Dual Banach algebras

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W*-algebras

- Recall that a *W*-algebra* is a C*-algebra $\mathcal{A}$ such that $\mathcal{A} = E'$ for some Banach space E;
- Then, automatically, the multiplication on $\mathcal{A}$ becomes separately weak*-continuous, and the involution becomes weak*-continuous;
- There always exists a weak*-continuous $*$-representation of $\mathcal{A}$ onto a *von Neumann* algebra inside $\mathcal{B}(H)$ for a suitable Hilbert space H;
- Furthermore, the E above is *isometrically unique*: if F is any other Banach space such that $\mathcal{A}$ is isometrically isomorphic to F', then E and F are isometrically isomorphic.

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Dual Banach algebras

- A *Dual Banach algebra* is a Banach algebra which is a dual space as a Banach space, and such that the multiplication becomes separately weak*-continuous.
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Algebras of operators

- Let E, F be Banach spaces, and form the *projective tensor product* $E\widehat{\otimes}F$ with norm

$$\|\tau\|_\pi = \inf\left\{\sum_{i=1}^{r}\|x_i\|\|y_i\| : \tau = \sum_{i=1}^{r} x_i \otimes y_i\right\} \quad (\tau \in E \otimes F).$$

- Then $(E\widehat{\otimes}F)' = \mathcal{B}(E, F')$, the space of all bounded linear operators from E to F', with duality given by

$$\langle T, x \otimes y\rangle = \langle T(x), y\rangle \quad (T \in \mathcal{B}(E, F'), x \in E, y \in F).$$

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Representing dual Banach algebras.

Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$.

- For each $\mu \in \mathcal{A}_*$, the map $\mathcal{A} \to \mathcal{A}_*$ given by $a \mapsto a \cdot \mu$ is weakly-compact.
- By interpolation space results, this map factors through a reflexive left $\mathcal{A}$-module E_μ.
- We can check that the resulting representation $\mathcal{A} \to \mathcal{B}(E_\mu)$ is actually weak*-continuous.
- Hence, if we let E be the l^2-direct sum of all such E_μ, we see that $\mathcal{A}$ is weak*-continuously isometric to a weak*-closed subalgebra of $\mathcal{B}(E)$.
- This looks very similar to the GNS construction for a W*-algebra.

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Derivations

- A *derivation* from a Banach algebra $\mathcal{A}$ to a Banach $\mathcal{A}$-bimodule E is a linear map d such that $d(ab) = d(a) \cdot b + a \cdot d(b)$.
- We say that an algebra is *contractable* if every derivation to every bimodule is *inner*, that is, $d(a) = a \cdot x - x \cdot a$ for some $x \in E$. It is conjectured that contractable algebras are finite-dimensional; this is true for C*-algebras, for example.
- An algebra is *amenable* if every derivation to every *dual* bimodule is inner. This is a richer class: for example, $L^1(G)$ is amenable if and only if the group is amenable. A C*-algebra is amenable if and only if it is *nuclear*.
- However, there are few amenable dual Banach algebras: $M(G)$ is amenable only when G is discrete (so that $M(G) = l^1(G)$) while an amenable von Neumann algebra is of the form

$$C(X) \otimes \bigoplus_{i=1}^{n} \mathbb{M}_{n_i}.$$

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Connes-amenability

- Let $\mathcal{A}$ be a dual Banach algebra, and let E be an $\mathcal{A}$-bimodule. Then E' is *normal* if the maps

$$\mathcal{A} \to E', \quad a \mapsto \begin{cases} a \cdot \mu, \\ \mu \cdot a, \end{cases}$$

are weak*-continuous, for each $\mu \in E'$.

- Then $\mathcal{A}$ is *Connes-amenable* if every weak*-continuous derivation from $\mathcal{A}$ to a normal dual bimodule is inner.
- Volker Runde has shown that then $M(G)$ is Connes-amenable if and only if G is amenable.
- If E is a reflexive Banach space with the approximation property, then $\mathcal{B}(E)$ is Connes-amenable if and only if $K(E)$, the algebra of compact operators, is amenable. So $\mathcal{B}(\ell^p)$ is Connes-amenable for $1 < p < \infty$.

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Injectivity

- Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and let $\mathcal{A}^c = \{a \in \mathcal{B}(H) : ab = ba \ (b \in \mathcal{A})\}$ be the commutant of $\mathcal{A}$ in $\mathcal{B}(H)$.
- An *expectation* for $\mathcal{A}^c$ is a norm-one projection $\mathcal{Q} : \mathcal{B}(H) \to \mathcal{A}^c$.
- A von Neumann algebra is *injective* if there is an expectation for $\mathcal{A}^c$.
- We can use the structure theorem for weak*-continuous $*$-isomorphisms to show that the definition of injectivity does not actually depend on the choice of representation $\mathcal{A} \subseteq \mathcal{B}(H)$.
- So this definition makes sense for W*-algebras.
- In fact, $\mathcal{A}$ is injective if and only if $\mathcal{A}$ is Connes-amenable.

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Injectivity for dual Banach algebras

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- A *quasi-expectation* for $\mathcal{A}^c$ is a bounded projection $\mathcal{Q} : \mathcal{B}(E) \to \mathcal{A}^c$ such that $\mathcal{Q}(aTb) = a\mathcal{Q}(T)b$ for $T \in \mathcal{B}(E)$ and $a, b \in \mathcal{A}^c$.
- An expectation is a quasi-expectation.
- If $\mathcal{A}$ is Connes-amenable, then whenever A is realised as a weak*-closed subalgebra of $\mathcal{B}(E)$, there is a quasi-expectation $\mathcal{B}(E) \to \mathcal{A}^c$.
- We say that $\mathcal{A}$ is *injective* if there is always a quasi-expectation for $\mathcal{A}^c$.
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The converse

- Building on work of Runde, and again using interpolation spaces extensively, the converse can be shown to hold.
- That is, a dual Banach algebra $\mathcal{A}$ is Connes-amenable if and only if whenever $\mathcal{A} \subseteq \mathcal{B}(E)$, there is a quasi-expectation for $\mathcal{A}^c$.
- However, unlike the von Neumann algebra case, we really do need to check for all E;
- For example, $\mathcal{B}(\ell^p \oplus \ell^q)$ is not Connes-amenable when $p, q \in (1, \infty) \setminus \{2\}$ are distinct. However, $\mathcal{B}(\ell^p \oplus \ell^q)$ obviously admits a quasi-expectation over itself.

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The converse

- Building on work of Runde, and again using interpolation spaces extensively, the converse can be shown to hold.
- That is, a dual Banach algebra $\mathcal{A}$ is Connes-amenable if and only if whenever $\mathcal{A} \subseteq \mathcal{B}(E)$, there is a quasi-expectation for $\mathcal{A}^c$.
- However, unlike the von Neumann algebra case, we really do need to check for all E;
- For example, $\mathcal{B}(\ell^p \oplus \ell^q)$ is not Connes-amenable when $p, q \in (1, \infty) \setminus \{2\}$ are distinct. However, $\mathcal{B}(\ell^p \oplus \ell^q)$ obviously admits a quasi-expectation over itself.

Conclusion

- This doesn't provide an "easy" proof that Connes-amenability and injectivity agree for W*-algebras, as we do not generate representations on Hilbert spaces;
- Indeed, which Banach spaces E *do* appear? What sort of weak*-representation $\mathcal{B}(\ell^p \oplus \ell^q) \rightarrow \mathcal{B}(E)$ doesn't allow a quasi-expectation?
- As above, $\mathcal{K}(E)$ is amenable if and only if $\mathcal{B}(E)$ is Connes-amenable (for "nice" E). This allows an "abtrast-nonsense" formulation of when $\mathcal{K}(E)$ is amenable, in terms of (necessarily rather pathological) tensor products of E. Can we use this to improve upon known results of when $\mathcal{K}(E)$ is and is not amenable?

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