A non-commutative notion of separate continuity

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Locally compact spaces

Let X be a locally compact (Hausdorff) space.

- $C_0(X)$ is the algebra of continuous functions "vanishing at infinity": $\{x \in X : |f(x)| \geq \epsilon\}$ is compact for all $\epsilon > 0$.
- We turn $C_0(X)$ into a *vector space* via pointwise operations.
- We turn $C_0(X)$ into an *algebra* via pointwise operations.
- We give $C_0(X)$ a *norm* via $\|f\| = \sup_{x \in X} |f(x)|$.
- Then $C_0(X)$ is *complete*.
- We give $C_0(X)$ an involution $f \mapsto f^*$ via pointwise complex conjugation.
- C^*-identity: $\|f^* f\| = \|f\|^2$.

Abstract C^*-Algebras

- A complex algebra A,
- which has a *norm*,
- which is *complete*,
- which satisfies the C^*-condition: $\|a^*a\| = \|a\|^2$.

Theorem (Gelfand)

Let A be a commutative C^*-algebra. Then there is a locally compact Hausdorff space X such that A is isomorphic to $C_0(X)$.

- "isomorphic" means all the structure is preserved.

Gelfand theory

- A *character* on A is a non-zero *homomorphism* $\phi : A \to \mathbb{C}$.
- Characters are always continuous, indeed, $\|\phi\| \leq 1$ always.
- The collection of all characters forms our space X, and we use the (relative) weak*-topology to turn X into a toplogical space.
- Little exercise: If X is compact, then every character on $C(X)$ is of the form: "evaluate at some point of X".

Example

Let X be a non-locally compact metric space. This is a "nice" space, and we can form $C_b(X)$ the algebra of *bounded* continuous functions. The "character space" of $C_b(X)$ is then the *Stone-Cech compactification* of X, the largest compact space containing a dense copy of X.

A little category theory

Suppose X and Y are compact, and $\alpha : X \to Y$ is a continuous map. Then we get an algebra homomorphism $\alpha^* : C(Y) \to C(X)$ given by

$$\alpha^*(f)(x) = f(\alpha(x)) \qquad (f \in C(X), x \in X).$$

Theorem

Let $\phi : C(Y) \to C(X)$ be a unital $*$-homomorphism. Then there is a continuous map $\alpha : X \to Y$ with $\phi = \alpha^*$.
In this way, the category of compact Hausdorff spaces and the opposite to the category of unital commutative C^*-algebras are isomorphic.

To construct α, just observe that ϕ, composed with evaluation at $x \in X$, gives a character on $C(Y)$, that is, a point $\alpha(x) \in Y$.

Locally compact case

Let $C^b(X)$ be the bounded continuous functions on X. Then $f : X \to Y$ induces a $*$-homomorphism $\theta : C_0(Y) \to C^b(X); a \mapsto a \circ f$.
Not every $*$-homomorphism arises in this way: an arbitrary $\theta : C_0(Y) \to C^b(X)$ gives a continuous map $f : X \to Y_\infty$ to the one-point compactification of Y.
To single out those maps which "never take the value ∞" you need to look at "non-degenerate $*$-homomorphisms":

$$\overline{\text{lin}}\{\theta(a)b : a \in C_0(Y), b \in C_0(X)\} = C_0(X).$$

Then we get:

The category of locally compact spaces with continuous maps	$\xleftrightarrow[\text{isomorphic}]{\text{anti}}$	The category of commutative C*-algebras and non-degenerate $*$-homomorphisms

Multiplier algebras

The *multiplier algebra* of a C*-algebra A is the largest C*-algebra B which contains A as a two-sided ideal, in an "essential" way:

$$\text{For } b \in B, \quad ab = ba = 0 \quad (a \in A) \implies b = 0.$$

Write $M(A)$ for the multiplier algebra (there are various constructions).

- If $A = C_0(X)$ then $M(A) = C^b(X)$.
- If $A = \mathcal{K}(H)$, compact operators on a Hilbert space, then $M(A) = \mathcal{B}(H)$, all operators on a Hilbert space.

A $*$-homomorphism $\theta : A \to M(B)$ is non-degenerate when

$$\overline{\text{lin}}\{\theta(a)b : a \in A, b \in B\} = B.$$

Then θ extends to a $*$-homomorphism $M(A) \to M(B)$ and in this way we can compose two non-degenerate $*$-homomorphisms, and get another non-degenerate $*$-homomorphism.

Intuition

- We say that a "morphism" (a la Woronowicz) $A \to B$ is a non-degenerate $*$-homomorphism $\theta : A \to M(B)$.
- Intuition: "This corresponds to a continuous function from the non-commutative space of B to the non-commutative space of A."

Motivation: semi-groups, compactifications

- A *semitopological semigroup* is a semigroup S which has a topology, such that the product map $S \times S \to S$ is separately continuous.
- For example: take $\mathbb{R}_\infty$ the one-point compactification of $\mathbb{R}$, with algebraic operations $\infty + t = t + \infty = \infty$.
- E.g. let S be a sub-semigroup of the semigroup of contractive linear maps on a Hilbert space.
- Or any reflexive Banach space.
- In fact, all compact semitopological semigroups arise in this way.

Motivation: A tiny look at quantum groups

Question

How do we fit a *group* into the "Gelfand" framework?

- Let G be a compact group; so have $G \times G \to G$.
- Same as a $*$-homomorphism $\Delta : C(G) \to C(G \times G) = C(G) \otimes C(G)$.
- The product is associative if and only if $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

Let $SC(S \times S)$ be the space of separately continuous functions on a compact space S. So for a compact semitopological semigroup, we can capture the product as a $*$-homomorphism $C(S) \to SC(S \times S)$.

Question

How can we think of $SC(S \times S)$ purely in terms of the commutative C^*-algebra $C(S)$?

Fubini

Fix a compact space X. Let $M(X) = C(X)^*$ be the space of regular finite Borel measures.

Theorem (Grothendieck)

Let $f \in SC(X \times X)$ and $\mu \in M(X)$. Then

$$\begin{array}{l} (\mathrm{id} \otimes \mu)f : s \mapsto \int_X f(s,t)\, d\mu(t) \\ (\mu \otimes \mathrm{id})f : s \mapsto \int_X f(t,s)\, d\mu(t) \end{array} \quad \text{are in } C(X).$$

For $\lambda \in M(X)$, we have $\lambda((\mathrm{id} \otimes \mu)(f)) = \mu((\lambda \otimes \mathrm{id})(f))$.

So each $f \in SC(X \times X)$ well-defines a bilinear map

$$M(X) \times M(X) \to \mathbb{C}.$$

Furthermore, this is separately weak*-continuous in each variable.

Abstract picture of $SC(X \times X)$, take 1

We can reverse this:

$$SC(X \times X) \cong Bil_\sigma(M(X), M(X); \mathbb{C})$$

the space of separately weak*-continuous, bilinear maps. (For the other implication, just evaluate at points.)

- The *projective tensor product* of Banach spaces E, F is a completion of the vector space $E \otimes F$.
- Universal property: $Bil(E, F; G) = \mathcal{B}(E \widehat{\otimes} F, G)$.
- If A, B are commutative C^*-algebras, then this norm agrees with the norm on $A^* \otimes B^*$ induced by pairing with $A \otimes B$, the minimal $C*$-tensor product.
- So, $(A^* \widehat{\otimes} B^*)^* = A^{**} \overline{\otimes} B^{**}$ (consult your favourite book on tensor products of von Neumann algebras, aka W^*-algebras.)

Abstract picture of $SC(X \times X)$, take 2

Setting $A = C(X)$,

$$SC(X \times X) = \{x \in A^{**}\overline{\otimes}A^{**} : (\mu \otimes \mathrm{id})x, (\mathrm{id} \otimes \mu)x \in A \ (\mu \in A^*)\}.$$

- The RHS makes sense for any C^*-algebra A.
- Do we win?
- What if $A = \mathcal{K}(H)$, compact operators?
- Then A^* is the trace-class operators, and $A^{**} = \mathcal{B}(H)$, all operators.
- So $A^{**}\overline{\otimes}A^{**} \cong \mathcal{B}(H \otimes H)$.
- Let $x \in \mathcal{B}(H \otimes H)$ be the "swap map".
- Then x slices into $\mathcal{K}(H)$, but $x^2 = 1$ does not.
- So RHS is not an algebra, in general.

From an idea from Ozawa

Let A be a unital (for convenience) C^*-algebra.
Write $SC(A \times A) = \{x \in A^{**}\overline{\otimes}A^{**} : (\mu \otimes \mathrm{id})x, (\mathrm{id} \otimes \mu)x \in A \ (\mu \in A^*)\}$.

Theorem (D. 2014)

Let $A \subseteq \mathcal{B}(H)$ be the universal representation, so also $A^{**} \subseteq \mathcal{B}(H)$. For $x \in A^{**}\overline{\otimes}A^{**}$, the following are equivalent:

1. $x, x^*x, xx^* \in SC(A \times A)$;
2. $x \in M(A \otimes \mathcal{K}(H)) \cap M(\mathcal{K}(H) \otimes A)$;
3. *pick o.n. basis $(e_i)_{i \in I}$ for H, so $\mathcal{B}(H) \cong \mathbb{M}_I$. Regarding $x \in A^{**}\overline{\otimes}\mathcal{B}(H) \cong \mathbb{M}_I(A^{**})$, we have that $x = (x_{ij}) \in \mathbb{M}_I(A)$, and that $\sum_i x_{ji}x_{ji}^*$ and $\sum_i x_{ij}^*x_{ij}$ converge in norm; and "the other way around".*

The collection of such x forms a C^*-subalgebra of $SC(A \times A)$, denoted $A \overset{\mathrm{SC}}{\otimes} A$, which contains all other C^*-subalgebras.

Sketch of the proof?

For von Neumann algebras

- A C^*-algebra which is a dual space;
- equivalently, closed in the SOT on $\mathcal{B}(H)$.
- Commutative examples: $L^\infty(\mu)$ for a measure μ.
- By Gelfand, $L^\infty(\mu) \cong C(K)$, for a *Hyperstonian* K.
- E.g. $\ell^\infty(\mathbb{N}) = C(\beta\mathbb{N})$ where $\beta\mathbb{N}$ is the Stone-Cech compactification.
- Problem: $SC(L^\infty(X) \times L^\infty(X)) \subseteq L^\infty(X)^{**}\overline{\otimes}L^\infty(X)^{**}$ which is "huge".
- Feels like $L^\infty(X \times X) = L^\infty(X)\overline{\otimes}L^\infty(X)$ should already be large enough to contain $SC(K \times K)$.
- (In fact, previous work shows it is, in the commutative case).

Pushing down

Let M be a von Neumann algebra, with predual M_*.

- L^∞ and L^1 duality; or $\mathcal{B}(H)$ and trace-class operators.
- Then $(M_*)^* = M$ and so M^* is the *bidual* of M_*.
- So there is the canonical map $M_* \to M^*$, from a Banach space to its bidual.
- You can check that the Banach space adjoint, $M^{**} \to M$, is a (weak*-weak*-continuous) $*$-homomorphism.
- So we get a (weak*-weak*-continuous) $*$-homomorphism $M^{**}\overline{\otimes}M^{**} \to M\overline{\otimes}M$.
- Restrict this to $\theta_{sc} : SC(M \times M) \to M\overline{\otimes}M$.

Some slicing

Given $x \in M\overline{\otimes}M$, we can always "slice" by members of M_*:

$$\langle (\mu \otimes \mathrm{id})(x), \lambda \rangle = \langle x, \mu \otimes \lambda \rangle = \langle (\mathrm{id} \otimes \lambda)(x), \mu \rangle.$$

This is analogous to integrating against one variable of an $L^\infty(X \times X)$ function.
We can do something similar for $\phi \in M^*$:

$$\langle (\phi \otimes \mathrm{id})(x), \mu \rangle := \langle \phi, (\mathrm{id} \otimes \mu)(x) \rangle \qquad (\mu \in M_*),$$

and similarly on the other side.
Finally, we define dual pairings between $M^* \widehat{\otimes} M^*$ and $M\overline{\otimes}M$:

$$\langle \phi \otimes_{\Box} \psi, x \rangle = \langle \phi, (\mathrm{id} \otimes \psi)(x) \rangle$$
$$\langle \phi \otimes_{\Diamond} \psi, x \rangle = \langle \psi, (\phi \otimes \mathrm{id})(x) \rangle$$

Links with weak compactness

For $x \in M \overline{\otimes} M$, consider the "orbit maps"

$$L_x, R_x : M_* \to M, \quad \mu \mapsto (\mu \otimes \mathrm{id})(x),\ (\mathrm{id} \otimes \mu)(x).$$

Theorem (Arens, folklore)

We have that $\langle \phi \otimes_\Box \psi, x \rangle = \langle \phi \otimes_\Diamond \psi, x \rangle$ for all ϕ, ψ if and only if L_x (equivalently, R_x) is a weakly compact operator. Write $\mathrm{wap}(M \overline{\otimes} M)$ for such x.

This is linked to the Arens products: how do we extend the product on a Banach algebra A to its bidual A^{**} such that $A \to A^{**}$ is a homomorphism, and we have some sort of one-sided weak*-continuity.

Links with *SC*

- Given $x \in M\overline{\otimes}M$ we might try to "lift" to some $y \in SC(M \times M)$ such that $\theta_{sc}(y) = x$.
- E.g. define $\langle y, \phi \otimes \psi \rangle = \langle \phi \otimes_\Box \psi, x \rangle$.
- Or use $\Diamond$?

Theorem (D.)

This idea works if and only if $x \in \mathrm{wap}(M\overline{\otimes}M)$. Indeed, θ_{sc} maps into $\mathrm{wap}(M\overline{\otimes}M)$ and is a bijection between $SC(M \times M)$ and $\mathrm{wap}(M\overline{\otimes}M)$.

We can of course restrict θ_{sc} to $M \overset{sc}{\otimes} M$ and so view this as the maximal subalgebra of $\mathrm{wap}(M\overline{\otimes}M)$.

Apply to $L^\infty(G)$

- Let G be a locally compact group and form $L^1(G)$

$$\int_G |f| < \infty \qquad (f * g)(s) = \int_G f(t)g(t^{-1}s)\, dt$$

  all with respect to the (left) Haar measure.

- Then $L^1(G)$ is a Banach algebra, and so the dual $L^1(G)^* = L^\infty(G)$ becomes an $L^1(G)$ module:

$$\langle f \cdot F, g\rangle = \langle F, g * f\rangle \qquad (F \in L^\infty(G), f, g \in L^1(G)).$$

- Classical theory: $F \in \mathrm{wap}(G)$ if and only if the orbit map $L^1(G) \to L^\infty(G); f \mapsto f \cdot F$ is weakly compact.
- Can equivalently use $F \cdot f$.

Into our framework

- We have that $L^\infty(G)\overline{\otimes}L^\infty(G) = L^\infty(G \times G)$.
- Define $\Delta : L^\infty(G) \to L^\infty(G \times G)$ by

$$\Delta(F)(s,t) = F(st) \qquad (F \in L^\infty(G), s,t \in G).$$

- Then $f \cdot F = (\mathrm{id} \otimes f)\Delta(F)$ and $F \cdot f = (f \otimes \mathrm{id})\Delta(F)$.
- So $F \in \mathrm{wap}(G)$ if and only if $\Delta(F) \in \mathrm{wap}(L^\infty(G)\overline{\otimes}L^\infty(G))$.
- In this classical case, this is already an algebra.
- My motivation was to study analogues of wap for non-commutative algebras.
- So we now have a definition; just have to study it for e.g. the Fourier algebra, quantum groups etc.