Quantum compactifications of the Fourier Algebra

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Bohr compactification

The *Bohr compactification* of a topological (semi)group G is a compact group $\mathfrak{b}G$ such that:

- there is a continuous (but not necessarily injective) group homomorphism $G \to \mathfrak{b}G$ which has dense range;
- given any compact group H and a continuous homomorphism $\phi : G \to H$, there exists a continuous homomorphism $\hat{\phi} : \mathfrak{b}G \to H$ such that

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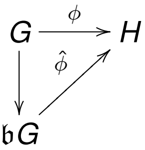

Almost periodic compactifications

Let G be a topological (semi)group and consider the commutative C*-algebra $C(G)$.

- We define AP(G) to be the collection of $f \in C(G)$ such that $\{f_s : s \in G\}$ is relatively compact in $C(G)$, where

$$f_s : G \to \mathbb{C}, \quad f_s(t) = f(ts) \qquad (t \in G).$$

  We call such functions *almost periodic*.
- Then AP(G) is a unital C*-subalgebra of $C(S)$, say with spectrum G^{AP}. Then G can be identified with a dense subspace of G^{AP}.

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Almost periodic compactifications (cont.)

- We can extend the multiplication of G to G^{AP}, turning G^{AP} into a *topological semigroup*.
- G^{AP} shares the same universality property as $\mathfrak{b}G$, in the category of topological semigroups.
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Locally compact groups

Now let G be a locally compact group, and consider the convolution algebra $L^1(G)$.

- $L^1(G)$ acts on its dual $L^\infty(G)$ by, for $f \in L^\infty(G)$,

$$\langle a \cdot f, b \rangle = \langle f, ba \rangle, \quad \langle f \cdot a, b \rangle = \langle f, ab \rangle \qquad (a, b \in L^1(G)).$$

- We say that $f \in L^\infty(G)$ is *almost periodic* if the map

$$L^1(G) \to L^\infty(G), \quad a \mapsto a \cdot f$$

  is a compact operator.
- Using the bounded approximate identity in $L^1(G)$, it is not hard to verify that $f \in L^\infty(G)$ is almost periodic if and only if $f \in C(G)$ and $f \in AP(G)$.

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Banach algebras

Of course, this definition of *almost periodic* makes sense on any Banach algebra $\mathcal{A}$, leading to a subspace $\mathrm{AP}(\mathcal{A})$ of $\mathcal{A}'$, the dual of $\mathcal{A}$.

- One can show that $\mathrm{AP}(A)$ is a closed submodule of $\mathcal{A}'$.
- The product on $\mathcal{A}$ extends to a product on $\mathrm{AP}(\mathcal{A})'$ such that the product on $\mathrm{AP}(\mathcal{A})'$ is jointly weak*-continuous on bounded sets.

(See the work of Lau and others).

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Group von Neumann algebras

Again let G be a locally compact group, and consider the left-regular representation of G on $L^2(G)$,

$$\lambda : G \to \mathcal{B}(L^2(G)), \quad \lambda(s)(f) : t \mapsto f(s^{-1}t),$$

for $s, t \in G, f \in L^2(G)$.

- Let $VN(G)$ be the *group von Neumann algebra*, which is generated by $\{\lambda(s) : s \in G\}$.
- We have the coassociative product

$$\Delta : VN(G) \to VN(G)\overline{\otimes}VN(G) = VN(G \times G),$$
$$\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s) \qquad (s \in G).$$

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Fourier algebras

- This is induced by the unitary operator

$$W : L^2(G \times G) \to L^2(G \times G), \quad Wf(s,t) = f(s,st).$$

  Then $\Delta(T) = W^*(T \otimes I)W$.

- $A(G)$, the *Fourier algebra*, is the predual of $VN(G)$. For each $a \in A(G)$, there exists $x, y \in L^2(G)$ such that

$$\langle T, a \rangle = (Tx|y) \qquad (T \in VN(G)).$$

- We identify $A(G)$ with a subspace of $C_0(G)$,

$$a(s) = (\lambda(s)x|y) = \int_G x(s^{-1}t)\overline{y(t)}\,dt \quad (s \in G).$$

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Fourier algebras and almost periodicity

- As Δ is weak*-continuous, it drops to a (completely) contractive, associative product on $A(G)$.
- This turns $A(G)$ into a (completely contractive) Banach algebra, which is a subalgebra of $C_0(G)$.
- When G is abelian, we get $A(G) = L^1(\hat{G})$.

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Special cases

Let $C^*_\delta(G)$ be the C*-algebra in $VN(G)$ generated by $\{\lambda(s) : s \in G\}$.

- If G is abelian, then $\mathrm{AP}(\hat{G}) = C^*_\delta(G)$.
- This makes sense: the "dual" idea to a compactification is a "discretisation".
- If G is amenable, and discrete, the also $\mathrm{AP}(\hat{G}) = C^*_\delta(G)$. (Dunkl, Ramirez, Granirer).
- In general, we don't even know if $\mathrm{AP}(\hat{G})$ need be a sub-C*-algebra of $VN(G)$.
- Chou studied when $\mathrm{AP}(\hat{G}) = C^*_\delta(G)$. When this occurs, we say that G has the *dual Bohr approximation property*.
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Quantum compactifications

Can we be sure that $C^*_\delta(G)$ is “correct”?

- A *compact quantum group* is a unital C*-algebra $\mathcal{A}$ with a coassociative product $\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\mathrm{min}} \mathcal{A}$, together with certain density conditions. (Woronowicz)
- Sołtan, by using finite dimensional (co)representations, defined a *Quantum bohr compactification*.
- Starting with a unital C*-algebra $\mathcal{A}$ with a coassociative product, Sołtan's methods produces a compact quantum group which has the same universal property as the classical Bohr compactification.
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For the Fourier Algebra

- We start with the *reduced group C^*-algebra* $C^*_\lambda(G)$.
- The left-regular representation extends to a contractive algebra homomorphism $\lambda : L^1(G) \to \mathcal{B}(L^2(G))$. Then $C^*_\lambda(G)$ is the closure of $\lambda(L^1(G))$.
- We apply Sołtan's method to $C^*_\lambda(G)$.
- This yields a C*-algebra inside the multiplier algebra of $C^*_\lambda(G)$.
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- This yields a C*-algebra inside the multiplier algebra of $C^*_\lambda(G)$.
- We can embed this into $VN(G)$, and we find that we get exactly $C^*_\delta(G)$.

Approximation properties

- Recall that $L^\infty(G)$, as a Banach space, has the *approximation property*.
- In particular, any operator $L^1(G) \to L^\infty(G)$ is compact if and only if it can be norm approximated by finite-rank operators.
- It is known that a von Neumann algebra has the approximation property if and only if it is nuclear, which is if and only if it is sub-homogeneous. It follows that $VN(G)$ has the approximation property only when G is abelian by finite.
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A first attempt

Instead, let us consider maps $A(G) \to VN(G)$ which are "close to" finite rank, in some sense.

- Actually, we are interested in the maps

$$\mathcal{R}_T : A(G) \to VN(G); \quad a \mapsto a \cdot T,$$

  for $T \in VN(G)$.
- Our original definition was

$$AP(\hat{G}) = \{T \in VN(G) : \mathcal{R}_T \text{ is compact}\}.$$

- Instead, consider the collection of $T \in VN(G)$ such that $\mathcal{R}_T$ can be norm-approximated by maps of the form $\mathcal{R}_S$, for $S \in VN(G)$ with $\mathcal{R}_S$ *finite-rank*.
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- So perhaps we should consider "approximable" to mean "can be approximated, in the *completely bounded norm*, by finite-rank maps".
- Indeed, if we consider the collection of $T \in VN(G)$ such that $\mathcal{R}_T$ can be approximated, in the cb-norm, by arbitrary finite-rank maps $A(G) \to VN(G)$, then...
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Proof for the discrete case

We identify the space of finite-rank operators $A(G) \to VN(G)$ with $VN(G) \otimes VN(G)$. The tensor $\tau = \sum_{j=1}^{n} T_j \otimes S_j$ induces the operator

$$\tau(a) = \sum_{j=1}^{n} \langle T_j, a \rangle S_j \qquad (a \in A(G)).$$

The completion we want is then simply the *operator space injective tensor product*

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Continued (1)

When G is discrete, let $(\delta_g)_{g\in G}$ denote the standard orthonormal basis for $\ell^2(G)$.

We have the norm-decreasing injection

$$VN(G) \to \ell^2(G), \quad T \mapsto T(\delta_{e_G}) = (t_g),$$

where I write e_G for the unit of G.

We recover T as the operator induced by convolution by (t_g).

This follows, as $VN(G)$ commutes with the right-regular representation.

So we identify $VN(G)$ with the subspace of $\ell^2(G)$ consisting precisely of those vectors $t = (t_g) \in \ell^2(G)$ such that

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Notice that for G discrete, $C^*_\delta(G) = C^*_\lambda(G)$, which is the closure of $\ell^1(G) \subseteq \ell^2(G)$ in $VN(G)$.
Define a map $\theta : \ell^2(G) \to \ell^2(G \times G)$ by

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Define a bilinear map $\star : VN(G) \times VN(G) \to VN(G)$ by

$$T \star S = \theta^*(T \otimes S)\theta.$$

We check that

$$\lambda(s) \star \lambda(t) = \delta_{s,t}\lambda(s),$$

so $\star$ does map into $VN(G)$ by normality.
We check that

$$\theta^*\Delta(T)\theta = \theta^* W^*(T \otimes I)W\theta = T \qquad (T \in VN(G)).$$

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Finally, we check that

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So once we identify $VN(G)$ with a subspace of $\ell^2(G)$, the operation $\star$ corresponds to the pointwise product. By the Cauchy-Schwarz inequality, $\ell^2(G) \cdot \ell^2(G) \subseteq \ell^1(G)$, and so $VN(G) \star VN(G) \subseteq C^*_\delta(G)$.
For $T \in VN(G)$,

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and so $\mathcal{R}_T$ is "approximable" means that $\Delta(T) \in VN(G)\check{\otimes}VN(G)$.
So in this case

$$T = \theta^*\Delta(T)\theta \in \theta^*(VN(G)\check{\otimes}VN(G))\theta \subseteq C^*_\delta(G).$$