Locally compact quantum groups
1. Locally compact groups from an (operator) algebra perspective

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Obligatory non-commutative topology

Theorem (Gelfand)

Let A be a unital commutative C^*-algebra, and let Φ_A be the collection of characters on A, given the relative weak*-topology. Then Φ_A is a compact Hausdorff space, and the map

$$\mathcal{G} : A \to C(\Phi_A); \quad \mathcal{G}(a)(\varphi) = \varphi(a),$$

is an isometric isomorphism.

Furthermore, a $*$-homomorphism $\theta : A \to B$ between unital C^*-algebras is always given by a continuous map $\phi : \Phi_B \to \Phi_A$ with

$$\mathcal{G}_B \circ \theta \circ \mathcal{G}_A^{-1}(f) = f \circ \phi \qquad (f \in C(\Phi_A)).$$

So, in principle, studying compact spaces and continuous maps between them is the same as studying commutative C^*-algebras.

Some (vague) motivation

- I'm going to come back to the ideas of the previous slide (repeatedly).
- But for now let's just take it as (vague) motivation for looking at various operator algebras.
- In particular, I'll look both a locally compact space G, for which we have a choice of $C_0(G)$ and $C^b(G)$;
- and at measured spaces (X, μ) where it's natural to look at $L^\infty(X)$.
- As the other talks in this series have looked at Banach algebras, I'll start instead there.

Locally compact groups

Let G be a locally compact group, and consider $C_0(G)$, $C^b(G)$ and $L^\infty(G)$ (left Haar measure). These are two C*-algebras and a von Neumann algebra: they depend only on the topological and measure space properties of G.

- For example, in the case when G is countable and discrete, these algebras capture nothing of interest about the *group*.

We turn $L^1(G)$ into a Banach algebra for the convolution product:

$$(f * g)(s) = \int_G f(t) g(t^{-1}s) \, dt.$$

This *does* remember the structure of G, in the following sense:

Theorem (Wendel)

If $L^1(G)$ and $L^1(H)$ are isometrically isomorphic as Banach algebras, then G is, as a topological group, isomorphic to H.

At the Operator algebra level

Can we equip $L^\infty(G)$ with "extra structure" so that it remembers G?

Define a map $\Delta : L^\infty(G) \to L^\infty(G \times G)$ by

$$\Delta(F)(s,t) = F(st) \qquad (F \in L^\infty(G), s, t \in G).$$

This is a unital, injective, $*$-homomorphism which is normal (weak*-continuous).

The pre-adjoint is a map $L^1(G \times G) \to L^1(G)$. As $L^1(G) \otimes L^1(G)$ embeds into $L^1(G \times G)$, we get a bilinear map on $L^1(G)$. This is actually the convolution product, as

$$\begin{aligned}\langle F, \Delta_*(f \otimes g)\rangle = \langle \Delta(F), f \otimes g\rangle &= \int_{G\times G} F(st) f(s) g(t) \; ds \; dt \\ &= \int_G F(t) \int_G f(s) g(s^{-1}t) \; ds \; dt = \langle F, f * g\rangle.\end{aligned}$$

Interpretation

- We can think of $(L^\infty(G), \Delta)$ as an object which remembers G.
- Indeed, Δ is "co-associative" in that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ as maps $L^\infty(G) \to L^\infty(G \times G \times G)$, as

$$(\Delta \otimes \mathrm{id})\Delta(F)(s,t,r) = F((st)r), \quad (\mathrm{id} \otimes \Delta)\Delta(F)(s,t,r) = F(s(tr)).$$

- A pair (M, Δ) with M a von Neumann algebra and $\Delta : M \to M \overline{\otimes} M$ coassociative is a "Hopf-von Neumann algebra".
- Not all commutative examples come from $L^\infty(G)$.
- Another interpretation is that $L^1(G)$ is a particularly nice Banach algebra: it's dual is a von Neumann algebra, and the dual of the product "respects" the structure of $L^\infty(G)$. Compare the notion of an "F-algebra" ("Lau-algebra").

Amenability

- A topologically left invariant mean on G is a state M on $L^\infty(G)$ with $M(f * F) = M(F)$ for $F \in L^\infty(G)$ and $f \in L^1(G)$ with $f \geq 0, \int f = 1$.
- Given $f \in L^1(G)$ let $\tilde{f}(s) = \nabla(s^{-1})f(s^{-1})$ with ∇ the modular function; then $f \mapsto \tilde{f}$ is an isometric linear anti-homomorphism on $L^1(G)$.
- We calculate:

$$f * F(s) = \int f(t)F(t^{-1}s)\ dt = \int f(t^{-1})\nabla(t^{-1})F(ts)\ dt = F \cdot \tilde{f},$$

  the module action of $L^1(G)$ on $L^\infty(G)$.
- Using Δ this is $(\tilde{f} \otimes \mathrm{id})\Delta(F)$.
- So M is a state with, for any $f \in L^1(G), F \in L^\infty(G)$,

$$\langle M, (f \otimes \mathrm{id})\Delta(F)\rangle = \langle M, F\rangle\langle 1, f\rangle \quad \Leftrightarrow \quad (\mathrm{id} \otimes M)\Delta(F) = \langle M, F\rangle 1.$$

- Non-commutative: Can't talk about points of course...

Towards the Fourier algebra: group algebras

We let G act on $L^2(G)$ by the left-regular representation:

$$\big(\lambda(s)\xi\big)(t) = \xi(s^{-1}t) \qquad (\xi \in L^2(G), s, t \in G).$$

The s^{-1} arises to make $G \mapsto B(H); s \mapsto \lambda(s)$ a group homomorphism.

We can integrate this to get a contractive homomorphism $\lambda : L^1(G) \to B(L^2(G))$. The action of $L^1(G)$ on $L^2(G)$ is just convolution:

$$\lambda(f)\xi(t) = \int_G f(s)\lambda(s)\xi(t) = \int_G f(s)\xi(s^{-1}t)\ ds.$$

Let the norm closure of $L^1(G)$ in $B(L^2(G))$ be $C_r^*(G)$, the (reduced) group C*-algebra. The weak-operator closure is $VN(G)$, the group von Neumann algebra. Equivalently, $VN(G)$ is $\{\lambda(s) : s \in G\}''$.

We can similarly form the right-regular representation $\rho(s)\xi(t) = \xi(ts)\nabla(s)^{1/2}$ leading to right group von Neumann algebra $VN_r(G)$. Then $VN(G)' = VN_r(G)$ and $VN_r(G)' = VN(G)$.
(Particularly short proofs of this may be sent to the speaker on a postcard.)

As a Hopf von Neumann algebra

We claim that there is a normal, unital injective $*$-homomorphism $\Delta : VN(G) \to VN(G \times G)$ satisfying

$$\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s) = \lambda(s, s).$$

Here we identify $VN(G)\overline{\otimes}VN(G)$ with $VN(G \times G)$. *If* Δ exists, then it's uniquely defined by this property.

Define $\hat{W} : L^2(G \times G) \to L^2(G \times G)$ by

$$\hat{W}\xi(s,t) = \xi(ts,t) \qquad (\xi \in L^2(G \times G), \xi, \eta \in G).$$

Then $\hat{W}$ is unitary, and

$$\begin{aligned} \big(\hat{W}^*(1 \otimes \lambda(r))\hat{W}\xi\big)(s,t) &= \big((1 \otimes \lambda(r))\hat{W}\xi\big)(t^{-1}s,t) \\ &= (\hat{W}\xi)(t^{-1}s, r^{-1}t) = \xi(r^{-1}tt^{-1}s, r^{-1}t) \\ &= (\lambda(r) \otimes \lambda(r))\xi(s,t). \end{aligned}$$

Definition of Δ

So we could *define* Δ by

$$\Delta(x) = \hat{W}^*(1 \otimes x)\hat{W} \qquad (x \in VN(G)).$$

Then obviously Δ is an injective, unital, normal $*$-homomorphism, and $\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s)$, so by normality, Δ must map into $VN(G \times G)$.

Obviously $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

So $(VN(G), \Delta)$ is a Hopf von Neumann algebra, and hence the pre-adjoint of Δ turns the predual of $VN(G)$ into a Banach algebra.

The Fourier Algebra

Let $A(G)$ be the predual of $VN(G)$.

- So $A(G)$ is the (unique) Banach space such that $A(G)^* = VN(G)$.
- As $\{\lambda(s) : s \in G\}$ has weak*-dense linear span in $VN(G)$, for $\omega \in A(G)$, the values
$$\omega(s) := \langle \lambda(s), \omega \rangle \qquad (s \in G)$$
completely determine ω.
- As $G \to VN(G); s \mapsto \lambda(s)$ is SOT continuous, $s \mapsto \omega(s)$ is continuous.
- We identify ω with this continuous function, and so realise $A(G)$ as a space of continuous functions.
- Another concrete realisation of the predual is as a quotient of the trace-class operators on $L^2(G)$. For $\xi, \eta \in L^2(G)$ let $\omega_{\xi,\eta}$ be the normal functional $VN(G) \ni x \mapsto (x\xi|\eta)$.
- Then
$$\omega_{\xi,\eta}(s) = (\lambda(s)\xi|\eta) = \int_G \xi(s^{-1}t)\overline{\eta(t)} \, dt \implies \omega_{\xi,\eta} \in C_0(G).$$

The Fourier *Algebra*

- So $A(G)$ is a subspace of $C_0(G)$.
- But the norm comes from $A(G)^* = VN(G)$; the map $A(G) \to C_0(G)$ is norm-decreasing and has dense range.
- We use the coproduct Δ to turn $A(G)$ into a Banach algebra

  $$\langle \lambda(s), \omega_1 \star \omega_2 \rangle := \langle \Delta(\lambda(s)), \omega_1 \otimes \omega_2 \rangle = \langle \lambda(s) \otimes \lambda(s), \omega_1 \otimes \omega_2 \rangle = \omega_1(s)\omega_2(s).$$

  Here I use "$\star$" for a product, not to denote convolution.
- Indeed, we see that the product is the point-wise product. $A(G) \to C_0(G)$ is also an algebra homomorphism.
- This is Eymard's Fourier algebra.
- [Walter] If $A(G)$ and $A(H)$ are isometrically isomorphic, then G is isomorphic to (maybe the opposite of) H. If we insist on *completely* isometric, we have that G is isomorphic to H.

For abelian groups

If G is abelian, we can form the Pontryagin dual $\hat{G}$:

- the collection of all continuous characters $G \to \mathbb{T}$;
- with group product the pointwise product $(\phi_1\phi_2)(s) = \phi_1(s)\phi_2(s)$.
- with topology given by uniform convergence on compacta.

We then have the Fourier transform:

$$\mathcal{F} : L^2(G) \to L^2(\hat{G}); \qquad \mathcal{F}(f)(\phi) = \int_G f(s)\overline{\phi(s)}\, ds$$

If we normalise the Haar measures correctly, $\mathcal{F}$ is unitary.

- the dual of $\mathbb{Z}$ is $\mathbb{T}$, where $\theta \in [0, 2\pi)$ parameterises the character $\mathbb{Z} \ni n \mapsto e^{in\theta}$;
- the dual of $\mathbb{R}$ is $\mathbb{R}$, where $x \in \mathbb{R}$ parameterises the character $\mathbb{R} \ni t \mapsto e^{itx}$. You need a 2π somewhere to get the normalisation correct.

The Fourier Transform

We regard $L^\infty(\hat{G})$ as also acting on $L^2(\hat{G})$, by multiplication.

Then we have a $*$-isomorphism

$$VN(G) \to L^\infty(\hat{G}) \qquad x \mapsto \mathcal{F} \circ x \circ \mathcal{F}^{-1},$$

(On integrable functions, this will reduce to (some variant of) the familiar Fourier transform formula.)

This $*$-isomorphism is normal, and so induces an isomorphism $A(G) \cong L^1(\hat{G})$.

Our intuition is that $A(G)$, even for non-abelian G, can be thought of as being the L^1 algebra on the "group" $\hat{G}$.

Amenability for $A(G)$

Theorem (Dunkl–Ramirez, Granirer, Renaud)

For any G there is a state $M \in VN(G)^*$ with $(\mathrm{id} \otimes M)\Delta(x) = \langle M, x\rangle 1$ for $x \in VN(G)$.

So $\hat{G}$ is always amenable.

Theorem (Leptin)

$A(G)$ has a bounded approximate identity if and only if G is amenable.

Of course, $L^1(G)$ always has a bounded approximate identity.

Duality between G and $\hat{G}$

- Given a homomorphism $G \to H$ we can define a homomorphism $\hat{H} \to \hat{G}$. These establishes an anti-equivalence of categories.
- Pontryagin duality: $\hat{\hat{G}} = G$ in a canonical fashion (biduality functor is naturally equivalent to the identity.)
- We have seen that $A(G)$ behaves "like" it is $L^1(\hat{G})$.
- Can we make this more precise? Single out a collection of objects, which include $A(G)$ and $L^1(G)$, which has a (bi)duality theory, and forms a category.
- Work of e.g. Takesaki, Tatsuuma, Stinespring, later Enock, Schwarz, Kac, Vainermann lead to "Kac algebras": Hopf von Neumann algebras (M, Δ) with many other "gadgets".
- While this works, it is complicated, and Woronowicz's notion of a *compact quantum group* does not fit into this framework: this is where we next look.

Unitary implementing the coproduct

In defining Δ on $VN(G)$ I made use of a unitary $\hat{W}$. Set

$$W = \sigma \hat{W}^* \sigma \implies W\xi(s,t) = \xi(s, s^{-1}t),$$

where $\sigma \in \mathcal{B}(L^2(G \times G))$ is the "swap map" $\sigma(\xi)(s,t) = \xi(t,s)$.

For $F \in L^\infty(G)$ acting on $L^2(G)$ by multiplication,

$$W^*(1 \otimes F)W\xi(s,t) = (1 \otimes F)W\xi(s,st) = F(st)W\xi(s,st) = F(st)\xi(s,t),$$

and so, again, $W^*(1 \otimes F)W = \Delta(F)$.

Where does W live?

$$W\xi(s,t) = \xi(s, s^{-1}t)$$

- Informally, given a von Neumann algebra M, we think of $L^\infty(G)\overline{\otimes}M$ as being bounded measurable functions $G \to M$.
- Then $s \mapsto \lambda(s)$ is even SOT continuous, so defines $\Lambda \in L^\infty(G)\overline{\otimes}VN(G)$ say, which acts on $\xi \otimes \eta$ as

$$\Lambda(\xi\otimes\eta)(s) = \xi(s)\lambda(s)\eta \text{ under } L^2(G\times G) = L^2(G, L^2(G)),$$
$$\implies \Lambda(\xi\otimes\eta)(s,t) = \xi(s)\eta(s^{-1}t) = W(\xi\otimes\eta)(s,t).$$

- So W "is" the left-regular representation, and $W \in L^\infty(G)\overline{\otimes}VN(G)$.
- More carefully, we could use Tomita's theorem and check that W commutes with $F \otimes \rho(s) \in L^\infty(G)\overline{\otimes}VN_r(G)$ so $W \in L^\infty(G)'\overline{\otimes}VN_r(G)' = L^\infty(G)\overline{\otimes}VN(G)$.

Using $W \in L^\infty(G)\overline{\otimes}VN(G)$

The map $\lambda : L^1(G) \to VN(G)$ is actually

$$\lambda(f) = (f \otimes \mathrm{id})(W) \qquad (f \in L^1(G)).$$

- This should be true given the informal thinking on the previous slide!

If $\xi, \eta \in L^2(G)$ and $f = \xi\overline{\eta} \in L^1(G)$, then f is $\omega_{\xi,\eta}$ restricted to $L^\infty(G) \subseteq \mathcal{B}(L^2(G))$ and

$$\begin{aligned}\big((\omega_{\xi,\eta} \otimes \mathrm{id})W\gamma\big|\delta\big) &= \big(W(\xi \otimes \gamma)\big|\eta \otimes \delta\big) = \int_{G\times G} \xi(s)\gamma(s^{-1}t)\overline{\eta(s)\delta(t)}\, ds\, dt \\ &= \int_{G\times G} f(s)\gamma(s^{-1}t)\overline{\delta(t)}\, ds\, dt = \big(f * \gamma\big|\delta\big).\end{aligned}$$

Thus indeed $(\omega_{\xi,\eta} \otimes \mathrm{id})W = \lambda(f)$.

For the dual

$$\hat{W}\xi(s,t) = \xi(ts,t)$$

Similarly, we calculate $(\omega_{\xi,\eta} \otimes \mathrm{id})(\hat{W})$:

$$\big((\omega_{\xi,\eta} \otimes \mathrm{id})(\hat{W})\gamma\big|\delta\big) = \big(\hat{W}(\xi \otimes \gamma)\big|\eta \otimes \delta\big)$$
$$= \int_{G\times G} \xi(ts)\gamma(t)\overline{\eta(s)\delta(t)}\ ds\ dt = \int_G (\lambda(t^{-1})\xi|\eta)\gamma(t)\overline{\delta(t)}\ dt.$$

- So $(\omega_{\xi,\eta} \otimes \mathrm{id})(\hat{W})$ is the operator on $L^2(G)$ of multiplication by the continuous function $t \mapsto \omega(t^{-1}) := (\lambda(t^{-1})\xi|\eta)$.
- So up to an inverse, this is the embedding of $A(G)$ into $C_0(G) \subseteq L^\infty(G)$.
- So W allows us to reconstruct $L^\infty(G), VN(G), L^1(G), A(G)$ their products and the maps between them.

Summary

- Introduced $L^1(G)$ and $A(G)$ from a von Neumann algebra perspective.
- Motivated, a little, that these are "dual" to each other:
    - Both from quite a "formal" level;
    - Also at the level of how proofs works.
- Saw how a single unitary operator essentially stores all the information.

What's next:

- We've focused on von Neumann algebras: but arguably the *topology* is more basic than the *measure theory*. So we should be looking at C^*-algebras.
- Haven't yet mentioned quantum groups.