simple.

Corollary

$\mathcal{A}/\mathcal{J}$ does not have purely infinite ultrapowers.

Proof.

It is purely infinite, but we found a non-bounded below homomorphism. □

Interesting (to me) that the example is rather "natural". We didn't "build in" to the algebra some "bad norm control".