Locally compact quantum groups

2. $C^*$-algebras and compact quantum groups

Matthew Daws

Leeds

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Theorem (Gelfand)

Let $A$ be a commutative $C^*$-algebra, and let $\Phi_A$ be the collection of characters on $A$, given the relative weak*-topology. Then $\Phi_A$ is a locally compact Hausdorff space, and the map

$$G : A \rightarrow C_0(\Phi_A); \quad G(a)(\varphi) = \varphi(a),$$

is an isometric isomorphism.

But how do we capture the notion of a continuous map between $\Phi_A$ and $\Phi_B$?

- $*$-homomorphisms $A \rightarrow B$ correspond to proper continuous maps $\Phi_B \rightarrow (\Phi_A)_\infty$, the one-point compactification of $\Phi_A$. 

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Multiplier algebras

Let $A$ be a $C^*$-algebra.

- Regard $A$ as acting non-degenerately (so $\text{lin}\{a(\xi) : a \in A, \xi \in H\}$ is dense in $H$) on $H$. Then
  
  $$M(A) = \{ T \in \mathcal{B}(H) : Ta, aT \in A \ (a \in A) \}.$$  

- Regard $A$ as a subalgebra of its bidual $A^{**}$; then
  
  $$M(A) = \{ x \in A^{**} : xa, ax \in A \ (a \in A) \}.$$  

- These are isomorphic (and independent of $H$).

An abstract way to think of $M(A)$ is as the pairs of maps $(L, R)$ from $A$ to $A$ with $aL(b) = R(a)b$. A little closed graph argument shows that $L$ and $R$ are bounded, and that

$$L(ab) = L(a)b, \quad R(ab) = aR(b) \quad (a, b \in A).$$

The involution in this picture is $(L, R)^* = (R^*, L^*)$ where $R^*(a) = R(a^*)^*$, $L^*(a) = L(a^*)^*$. You can move between these pictures by a bounded approximate identity argument.
Multiplier algebras 2

- $M(A)$ is the largest $C^*$-algebra containing $A$ as an essential ideal: if $x \in M(A)$ and $axb = 0$ for all $a, b \in A$, then $x = 0$.
- So $M(A)$ is the largest (sensible) unitisation of $A$.

Applied to $C_0(X)$, unitisations correspond to compactifications of $X$.

- Indeed, $M(C_0(X))$ is isomorphic to $C^b(X)$ the algebra of all bounded continuous functions on $X$.
- The character space of $C^b(X)$ is $\beta X$, the Stone-Čech compactification.
Morphisms

A morphism $A \to B$ between $C^*$-algebras is a non-degenerate $\ast$-homomorphism $\theta : A \to M(B)$.

- $\theta$ is non-degenerate if $\{\theta(a)b : a \in A, b \in B\}$ is linearly dense in $B$.

The strict topology on $M(B)$ is:

$$x_\alpha \to x \iff x_\alpha b \to xb, \ bx_\alpha \to bx \ (b \in B).$$

Non-degeneracy is equivalent to:

- For any (or all) bounded approximate identity $(e_\alpha)$ in $A$, the net $(\theta(e_\alpha))$ converges strictly to $1 \in M(B)$;

- $\theta$ is the restriction of a strictly continuous $\ast$-homomorphism $\tilde{\theta} : M(A) \to M(B)$.

We can construct the extension: $\tilde{\theta}(x)\theta(a)b = \theta(xa)b$ and so forth.
Theorem

Let $X, Y$ be locally compact spaces.

- Given a continuous map $\phi : Y \to X$, the map $\theta : C_0(X) \to C^b(Y); f \mapsto f \circ \phi$ is a morphism.

- Any morphism $C_0(X) \to C_0(Y)$ is induced in this way.

So we have some machinery: but it captures exactly what we want!
Compact quantum groups

Let $G$ be a compact semigroup (associative, continuous product).

- Define $\Delta : C(G) \to C(G \times G); \Delta(f)(s, t) = f(st)$ which is a unital $^*$-homomorphism;
- again this is coassociative $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$;
- Every coassociative $\Delta : C(G) \to C(G \times G)$ arises in this way (from some product on $G$).

How do we capture the notion of a group?

- Write down the identity and inverse, as maps on $C(G)$?
- Inelegant; doesn’t generalise.

**Theorem**

A compact semigroup $G$ is a group if and only if satisfies cancellation:

\[
\begin{align*}
st = sr & \implies t = r, \\
ts = rs & \implies t = r.
\end{align*}
\]

If you’re bored: prove this.
Cancellation as density

**Theorem**

*G satisfies cancellation if and only if*

\[
\text{lin}\{(a \otimes 1)\Delta(b) : a, b \in C(G)\}, \quad \text{lin}\{(1 \otimes a)\Delta(b) : a, b \in C(G)\}
\]

*are dense in* \(C(G \times G) = C(G) \otimes C(G)\).

**Sketch proof.**

- Commutative, so these are \(*\)-subalgebras, so can apply Stone-Weierstrauss:
  dense if and only if they separate points;
- \((a \otimes 1)\Delta(b)(s, t) = a(s)b(st)\);
- so \(st = sr\) if and only if \(f(s, t) = f(s, r)\) for all \(f\) in the 1st set;
- so separates points if and only if cancellation.
Compact quantum groups

Definition (Woronowicz)

A compact quantum group is a unital C*-algebra $A$ with a coassociative unital $*$-homomorphism $\Delta : A \to A \otimes A$ with

$$ \{(a \otimes 1)\Delta(b) : a, b \in A\}, \quad \{(1 \otimes a)\Delta(b) : a, b \in A\} $$

linearly dense in $A \otimes A$.

So if $A$ is commutative, we exactly capture the notion of a compact group.

Let $\Gamma$ be a discrete group, and $A = C_r^*(\Gamma)$ the reduced group C*-algebra, say generated by $\{\lambda(s) : s \in \Gamma\}$.

- Exactly as in the last lecture, can construct a coproduct $\Delta : \lambda(s) \mapsto \lambda(s) \otimes \lambda(s)$.
- Cancellation is easy to verify: $(\lambda(st^{-1}) \otimes 1)\Delta(\lambda(t)) = \lambda(s) \otimes \lambda(t)$.
- Every cocommutative ($\Delta = \sigma\Delta$) compact quantum group is of this form.
Construction of Haar state

- From now on, \((A, \Delta)\) is a compact quantum group.
- Turn \(A^*\) into a (completely contractive) Banach algebra:
  \[
  \langle \mu * \lambda, a \rangle = \langle \mu \otimes \lambda, \Delta(a) \rangle \quad (\mu, \lambda \in A^*, a \in A).
  \]

**Theorem**

*There is a unique state \(\varphi\) with \((\varphi \otimes \text{id})\Delta(a) = (\text{id} \otimes \varphi)\Delta(a) = \langle \varphi, a \rangle 1.\)*

**Very sketch proof.**

- Equivalent to \(\varphi * \mu = \mu * \varphi = \langle \mu, 1 \rangle \varphi\) for all \(\mu \in A^*\).
- If want this for one state \(\mu\) then \(\varphi = \lim \frac{1}{n}(\mu + \mu^2 + \cdots + \mu^n)\).

See van Daele, PAMS 1995.

For \(a \in C(G)\):

\[
(id \otimes \varphi)\Delta(a)(t) = \int_G a(ts) \, d\varphi(s), \quad \langle \varphi, a \rangle 1(t) = \int_G a(s) \, d\varphi(s).
\]
Regular representation

Let $\mathcal{G}$ be the “object” which is our compact quantum group.

- Let $L^2(\mathcal{G})$ be the GNS space for the Haar state $\varphi$. Let $\pi_{\varphi}, \xi_\varphi$ be the representation and the cyclic vector.

Let $\pi : A \to B(K)$ be some auxiliary non-degenerate $*$-representation.

**Theorem**

There is a unitary $U \in B(K \otimes L^2(\mathcal{G}))$ with

$$U^*(\xi \otimes \pi_{\varphi}(a)\xi_\varphi) = (\pi \otimes \pi_{\varphi})(\Delta(a))(\xi \otimes \xi_\varphi).$$

(All this theory is due to Woronowicz; some presentation motivated by Maes, van Daele, Timmermann.)
We have that $U$ is a multiplier of $\pi(A) \otimes B_0(L^2(G))$. 

$B_0(L^2(G))$ is the compact operators on $L^2(G)$.

Also $(\pi \otimes \pi \varphi)\Delta(a) = U^*(1 \otimes \pi \varphi(a))U$.

A SOT continuous unitary representation $\pi$ of a compact group $G$ gives a map 

$$G \to B(H) = M(B_0(H)); \quad s \mapsto \pi(s).$$

This is continuous for the strict topology; given $f \in C_0(G, B_0(H))$ the map 

$$G \to B_0(H); \quad s \mapsto \pi(s)f(s)$$

is continuous. So 

$$(\pi(s))_{s \in G} \in M(C_0(G) \otimes B_0(H)).$$

Given $V \in M(C_0(G) \otimes B_0(H))$ how do we recognise that it’s a representation?
Representations continued

\[ C^b_{str}(G, \mathcal{B}_0(H)) \cong M(C_0(G) \otimes \mathcal{B}_0(H)) \]
\[ (\pi(s)) \leftrightarrow V \quad (s \mapsto f(s)\pi(s)\xi) \leftrightarrow V(f \otimes \xi) \quad (f \in C_0(G), \xi \in H). \]

- \(\pi(s)\) unitary for all \(s\) corresponds to \(V\) being a unitary operator.

- A representation means:

\[ (\Delta \otimes \text{id})V \leftrightarrow (\pi(st))_{(s,t) \in G \times G} = (\pi(s)\pi(t))_{(s,t) \in G \times G} \leftrightarrow V_{13}V_{23}. \]

- This is “leg-numbering notation”: \(V_{23} = 1 \otimes V\) acts on the 2nd/3rd components; \(V_{13} = \sigma_{12}V_{23}\sigma_{12}\).

**Definition**

A corepresentation of \((A, \Delta)\) is \(V \in M(A \otimes \mathcal{B}_0(H))\) with \((\Delta \otimes \text{id})(V) = V_{13}V_{23}\. 
Left regular representation

**Theorem**

If $\pi : A \to \mathcal{B}(K)$ is faithful, then $U \in M(\pi(A) \otimes \mathcal{B}_0(L^2(G)))$ is a corepresentation.

- $\pi$ faithful, so $M(\pi(A) \otimes \mathcal{B}_0(L^2(G))) \cong M(A \otimes \mathcal{B}_0(L^2(G)))$.

**Theorem**

For $a, b \in A$ set $\xi = \pi_\phi(a)\xi_\phi$, $\eta = \pi_\phi(b)\xi_\phi$. Then

$$(\text{id} \otimes \omega_{\xi,\eta})(U) = (\text{id} \otimes \phi)(\Delta(b^*)(1 \otimes a))$$

$$(\text{id} \otimes \omega_{\xi,\eta})(U^*) = (\text{id} \otimes \phi)((1 \otimes b^*)\Delta(a))$$

(Here I suppress the $\pi$).

- By cancellation, such slices are hence dense in $A$. 
Finite dimensional corepresentations

- If $H$ finite dimensional then pick a basis, $H \cong \mathbb{C}^n$.
- $\mathcal{B}_0(H) \cong \mathbb{M}_n$ and $M(A \otimes \mathcal{B}_0(H)) = A \otimes \mathcal{B}_0(H) \cong \mathbb{M}_n(A)$.
- A unitary $V = (V_{ij})$ is a corepresentation if and only if
  \[ \Delta(V_{ij}) = \sum_{k=1}^{n} V_{ik} \otimes V_{kj}. \]
- A subspace $K \subseteq H$ is invariant for $V$ if
  \[ V(1 \otimes p) = (1 \otimes p)V(1 \otimes p) \]
  for $p : H \to K$ the orthogonal projection.
- Given $V \in M(A \otimes \mathcal{B}_0(H_V))$ and $W \in M(A \otimes \mathcal{B}_0(H_W))$ an operator $T : H_V \to H_W$ is an intertwiner if $W(1 \otimes T) = (1 \otimes T)V$.
- Hence have notions of being irreducible, a subcorepresentation, (unitary) equivalence and so forth.
Theorem (Schur’s Lemma)

Let $x$ intertwine corepresentations $W, V$. The kernel, and the closure of the image, of $x$ are invariant subspaces of $W$, respectively, $V$. If

- $W$ and $V$ are irreducible; or
- $W$ and $V$ are finite-dimensional of the same dimension and one is irreducible,

then $x = 0$ if $W, V$ are not equivalent; if $x \neq 0$ then $x$ is invertible. Then span of such invertibles is one-dimensional.
Averaging with the Haar state

**Theorem**

Let $W, V$ be corepresentations, and let $x \in \mathcal{B}(H_W, H_V)$. Then

$$y = (\varphi \otimes \text{id})(V^*(1 \otimes x)W) \in \mathcal{B}(H_W, H_V)$$

satisfies $V^*(1 \otimes y)W = 1 \otimes y$. If $x$ compact, so is $y$.

**Proof.**

Using $(\varphi \otimes \text{id})\Delta(\cdot) = \varphi(\cdot)1$,

$$(\varphi \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})(V^*(1 \otimes x)W) = 1 \otimes (\varphi \otimes \text{id})(V^*(1 \otimes x)W) = 1 \otimes y$$

$$(\Delta \otimes \text{id})(V^*(1 \otimes x)W) = V_{23}^* V_{13}^*(1 \otimes 1 \otimes x)W_{13}W_{23}$$

$$(\varphi \otimes \text{id} \otimes \text{id})(V_{23}^* V_{13}^*(1 \otimes 1 \otimes x)W_{13}W_{23}) = V^*(1 \otimes y)W.$$ 

If $V$ is unitary then $(1 \otimes y)W = V(1 \otimes y)$ so we have an intertwiner.
Applications 1

**Theorem**

An irreducible unitary corepresentation is finite-dimensional.

**Proof.**

Let $V$ be the corepresentation.

- Pick a compact $x \in \mathcal{B}_0(H_V)$ and average to a compact intertwiner

$$y = (\varphi \otimes \text{id})(V^*(1 \otimes x)V) \in \mathcal{B}(H_V)$$

- By Schur, $y = 0$ or $y \in \mathbb{C}1$.

- $y$ is compact, so if $y = t1$ for $t \neq 0$ we’re done.

- Let $x$ vary through a net of finite-dimensional orthogonal projections to see that $y$ must be non-zero for some choice.
Applications 2

Theorem

Any unitary corepresentation $V$ decomposes as the direct sum of irreducibles.

Sketch proof.

- If $V$ is unitary then if $K$ is an invariant subspace for $V$ so is $K^\perp$.
- So the collection of intertwiners from $V$ to itself is a $C^*$-algebra $B$ say.
- The previous averaging argument shows that we can find a bounded approximate identity in $B$ consisting of compact operators.
- So $B$ is the direct sum of matrix algebras.
- So $V$ decomposes as finite-dimensional corepresentations.
- Can obviously decompose finite-dimensional corepresentations into irreducibles.
Theorem

Let $V$ be an irreducible unitary corepresentation of $(A, \Delta)$. Then $V$ is equivalent to a subrepresentation of $U$.

Proof.

- Pick any $x \in \mathcal{B}(L^2(G), H_V)$ and average to an intertwiner

$$y = (\varphi \otimes \text{id})(V^*(1 \otimes x)U).$$

- If $y$ is non-zero, use Schur to conclude $y$ is onto.

- As $V, U$ are unitary, it follows that $y^*$ is also an intertwiner, injective by Schur, so gives required equivalence.
Continued proof

\[ y = (\varphi \otimes \text{id})(V^*(1 \otimes x)U). \]

- Maybe \( y = 0 \) for all \( x \), so test on rank-one maps \( x = \theta_{\xi,a\xi,\varphi} \), giving
  \[
  0 = (yb_{\xi,\varphi}|\eta) = \langle \varphi \otimes \omega_{b_{\xi,\varphi}}, V^*(1 \otimes \theta_{\xi,a\xi,\varphi})U \rangle \\
  = \varphi((\text{id} \otimes \omega_{\xi,\eta})(V^*)(\text{id} \otimes \omega_{b_{\xi,\varphi},a_{\xi,\varphi}})(U)) \\
  = \varphi((\text{id} \otimes \omega_{\xi,\eta})(V^*)(\text{id} \otimes \varphi)(\Delta(a^*)(1 \otimes b)))
  \]

- Think of \( V = (V_{ij}) \in \mathbb{M}_n(A) \).

- By cancellation, and taking \( \xi, \eta \) to be basis vectors, conclude that
  \( 0 = \varphi(V_{ij}^*a) \) for all \( a \in A \).

- But \( V \) is unitary, so taking \( a = V_{ij} \) gives
  \[ 0 = \sum_i \varphi(V_{ij}^*V_{ij}) = \varphi(1) = 1. \]
Algebra of “matrix elements”

**Definition**

Let $A_0 \subseteq A$ be the linear span of matrix elements $V_{ij}$ arising from all finite-dimensional (irreducible) unitary corepresentations $V = (V_{ij})$.

- $U$ decomposes as a direct sum of (all the) irreducible (finite-dimensional) corepresentations.
- So also $L^2(G)$ decomposes as (finite-dimensional) invariant subspaces.
- Given $\xi, \eta \in L^2(G)$, approximate by vectors with “finite-support”.
- So can approximate $(\mathrm{id} \otimes \omega_{\xi, \eta})(U)$ by linear combination of matrix elements.
- So $A_0$ dense in $A$.
- $A_0$ is an algebra: tensor product of corepresentations ($V \biguplus W = V_{12} W_{13}$).
- Is $A_0$ a $*$-algebra?