# The Fourier Algebra and homomorphisms

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# Outline

1. The Fourier Algebra – Finite groups
2. For general groups
3. Homomorphisms

# Group algebras

Let $G$ be a finite group, and consider the group algebra $\mathbb{C}[G]$. That is, $G$ forms a basis for a $\mathbb{C}$ vector space, with convolution as the product:

$$\Big(\sum_{s\in G}\lambda_s s\Big)\Big(\sum_{t\in G}\mu_t t\Big) = \sum_{s,t}\lambda_s\mu_t st = \sum_s\Big(\sum_r \lambda_r\mu_{r^{-1}s}\Big)s.$$

Endow $\mathbb{C}[G]$ with the usual inner product

$$\Big\langle \sum_s \lambda_s s, \sum_t \mu_t t\Big\rangle = \sum_s \lambda_s\overline{\mu_s}.$$

We write $\ell^2(G)$ for the resulting (finite dimensional) Hilbert space. Then $\mathbb{C}[G]$ acts on $\ell^2(G)$ by left multiplication (again, convolution). Notice that the action of $s\in G$ gives a surjective isometry on $\ell^2(G)$: so is a *unitary* map. So this is a *unitary representation* of the group $G$.

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# C*-algebras

We can identify $\mathbb{C}[G]$ as an algebra of linear maps on $\ell^2(G)$ (so, if we like, $G \times G$ matrices). This induces the operator norm on $\mathbb{C}[G]$:

$$\|x\| = \sup\left\{\|x\xi\| = (x\xi|x\xi)^{1/2} : \xi \in \ell^2(G), \|\xi\| \leq 1\right\}.$$

As we're acting on a Hilbert space, an operator has an adjoint which satisfies $(x\xi|\eta) = (\xi|x^*\eta)$. (Thinking of $x$ as a matrix, $x^*$ is the hermitian transpose). Then it's possible to show that $\|x\|^2 = \|x^*x\|$: this is the C*-condition.

For us,

$$x = \sum_s \lambda_s s \implies x^* = \sum_s \overline{\lambda_s} s^{-1}.$$

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Fix $\xi, \eta \in \ell^2(G)$. We can define a linear functional

$$\omega = \omega_{\xi,\eta} : C_r^*(G) \to \mathbb{C}; \quad \omega(x) = (x\xi|\eta).$$

Let $\xi = \sum_s \xi_s s$ and $\eta = \sum_t \eta_t t$. Then

$$\omega(r) = (r\xi|\eta) = \sum_{s,t} \xi_s \overline{\eta_t}(rs|t) = \sum_s \xi_s \overline{\eta_{rs}}.$$

As $\mathbb{C}[G]$, and hence $C_r^*(G)$, are the span of $G$, it follows that $\{\omega(r) : r \in G\}$ determines $\omega$. So we can think of $\omega$ as being a function $G \to \mathbb{C}$.

The *Fourier algebra* $A(G)$ is the subset of $\mathbb{C}^G$ formed by $\{\omega_{\xi,\eta} : \xi, \eta \in \ell^2(G)\}$.

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# Why an algebra?

So why is $A(G)$ an algebra? I want to build a bit of theory here.
Define a map $\Delta : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G] = \mathbb{C}[G \times G]$ by

$$\Delta(s) = s \otimes s,$$

and extend by linearity. Then $\Delta$ is a homomorphism, and also $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$, so $\Delta$ is *co-associative*.
Actually $\Delta$ gives a isometry $C_r^*(G) \to C_r^*(G \times G)$. (This is automatic by some C$^*$-algebra theory, but...) Define $W : \ell^2(G \times G) \to \ell^2(G \times G)$ by

$$W(s \otimes t) = t^{-1}s \otimes t.$$

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# The dual becomes an algebra

Let $C_r^*(G)^*$ be the space of all linear functionals $C_r^*(G) \to \mathbb{C}$. Then $\Delta$ induces an algebra product on $C_r^*(G)^*$ by

$$(\omega_1 \cdot \omega_2)(x) = (\omega_1 \otimes \omega_2)\Delta(x) \qquad (\omega_1, \omega_2 \in C_r^*(G)^*, x \in C_r^*(G)).$$

Every member of $C_r^*(G)^*$ arises as $\omega_{\xi,\eta}$ for some $\xi, \eta \in \ell^2(G)$.
So $A(G) = C_r^*(G)^*$. The product is then

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so we do just get the pointwise product.
(Mention Hopf algebras).

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# Norms

- Actually, as $G$ is finite, really $A(G) = \mathbb{C}^G$.
- However, $A(G)$ carries a natural norm as the dual of $C_r^*(G)$.
- The previous construction shows that this norm is an algebra norm: $\|\omega_1 \cdot \omega_2\| \leq \|\omega_1\|\|\omega_2\|$.

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To get a handle on this norm, let's look at some representation theory.

# Abelian case

Firstly, what if $G$ is abelian? Then every irreducible representation is one dimensional, and the collection of irreps forms a group: the *dual group* of $G$:

$$\hat{G} = \{\chi : G \to \mathbb{T} \text{ is a homomorphism } \}.$$

We also have the Fourier transform

$$\mathcal{F} : \ell^2(G) \to \ell^2(\hat{G}); \quad s \mapsto \sum_{\chi\in\hat{G}} \chi(s)\chi.$$

We can also interpret this as a map $\mathbb{C}[G] \to \mathbb{C}^{\hat{G}}$; then we get an isometry from $C_r^*(G)$ to $C(\hat{G})$, the space of continuous functions of $\hat{G}$ with the supremum (maximum) norm.

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# Abelian case cont.

So if $C_r^*(G) \cong C(\hat{G})$, then the duals are also isometric

$$A(G) = C_r^*(G)^* \cong C(\hat{G})^*.$$

What is the dual of $C(\hat{G})$? As $\hat{G}$ is finite, it is just functions $\hat{G} \to \mathbb{C}$ with the 1-norm:

$$\Big\| \sum_{\chi \in \hat{G}} \lambda_\chi \chi \Big\|_1 = \sum_\chi |\lambda_\chi|.$$

We denote this normed space by $\ell^1(\hat{G})$.
We can identify $\ell^1(\hat{G})$ with $\mathbb{C}[\hat{G}]$; then the 1-norm is an algebra norm.
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# General case

Now let $\hat{G}$ be the collection of (isomorphism classes) of irreducible representations of $G$; this is no longer a group in general.

- We have the decomposition

$$\mathbb{C}[G] \cong \bigoplus_{\pi \in \hat{G}} n_\pi \pi,$$

- Here $\pi$ is a representation of $G$ on a finite dimensional Hilbert space $H_\pi$, and the notation $n_\pi \pi$ means that $\pi$ occurs with multiplicity $n_\pi := \dim(H_\pi)$.
- So we find that

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# Interlude: Dual spaces of matrices

We define a (bilinear) dual pairing between $\mathbb{M}_n$ and $\mathbb{M}_n$ by "trace duality":

$$\langle x, y \rangle = \mathrm{Tr}(xy) \qquad (x, y \in \mathbb{M}_n).$$

- So $\mathbb{M}_n^* \cong \mathbb{M}_n$.
- We give $\mathbb{M}_n$ the operator norm: $\|x\|^2 = \|x^*x\|$. Then $x^*x$ is positive (semi) definite, so it has positive eigenvalues, and so

$$\|x\| = \|x^*x\|^{1/2} = \max\left\{\lambda^{1/2} : \lambda \text{ is an eigenvalue of } x^*x\right\}.$$

- It turns out that the dual norm induced on $\mathbb{M}_n$ is

$$\begin{aligned}\|y\|^* &:= \sup\left\{\mathrm{Tr}(xy) : \|x\| \leq 1\right\} \\ &= \sum\left\{\lambda^{1/2} : \lambda \text{ is an eigenvalue of } y^*y\right\}.\end{aligned}$$

- Write $\mathbb{T}_n$ for $\mathbb{M}_n$ with this norm: the "trace class" norm.

# Interlude: Dual spaces of matrices

We define a (bilinear) dual pairing between $\mathbb{M}_n$ and $\mathbb{M}_n$ by "trace duality":

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# Fourier algebra

We saw that

$$C_r^*(G) \cong \bigoplus_{\pi\in\hat{G}} \mathbb{M}_{n_\pi}.$$

- Thus

$$A(G) = C_r^*(G)^* \cong \bigoplus_{\pi\in\hat{G}} n_\pi \mathbb{T}_{n_\pi}.$$

- Notice that here we *do* need to worry about the multiplicities, as we have a "sum" norm, not a "max" norm.
- To be exact, $n\mathbb{T}_n$ is the space $\mathbb{T}_n$, but with the norm multiplied by $n$.
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# The product

Suppose $\omega_1 \in A(G)$ is given by a single irreducible $\pi_1 \in \hat{G}$, say

$$\omega_1(s) = (\pi_1(s)\xi_1|\eta_1) \qquad (s \in G, \xi_1, \eta_1 \in H_{\pi_1}).$$

Similarly $\pi_2$.

- Then

$$(\omega_1 \cdot \omega_2)(s) = \omega_1(s)\omega_2(s) = \big((\pi_1 \otimes \pi_2)(s)\xi_1 \otimes \xi_2 \big| \eta_1 \otimes \eta_2\big).$$

- So to understand the product $\omega_1 \cdot \omega_2$, we need to understand how to write $\pi_1 \otimes \pi_2$ as a sum of irreducibles. This can be done using fusion rules etc.
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# Locally compact groups

A *locally compact group* is a locally compact (Hausdorff) topological space which is also a group, and with the group operations being continuous.

Locally compact groups are essentially characterised as being those groups which admit an invariant measure: the Haar measure:

$$\mu(A) = \mu(sA) \qquad (s \in G, A \subseteq G).$$

- Any group with the discrete topology, and the counting measure.
- Any compact group: $\mathbb{T}, SU(n), O(n)$ etc. Haar measure is a probability measure.
- Any Lie group: $\mathbb{R}, SL_n(\mathbb{R})$ etc.
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# Various group algebras

Let $L^1(G)$ be the (equivalence classes) of integrable functions on $G$, which becomes a Banach algebra for the convolution product:

$$f * g(s) = \int_G f(t)g(t^{-1}s)\,dt \qquad (f,g \in L^1(G), s \in G).$$

- There is a natural representation of $L^1(G)$ on $L^2(G)$ given by left convolution: the norm closure of the image is $C_r^*(G)$, the (reduced) group C$^*$-algebra.
- To look at $C_r^*(G)^*$ would give too large an algebra.
- Instead, we take the *weak operator topology* closure of $L^1(G)$ acting on $L^2(G)$– this gives $VN(G)$ the group von Neumann algebra.
- Write $\lambda(s)$ for the left translation operator given by $s \in G$.
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# The Fourier algebra

- We now restrict attention to $VN(G)_*$, the functionals on $VN(G)$ which are weak operator topology continuous. So we set $A(G) = VN(G)_*$.
- The operators $\{\lambda(s) : s \in G\}$ generate $VN(G)$ for the weak operator topology. So for $\omega \in A(G)$, the values

$$\omega(s) = \langle \lambda(s), \omega \rangle \qquad (s \in G),$$

  completely determine $\omega$. Hence we can think of $A(G)$ as a space of functions $G \to \mathbb{C}$.
- As before, we have $\Delta : VN(G) \to VN(G \times G)$ given by $\lambda(s) \mapsto \lambda(s) \otimes \lambda(s)$.
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# The Fourier algebra: which functions?

A bit of machinery shows that each $\omega \in A(G)$ is of the form

$$\omega = \omega_{\xi,\eta} : \quad x \mapsto (x\xi|\eta) \qquad (x \in VN(G)),$$

for some $\xi, \eta \in L^2(G)$. (Not obvious why we don't need linear combinations etc.)

- For $s \in G$ we calculate

$$\omega(s) = (\lambda(s)\xi|\eta) = \int_G \xi(s^{-1}t)\overline{\eta(t)}\, dt = \int_G \overline{\eta(t)}\check{\xi}(t^{-1}s)\, dt = (\overline{\eta}*\check{\xi})(s),$$

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# The Fourier algebra: which functions?

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- Any member of $L^1(G)$ is the pointwise product of two $L^2$ functions, and $\mathcal{F}$ turns the pointwise product into the convolution product.
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Remember that the representation theory of compact groups is very similar to that for finite groups.

- Each irreducible representation is finite dimensional, and we get the isomorphisms

$$C_r^*(G) \cong \bigoplus_{\pi\in\hat{G}} \mathbb{M}_{n_\pi}, \quad VN(G) \cong \prod_{\pi\in\hat{G}} \mathbb{M}_{n_\pi}, \quad A(G) \cong \bigoplus_{\pi\in\hat{G}} \mathbb{T}_{n_\pi}.$$

- Again, the multiplication comes from tensoring irreps.
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- Let $G$ and $H$ be finite groups. When are $A(G)$ and $A(H)$ isomorphism algebras? Well, $A(G) \cong \mathbb{C}^G$ and $A(H) \cong \mathbb{C}^H$, so $A(G) \cong A(H)$ if and only if $|G| = |H|$.
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- Le Pham (2010) extended this in various ways. For example, if $\theta : A(G) \to A(H)$ is a *contractive* homomorphism, then
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# Annoying anti-homomorphisms

To remove the possibility of an anti-homomorphism, we need a more "rigid" sense of isometry.

- Given a bounded linear map $\theta : A(G) \to A(H)$, the adjoint gives a map $\theta^* : VN(H) \to VN(G)$.
- We identify $\mathbb{M}_n \otimes VN(H)$ with $n \times n$ matrices of elements of $VN(H)$. This acts naturally on $L^2(H) \oplus \cdots \oplus L^2(H)$ ($n$ times) and so $\mathbb{M}_n \otimes VN(H)$ is again a C$^*$-algebra.
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# A little bit of philosophy

- Recall that if $G$ is abelian, then $A(G) \cong L^1(\hat{G})$.
- So even if $G$ is not abelian, we can still think of $A(G)$ as being the algebra $L^1(\hat{G})$, even though $\hat{G}$ doesn't exist. (And we saw that in the compact case, this is not insane).
- An interesting thing to do with $L^1(G)$ is to study homomorphisms $\theta : L^1(G) \to \mathcal{B}(H)$, where $\mathcal{B}(H)$ is the algebra of operators on a Hilbert space $H$.
- There is a one-one correspondence between such (non-degenerate) homomorphisms and representations $\pi : G \to \mathcal{B}(H)$, where

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# Representations on a Hilbert space

- An even more interesting thing to study is $*$-homomorphisms $L^1(G) \to \mathcal{B}(H)$; these correspond to looking at unitary representations of $G$.
- If $G$ is finite, then given any representation of $G$ on $H$, we can always choose an invariant inner-product making the representation unitary.
- This corresponds to the following: if $\theta : L^1(G) \to \mathcal{B}(H)$ is a homomorphism, then there is an invertible $T \in \mathcal{B}(H)$ with

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# For the Fourier algebra

So, we might look at homomorphisms $\theta : A(G) \to \mathcal{B}(H)$.

- The involution on $A(G)$ is just pointwise conjugation of functions.
- If $\theta : A(G) \to \mathcal{B}(H)$ is a $*$-homomorphism, then you can continuously extend it to a $*$-homomorphism $C_0(G) \to \mathcal{B}(H)$, and such things are well-understood.
- So, we ask again: when is $\theta : A(G) \to \mathcal{B}(H)$ similar to a $*$-homomorphism? This seems hopeless…
- Instead, we restrict again to those $\theta$ such that the dilations

$$\iota \otimes \theta : \mathbb{T}_n \otimes A(G) \to \mathbb{M}_n \otimes \mathcal{B}(H)$$

are uniformly bounded in $n$. We say that $\theta$ is *completely bounded*.

# For the Fourier algebra

So, we might look at homomorphisms $\theta : A(G) \to \mathcal{B}(H)$.

- The involution on $A(G)$ is just pointwise conjugation of functions.
- If $\theta : A(G) \to \mathcal{B}(H)$ is a $*$-homomorphism, then you can continuously extend it to a $*$-homomorphism $C_0(G) \to \mathcal{B}(H)$, and such things are well-understood.
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Still looking at a homomorphism $\theta : A(G) \to \mathcal{B}(H)$.

- For technical reasons, introduce $\check{\theta} : A(G) \to \mathcal{B}(H)$ defined by $\check{\theta}(\omega) = \theta(\check{\omega})$. (Remember that $\check{\omega}(s) = \omega(s^{-1})$).
- Brannan and Samei (2010) showed that $\theta$ is similar to a $*$-homomorphism if, and only if, both $\theta$ and $\check{\theta}$ are completely bounded.
- Furthermore, if $G$ is discrete (or more generally a SIN group) then you don't need to consider $\check{\theta}$.
- Conjecture: this is true for all $G$.

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