Locally compact quantum groups
3. Further aspects of Compact Quantum Groups

Matthew Daws

Leeds

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CQGs: Recap

- Unital C^*-algebra A with coproduct Δ, satisfying "cancellation":

$$\overline{\text{lin}}\{(a \otimes 1)\Delta(b) : a, b \in A\} = \overline{\text{lin}}\{(1 \otimes a)\Delta(b) : a, b \in A\} = A \otimes A.$$

- There exists an invariant Haar state φ with GNS $(L^2(\mathbb{G}), \pi_\varphi, \xi_\varphi)$.
- Formed "left-regular corepresentation" $U \in M(A \otimes \mathcal{B}_0(L^2(\mathbb{G})))$:

$$U^*(\xi \otimes \pi_\varphi(a)\xi_\varphi) = (\pi \otimes \pi_\varphi)\big(\Delta(a)\big)(\xi \otimes \xi_\varphi)$$

- Studied category of corepresentations.
- U decomposes as direct sum of all the irreducibles.
- $A_0 \subseteq A$ algebra of matrix coefficients.

Is A_0 a $*$-algebra?

- Typical element $V_{ij} \in A_0$; so is $V_{ij}^* \in A_0$?
- Motivates looking at $\overline{V} := (V_{ij}^*)$. Still a corepresentation:

$$\Delta(V_{ij}^*) = \Delta(V_{ij})^* = \Big(\sum_k V_{ik} \otimes V_{kj} \Big)^* = \sum_k V_{ik}^* \otimes V_{kj}^*.$$

Theorem

Let V be an irreducible corepresentation. Then $\overline{V}$ is equivalent to a unitary corepresentation. In particular, $V_{ij}^* \in A_0$.

Proof.

Show that $\overline{V}$ is a sub-corepresentation of U. Same game: choose $x \in \mathcal{B}(L^2(\mathbb{G}), H_V)$ and set

$$y = (\varphi \otimes \mathrm{id})(\overline{V}^* (1 \otimes x) U),$$

argue that if $y \neq 0$ then y^* implements an isomorphism; if $y = 0$ for all x then derive contradiction. □

"F-matrices"

Let $\mathrm{Irr}(\mathbb{G})$ be the collection of equivalence classes of irreducible representations of (A,Δ). Choose representatives u^α.

Theorem

For each α there is a positive, invertible, trace 1 matrix F^α with

$$\varphi\big((u^\beta_{ip})^* u^\alpha_{jq}\big) = \begin{cases} F^\alpha_{ji} & : \alpha=\beta, p=q, \\ 0 & : \textit{otherwise.} \end{cases}$$

Sketch proof.

We apply our averaging argument to $x = e_{ij}$ a matrix unit:

$$y = (\varphi\otimes\mathrm{id})\big((u^\beta)^*(1\otimes x)u^\alpha\big) = \cdots = \sum_{p,q} \varphi\big((u^\beta_{ip})^* u^\alpha_{jq}\big) e_{pq}.$$

Then y intertwines u^α, u^β so is 0 if $\alpha\neq\beta$; otherwise $y = F^\alpha_{ji}1$. Then ... □

Application: A basis

$$\varphi\big((u^{\beta}_{ip})^* u^{\alpha}_{jq}\big) = \delta_{\alpha,\beta}\delta_{p,q}F^{\alpha}_{ji}.$$

Theorem

The set $\{u^{\alpha}_{ij} : \alpha \in \mathrm{Irr}(\mathbb{G}), 1 \leq i,j \leq n_{\alpha}\}$ *is a basis for* A_0.

Proof.

By definition this spans A_0. If $\sum t^{\alpha}_{ij} u^{\alpha}_{ij} = 0$ for some scalars (t^{α}_{ij}) then for any β, p, q,

$$0 = \sum_{\alpha,i,j} t^{\alpha}_{ij}\varphi((u^{\beta}_{pq})^* u^{\alpha}_{ij}) = \sum_{i} F^{\beta}_{ip} t^{\beta}_{iq}.$$

As F^{β} is invertible, this implies that $t^{\beta}_{iq} = 0$ for all i, q, β, as required. □

A Hopf $*$-algebra

We define $\epsilon : A_0 \to \mathbb{C}$ and $S : A_0 \to A_0$ by

$$\epsilon(u^\alpha_{ij}) = \delta_{i,j}, \qquad S(u^\alpha_{ij}) = (u^\alpha_{ji})^*.$$

Or equivalently, for any (finite-dimensional) unitary corepresentation V,

$$(S \otimes \mathrm{id})(V) = V^*, \qquad (\epsilon \otimes \mathrm{id})(V) = I.$$

Theorem

Then $(A_0, \Delta, \epsilon, S)$ is a Hopf $*$-algebra.

This gives a purely *algebraic* approach to compact quantum groups: the Hopf $*$-algebras which can arise are exactly those which are spanned by matrix coefficients of *unitary* corepresentations.

What happens in the commutative case?

V corresponds to a unitary group representation $\pi : G \to \mathbb{M}_n$:

$$V \in C(G) \otimes \mathbb{M}_n \cong C(G, \mathbb{M}_n), \qquad V = (\pi(s))_{s\in G}.$$
$$(\mathrm{id} \otimes \omega_{\xi,\eta})(V) = \big((\pi(s)\xi|\eta)\big)_{s\in G} \in C(G),$$
$$(\mathrm{id} \otimes \omega_{\xi,\eta})(V^*) = \big((\pi(s^{-1})\xi|\eta)\big)_{s\in G} \in C(G).$$

Such continuous functions are linearly dense in $C(G)$.

$$(\epsilon \otimes \mathrm{id})(V) = I \;\Leftrightarrow\; \langle \epsilon, (\pi(s)\xi|\eta)_{s\in G}\rangle = (\xi|\eta)$$

so we conclude that $\epsilon \in C(G)^*$ is the functional: "evaluate at the group identity".

$$(S \otimes \mathrm{id})(V) = V^* \;\Leftrightarrow\; S\big((\pi(s)\xi|\eta)_{s\in G}\big) = (\pi(s^{-1})\xi|\eta)_{s\in G}$$

so $S : C(G) \to C(G)$ is the $*$-homomorphism induced by the group inverse.
In general ϵ and S are unbounded.

Characters

Theorem

$$\varphi\big(u^\alpha_{ip}(u^\beta_{jq})^*\big) = \delta_{\alpha,\beta}\delta_{i,j}\frac{(F^\alpha)^{-1}_{qp}}{\mathrm{Tr}((F^\alpha)^{-1})}.$$

Set $t_\alpha = \mathrm{Tr}((F^\alpha)^{-1}) > 0$ and define a linear map by

$$f_z : A_0 \to \mathbb{C}; \qquad u^\alpha_{ij} \mapsto ((F^\alpha)^{-z})_{ij} t_\alpha^{-z/2}.$$

Turn A_0^* into an algebra via $\langle \mu \star \lambda, a\rangle = \langle \mu \otimes \lambda, \Delta(a)\rangle$.

Theorem

Each f_z is a character on A_0, $f_0 = \epsilon$, $f_z(a^*) = f_{\overline{z}}(a)^*$ and $f_z \star f_w = f_{z+w}$. If we define

$$\sigma(a) = f_i \star a \star f_i := (f_i \otimes \mathrm{id} \otimes f_i)\Delta^2(a) \qquad (a \in A_0),$$

then $\varphi(ab) = \varphi(b\sigma(a))$. (Note: $\Delta^2 = (\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$).

φ is not a *trace* but it nearly is.

Properties of Haar state on A

Theorem

φ is "faithful" on A_0 ($\varphi(a^*a) = 0 \implies a = 0$).

Proof.

If $\varphi(a^*a) = 0$ then $\varphi(a^*b) = 0$ for all $b \in A_0$ (Cauchy-Schwarz). Set $b = u^\beta_{pq}$ and use an F-matrix argument again. □

Theorem

For $a \in A$, $\varphi(a^*a) = 0 \Leftrightarrow \varphi(aa^*) = 0$.

Proof.

- Cauchy-Schwarz $\implies$ $\varphi(a^*b) = 0$ for all $b \in A$.
- Find $(a_n) \subseteq A_0$ converging to a in norm.
- Recall automorphism σ; then $0 = \lim_n \varphi(a_n^*\sigma(b)) = \lim_n \varphi(ba_n^*) = \varphi(ba^*)$.

□

Further conclusions

Theorem

$N_\varphi = \{a \in A : \varphi(a^* a) = 0\}$ is a two-sided ideal in A. If $\Lambda : A \to L^2(\mathbb{G}); a \mapsto \pi_\varphi(a)\xi_\varphi$ is the GNS map, then $\ker \Lambda = \ker \pi_\varphi = N_\varphi$.

Proof.

- Standard C^*-theory: N_φ is a left ideal.
- Previous theorem shows N_φ self-adjoint, so an ideal.
- By definition $\ker \Lambda = N_\varphi$ and $\ker \pi_\varphi \subseteq \ker \Lambda$.
- $a \in N_\varphi \implies b^* a \in N_\varphi \implies a^* b \in N_\varphi \implies \pi_\varphi(a^*) = 0 \implies \pi_\varphi(a) = 0$.

φ really "looks like" it is a trace! □

“Reduced” C^*-algebras

$$\ker \Lambda = \ker \pi_\varphi = \ker \varphi = N_\varphi.$$

Let $C(\mathbb{G}) = A/N_\varphi$ a C^*-algebra; φ drops to $C(\mathbb{G})$ and is faithful.

Theorem

The GNS space for φ on $C(\mathbb{G})$ is isomorphic to $L^2(\mathbb{G})$, and $C(\mathbb{G}) \cong \pi_\varphi(A)$. There is a unital $*$-homomorphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ turning $C(\mathbb{G})$ into a compact quantum group.

Proof.

Form the left-regular representation, but this time use $\pi = \pi_\varphi$ to get $W \in M(\pi_\varphi(A) \otimes \mathcal{B}_0(L^2(\mathbb{G}))) = M(C(\mathbb{G}) \otimes \mathcal{B}_0(L^2(\mathbb{G})))$ with

$$W^*(1 \otimes \pi_\varphi(a))W = (\pi_\varphi \otimes \pi_\varphi)\Delta(a) \qquad (a \in A).$$

So *define* Δ on $C(\mathbb{G})$ by $\Delta(x) = W^*(1 \otimes x)W$. Density of A_0 in $C(\mathbb{G})$ shows that Δ does map to $C(\mathbb{G}) \otimes C(\mathbb{G})$; similarly cancellation holds for $C(\mathbb{G})$. □

von Neumann algebra

Let $L^\infty(\mathbb{G}) = C(\mathbb{G})''$ in $\mathcal{B}(L^2(\mathbb{G}))$. Again define

$$\Delta(x) = W^*(1 \otimes x)W \qquad (x \in L^\infty(\mathbb{G})),$$

which by weak*-continuity maps into $L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$.

Theorem

The normal extension of φ to $L^\infty(\mathbb{G})$ is faithful.

Proof.

- Let $\varphi(x^*x) = 0$ so $x\varphi_\xi = 0$.
- Kaplansky Density: bounded net (a_i) in $C(\mathbb{G})$ with converges strongly to x. For $b, c \in A_0$,

$$(x\sigma(b)\xi_\varphi | c\xi_\varphi) = \lim_i \varphi(c^* a_i \sigma(b)) = \lim_i \varphi(bc^* a_i) = \lim_i (a_i \xi_\varphi | cb^* \xi_\varphi)$$
$$= (x\xi_\varphi | cb^* \xi_\varphi) = 0.$$

- Density: $(x\xi|\eta) = 0$ for $\xi, \eta \in L^2(\mathbb{G})$, so $x = 0$.

□

Discussion of amenability and $C^*(\Gamma)$

Let Γ be a discrete group, so $\widehat{\Gamma} := C_r^*(\Gamma)$ is a compact quantum group, $\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s)$

$$\varphi(\lambda(s)) = \delta_{s,e} \implies L^2(\widehat{\Gamma}) = \ell^2(\Gamma).$$

- Could also work with $C^*(\Gamma)$
- Existence of Δ follows from universal property, as $s \mapsto \lambda(s) \otimes \lambda(s)$ is a unitary representation.
- φ is now faithful if and only if Γ is *amenable*.
- $C_r^*(\Gamma) = C^*(\Gamma)$ if and only if Γ is amenable.
- $A_0 = \mathbb{C}[\Gamma]$ and $\epsilon : \lambda(s) \mapsto 1$ is bounded on $C^*(\Gamma)$.
- ϵ bounded on $C_r^*(\Gamma)$ if and only if Γ is amenable.

Duality

As $\Delta(\cdot) = W^*(1 \otimes \cdot)W$ and $(\Delta \otimes \mathrm{id})(W) = W_{13}W_{23}$,

$$W_{12}^* W_{23} W_{12} = W_{13} W_{23} \implies W_{23} W_{12} = W_{12} W_{13} W_{23}.$$

- This says that W is *multiplicative*.
- See Baaj–Skandalis, Woronowicz and Sołtan–Woronowicz.
- $\widehat{W} := \sigma W^* \sigma$ is also multiplicative.

$$c_0(\widehat{\mathbb{G}}) = \overline{\{(\omega \otimes \mathrm{id})(W)\}}^{\|\cdot\|} = \overline{\{(\mathrm{id} \otimes \omega)(\widehat{W})\}}^{\|\cdot\|} \qquad \ell^\infty(\widehat{\mathbb{G}}) = c_0(\widehat{\mathbb{G}})''$$

are a C^*-algebra and a von Neumann algbera with a coproduct

$$\widehat{\Delta}(x) = \widehat{W}^*(1 \otimes x)\widehat{W} \qquad (x \in c_0(\mathbb{G}), \ell^\infty(\mathbb{G})).$$

But here $\widehat{\Delta} : c_0(\widehat{\mathbb{G}}) \to M(c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{G}}))$ is a morphism.

$$W \in L^\infty(\mathbb{G}) \overline{\otimes} \ell^\infty(\widehat{\mathbb{G}}) \qquad W \in M(C(\mathbb{G}) \otimes c_0(\widehat{\mathbb{G}})).$$

Identifying $c_0(\widehat{\mathbb{G}})$

$$\varphi\big((u^{\beta}_{ip})^* u^{\alpha}_{jq}\big) = \delta_{\alpha,\beta}\delta_{p,q}F^{\alpha}_{ji} \quad \Longrightarrow \quad \big(u^{\alpha}_{jq}\xi_\varphi \big| u^{\beta}_{ip}\xi_\varphi\big) = \delta_{\alpha,\beta}\delta_{p,q}F^{\alpha}_{ji}.$$

- For fixed α, $\mathrm{lin}\{u^{\alpha}_{jq}\xi_\varphi\}$ is isomorphic to $\mathbb{C}^{n_\alpha} \otimes \mathbb{C}^{n_\alpha}$.
- So $L^2(\mathbb{G}) \cong \bigoplus_{\alpha\in \mathrm{Irr}(\mathbb{G})} \mathbb{C}^{n_\alpha} \otimes \mathbb{C}^{n_\alpha}$.
- Under this isomorphism,

$$W = \sum_{\alpha} \sum_{i,j} u^{\alpha}_{ij} \otimes e^{\alpha}_{ij}$$

  where $e^{\alpha}_{ij} \in \mathbb{M}_{n_\alpha}$ acts on the (e.g.) first variable of $\mathbb{C}^{n_\alpha} \otimes \mathbb{C}^{n_\alpha}$.
- Now easy to see that $c_0(\widehat{\mathbb{G}}) = \big\{(\omega \otimes \mathrm{id})(W)\big\}^{\|\cdot\|}$ is isomorphic to $\bigoplus_\alpha \mathbb{M}_{n_\alpha}$.
- So as an algebra $c_0(\widehat{\mathbb{G}})$ is easy; but $\widehat{\Delta}$ is complicated (essentially encodes how $u^{\alpha} \circledT u^{\beta}$ is written as irreducibles.)

Discrete/Compact duality

- $\widehat{\mathbb{G}}$ is a *discrete quantum group*. (van Daele: axiomatisation not in terms of compact $\mathbb{G}$.)
- There are *weights* $\widehat{\varphi}, \widehat{\psi}$ on $\ell^\infty(\widehat{\mathbb{G}})$

$$(\mathrm{id} \otimes \widehat{\varphi})\widehat{\Delta}(x) = \widehat{\varphi}(x)1, \qquad (\widehat{\psi} \otimes \mathrm{id})\widehat{\Delta}(x) = \widehat{\psi}(x)1.$$

- For $x = (x^\alpha) \in \ell^\infty(\widehat{\mathbb{G}}) = \prod_\alpha \mathbb{M}_{n_\alpha}$,

$$\widehat{\varphi}(x) = \sum_\alpha \Lambda_\alpha^2 \mathrm{Tr}_\alpha(F^\alpha x^\alpha)$$

  where $\Lambda_\alpha^2 = \mathrm{Tr}((F^\alpha)^{-1})$.
- Tomita-Takesaki theory: $\widehat{\nabla}$ on $L^2(\mathbb{G})$ implements the modular automorphism group $\widehat{\sigma}_t(x) = \widehat{\nabla}^{-it} x \widehat{\nabla}^{it}$ and conjugation $\ell^\infty(\widehat{\mathbb{G}}) \to \ell^\infty(\widehat{\mathbb{G}})'; x \mapsto \widehat{J} x^* \widehat{J}$. (Generalises modular function on G and behaviour of $VN(G)$).

Antipode

- The map $x \mapsto \widehat{\nabla}^{-it} x \widehat{\nabla}^{it}$ also maps $C(\mathbb{G})$ into itself, and implements a continuous automorphism group (τ_t), the *scaling group*.
- On A_0 we can express this using the characters f_{it}.
- Recall the antipode

$$S\big((\mathrm{id} \otimes \omega)(W)\big) = (\mathrm{id} \otimes \omega)(W^*).$$

- Define $R(x) = \hat{J} x^* \hat{J}$ for $x \in C(\mathbb{G})$, which also maps $C(\mathbb{G})$ into itself. An anti-$*$-homomorphism which commutes with (τ_t).
- We get an (unbounded) analytic extension $\tau_{-i/2}$ and $S = R\tau_{-i/2}$.
- $R = S$ iff $\tau_t = \mathrm{id}$ iff $\widehat{\varphi} = \widehat{\psi}$ iff φ is tracial iff $\mathbb{G}$ is a Kac algebra.

Examples/Buzzwords

- Deformations of compact Lie groups: $SU_q(2)$ (Woronowicz). Non-Kac type.
- Quantum permutation groups S_n^+ and quantum orthogonal groups O_n^+ (Wang).
- "Universal quantum groups". (Wang, van Daele).
- Liberation of quantum groups; Easy quantum Groups $S_n \subseteq \mathbb{G} \subseteq O_n^+$ (Banica, Speicher).
- Easy quantum groups now well classified (e.g. Curran, Weber, Raum, Freslon).
- Key tool is to study the representation category $\mathrm{Irr}(\mathbb{G})$ and Woronowicz's generalisation of Tannaka-Krein duality.
- Mostly of Kac type: $L^\infty(\mathbb{G})$ finite von Neumann algebra, lots of work on von Neumann algebra properties of $L^\infty(\mathbb{G})$. (e.g. Brannan, Freslon).
- Next time: what can we say for $L^1(\mathbb{G})$?

Time allowing: S_n^+

Let $(a_{ij})_{i,j=1}^n$ be a matrix of functions on some space X with:

- $a_{ij} = a_{ij}^* = a_{ij}^2$ (so a_{ij} is $0,1$-valued);
- for all i, $\sum_j a_{ij} = 1$ and for all j, $\sum_i a_{ij} = 1$ (so at each point of X, if we evaluate, we get a permutation matrix).

The maximal commutative C^*-algebra generated by such matrices is just the collection of all permutation matrices, i.e. $C(S_n)$.

- Let $C(S_n^+)$ be the non-commutative C^*-algebra generated by such matrices.
- Universal property: if A any C^*-algebra and $\hat{a}_{ij} \in A$ elements with the relations, there is a unique $*$-homomorphism $\theta : C(S_n^+) \to A$ with $\theta(a_{ij}) = \hat{a}_{ij}$.
- Apply with $A = C(S_n^+) \otimes C(S_n^+)$ and $\hat{a}_{ij} = \sum_k a_{ik} \otimes a_{kj}$.
- Gives $\Delta : A \to A \otimes A$ coproduct.
- Can manually check the cancellation conditions.