Kaplansky Density for automorphism groups

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Outline

1. Operator algebras
2. One parameter automorphism groups
3. Interlude: Motivation
4. Kaplansky density for automorphism groups

Operator algebras

A C^*-*algebra* is either:

- A norm closed, self-adjoint, subalgebra A of $\mathcal{B}(H)$ (algebra of bounded operators on a Hilbert space).
- A Banach algebra A with an involution $*$ with $\|a^*a\| = \|a\|^2$ for $a \in A$.

A *von Neumann algebra* is either:

- A SOT closed, self-adjoint, subalgebra M of $\mathcal{B}(H)$.
  So if (x_i) a net in M, and $x \in \mathcal{B}(H)$, with $\|x_i(\xi) - x(\xi)\| \to 0$ for $\xi \in H$, then $x \in M$.
- A C^*-algebra M which is isometrically isomorphic to the dual of some Banach space M_*.

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Trace class operators

Let $\mathcal{T}(H)$ be the space of trace-class operators on H: those $x \in \mathcal{B}(H)$ for which $|x|$ has finite trace, $\mathrm{tr}(|x|) < \infty$.

For $\xi, \eta \in H$ let $\theta_{\xi,\eta} \in \mathcal{T}(H)$ be the rank-one operator

$$\theta_{\xi,\eta}(\gamma) = (\gamma|\eta)\xi \qquad (\gamma \in H).$$

There is a dual pairing between $\mathcal{T}(H)$ and $\mathcal{B}(H)$:

$$\langle x, y \rangle = \mathrm{tr}(xy) \qquad (x \in \mathcal{B}(H), y \in \mathcal{T}(H)).$$

- Under this, $\mathcal{B}(H)$ is the dual space of $\mathcal{T}(H)$.
- Under this, $\theta_{\xi,\eta}$ induces the "vector functional" $\omega_{\xi,\eta}$ on $\mathcal{B}(H)$:

$$\langle x, \omega_{\xi,\eta} \rangle = \mathrm{tr}(x\theta_{\xi,\eta}) = (\eta|x(\xi)) \qquad (x \in \mathcal{B}(H)).$$

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Preduals

- We often write $\mathcal{B}(H)_*$ for $\mathcal{T}(H)$ as $\mathcal{T}(H)$ is the *predual* of $\mathcal{B}(H)$.
- Given a von Neumann algebra $M \subseteq \mathcal{B}(H)$, that M is SOT closed means that...
- M is closed in $\mathcal{B}(H)$ for the weak*-topology induced by $\mathcal{B}(H)_*$.
- Equivalently, $M = ({}^\perp M)^\perp$ where

$$ {}^\perp M = \{\omega \in \mathcal{B}(H)_* : \langle x, \omega \rangle = 0 \ (x \in M)\}. $$

- Equivalently (Hahn-Banach) the quotient $M_* = \mathcal{B}(H)_* / {}^\perp M$ is the predual of M:

$$ \big(\mathcal{B}(H)_* / {}^\perp M\big)^* = ({}^\perp M)^\perp = M. $$

- Conversely, if M is a C^*-algebra with a predual M_*, a GNS type argument shows that there is H with $M \subseteq \mathcal{B}(H)$ and $M_* \cong \mathcal{B}(H)_* / {}^\perp M$.

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Kaplansky Density

Theorem (Kaplansky)

Let M be a von Neumann algebra, and $A \subseteq M$ be a C^*-algebra which is weak-dense in M. Then the unit ball of A is weak*-dense in the unit ball of M.*

How could this fail?

Consider a Hilbert space H with orthonormal basis (e_n). Think of $x \in \mathcal{B}(H)$ as an infinite matrix (x_{ij}). Let ω be a state on $\mathcal{B}(H)$ which annihilates all compact operators. Finally, set

$$X = \{x \in \mathcal{B}(H) : 2x_{11} = \omega(x)\}.$$

Claim

The weak-closure of X equals all of $\mathcal{B}(H)$.*

Sketch.

The compacts are weak*-dense in $\mathcal{B}(H)$, so approximate $x \in \mathcal{B}(H)$ by a compact. Then fiddle what happens to the $(1,1)$ matrix entry, by adding a multiple of the identity, to get inside X. □

How could this fail, cont.

$$X = \{x \in \mathcal{B}(H) : 2x_{11} = \omega(x)\}.$$

- If x is in the unit ball of X then $2|x_{11}| = |\omega(x)| \leq \|x\| \leq 1$ (as ω is a state). So $|x_{11}| \leq 1/2$.
- As evaluating a matrix entry is weak*-continuous, any x in the weak*-closure of the unit ball of X has $|x_{11}| \leq 1/2$.
- Thus the unit ball of X is not weak*-dense in the unit ball of $\mathcal{B}(H)$.
- But there is some sort of norm control. Q: Is this necessary?

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Algebra example

For any subspace $Y \subseteq \mathcal{B}(H)$ let

$$S_Y = \left\{ \begin{pmatrix} \alpha & x \\ 0 & \alpha \end{pmatrix} : \alpha \in \mathbb{C}, x \in Y \right\} \subseteq \mathcal{B}(H \oplus H) = M_2(\mathcal{B}(H)).$$

- This is a subalgebra, but not self-adjoint.
- The weak*-closure of S_Y is $S_{\overline{Y}}$, where $\overline{Y}$ is the weak*-closure of Y in $\mathcal{B}(H)$.
- So S_X is weak*-dense in $S_{\mathcal{B}(H)}$.
- If $\begin{pmatrix} \alpha & x \\ 0 & \alpha \end{pmatrix}$ is in the unit ball of S_X then $\|x\| \leq 1$. And so $|x_{11}| \leq 1/2$.
- So the weak*-closure of the unit ball of S_X does not contain $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, for example.

Outline

Automorphism groups

Definition

Let E be a Banach space. A one-parameter group of isometries of E is a family $(\alpha_t)_{t\in\mathbb{R}}$ with:

- Each α_t is a contraction in $\mathcal{B}(E)$;
- $\alpha_0 = 1$;
- $\alpha_{t+s} = \alpha_t \circ \alpha_s$ for $s, t \in \mathbb{R}$.

Then $\alpha_{-t} \circ \alpha_t = \alpha_t \circ \alpha_{-t} = \alpha_0 = 1$ so each α_t is a bijective isometry.

Say that (α_t) is *strongly-continuous* or a *C_0-group* if

$$\lim_{t\to 0} \|\alpha_t(x) - x\| = 0 \qquad (x \in E).$$

Equivalently, $\mathbb{R} \to E, t \mapsto \alpha_t(x)$ is (norm) continuous.

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Examples

Let $E = H$ a Hilbert space, so that each α_t is a unitary on H.

Theorem (Stone)

There is an (unbounded) self-adjoint operator T with $\alpha_t = \exp(iTt)$ for $t \in \mathbb{R}$.

Let $T \in \mathbb{M}_n$ be self-adjoint, so $u_t = \exp(iTt)$ forms a 1-parameter unitary group on $\mathbb{C}^n$. For $x \in \mathbb{M}_n$ define

$$\alpha_t(x) = u_t x u_{-t} = e^{iTt} x e^{-iTt} \qquad (x \in \mathbb{M}_n).$$

- Each α_t is an isometry for the operator norm.
- (α_t) is a 1-parameter group.
- Each α_t is a $*$-automorphism of the algebra $\mathbb{M}_n$.

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Examples cont.

Consider $C_0(\mathbb{R})$, the C^*-algebra of continuous functions $f : \mathbb{R} \to \mathbb{C}$ with $\lim_{|t|\to\infty} f(t) = 0$.

- Define $\alpha_t(f)$ to be the function $s \mapsto f(s-t)$.
- Then (α_t) is a 1-parameter group of $*$-automorphisms of $C_0(\mathbb{R})$.

Let $L^\infty(\mathbb{R})$ be the von Neumann algebra of (equivalence classes) of (essentially) bounded measurable functions $f : \mathbb{R} \to \mathbb{C}$.

- Define $\alpha_t(f)$ to be the function $s \mapsto f(s-t)$.
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Notice that $C_0(\mathbb{R})$ is weak*-dense in $L^\infty(\mathbb{R})$, and that the automorphism groups are compatible with this inclusion.

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Analytic generators: Holomorphic functions

Let E be a Banach space, $D \subseteq \mathbb{C}$ a domain, and $f : D \to E$ a function. The following are equivalent:

- f is *analytic* in the sense that for each $\alpha \in D$ there is an absolutely convergence power series for f, near α:

$$f(z) = \sum_{n \geq 0} a_n (z - \alpha)^n \qquad |z - \alpha| < r.$$

- f is holomorphic, in the sense that there is $F \subseteq E^*$ norming, with $D \to \mathbb{C}; z \mapsto \phi(f(z))$ is differentiable, for each $\phi \in F$.

Here *norming* means that

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In particular, "weakly holomorphic" or "weak*-holomorphic" imply "norm analytic".

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Analytic generators: Regular functions

Given $\alpha \in \mathbb{C}$ let

$$S(\alpha) = \left\{ z \in \mathbb{C} : \begin{matrix} 0 \leq \mathrm{Im}(z) \leq \mathrm{Im}(\alpha) & \text{if } \mathrm{Im}(\alpha) \geq 0 \\ 0 \geq \mathrm{Im}(z) \geq \mathrm{Im}(\alpha) & \text{if } \mathrm{Im}(\alpha) \leq 0 \end{matrix} \right\}.$$

That is, the closed horizontal strip bounded by $\mathbb{R}$ and $\mathbb{R} + \alpha$.

A function $f : S(\alpha) \to E$ is *regular* if f is continuous, analytic in the interior of $S(\alpha)$, and bounded on $\mathbb{R}$ and $\mathbb{R} + \alpha$:

$$M := \sup_{t \in \mathbb{R}} \max\big(\|f(t)\|, \|f(\alpha + t)\| \big) < \infty.$$

The 3-Lines Theorem shows that then $\|f(z)\| \leq M$ for all $z \in S(\alpha)$.

Some link with complex interpolation?

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$$x \in D(\alpha_z) \text{ when there is } f : S(z) \to E \text{ regular with } f(t) = \alpha_t(x) \ (t \in \mathbb{R}).$$

Then we set $\alpha_z(x) = f(z)$.

- Morera's Theorem and the Reflection Principle imply that such an f is unique. So α_z is well-defined.
- Think of α_z as an "analytic extension" of the mapping $t \mapsto \alpha_t(x)$.
- Can show that $D(\alpha_z)$ is dense in E and that α_z is *closed*.
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- Morera's Theorem and the Reflection Principle imply that such an f is unique. So α_z is well-defined.
- Think of α_z as an "analytic extension" of the mapping $t \mapsto \alpha_t(x)$.
- Can show that $D(\alpha_z)$ is dense in E and that α_z is *closed*.
- Then α_{-i} is the *analytic generator* of (α_t).

Analytic generators

Given (α_t), a 1-parameter group on E, and $z \in \mathbb{C}$, define an operator $D(\alpha_z) \to E$ by

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Examples

When (α_t) is a continuous unitary group on a Hilbert space H, with $\alpha_t = \exp(iTt)$, then

$$\alpha_{-i} = \exp(T).$$

Define $\exp(T)$ by functional calculus. The equality means with equality of domains. (Of course formally obvious; but the LHS and RHS have different definitions.)

If (α_t) on $\mathbb{M}_n$ is

$$\alpha_t(x) = u_t x u_{-t} = e^{iTt} x e^{-iTt},$$

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α_z is *closed* in the sense that the *graph*

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is closed.

Recall how to compose two unbounded operators $T : D(T) \to E, S : D(S) \to E$:

$$D(ST) = \{x \in D(T) : T(x) \in D(S)\}; \quad ST : D(ST) \ni x \mapsto S(T(x)).$$

Then $S = T$ means $\mathcal{G}(S) = \mathcal{G}(T)$; and $S \subseteq T$ means $\mathcal{G}(S) \subseteq \mathcal{G}(T)$. As closed operators, we have that

- $\alpha_t \circ \alpha_z = \alpha_z \circ \alpha_t = \alpha_{t+z}$
- If z, w lie on the same side of the real axis, then $\alpha_z \alpha_w = \alpha_{z+w}$
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$$\alpha_t(f)(s) = f(s-t) \qquad (s,t \in \mathbb{R}, f \in C_0(\mathbb{R})).$$

- Let $f \in D(\alpha_{-i})$;
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- Define $g : S(i) \to \mathbb{C}$ by $g(z) = F(-z)(0)$.
- Then $g(t) = F(-t)(0) = \alpha_{-t}(f)(0) = f(t)$.
- Also g is regular.
- Can reverse this: given regular $g : S(i) \to \mathbb{C}$ then define $F : S(-i) \to C_0(\mathbb{R})$ by $F(z)(t) = g(t-z)$, so that F becomes a $C_0(\mathbb{R})$-valued regular function.

So f itself analytically extends to $S(i)$, and $F(-i)$ is this extension of f, evaluated on $\mathbb{R} + i$.

(Somehow like a Hardy space. . .)

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Some properties: C^*-algebra case

Now suppose $E = A$ is a C*-algebra and each α_t is a $*$-automorphism. Given $a, b \in D(\alpha_z)$ with associated regular functions

$$F_a, F_b : S(z) \to A$$

we can pointwise multiply to obtain

$$F : S(z) \to A; \quad w \mapsto F_a(w)F_b(w).$$

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Outline

Locally compact quantum groups

The Operator algebraic approach to Quantum Groups uses C^* and von Neumann algebras to generalise the notion of a locally compact group, and Pontryagin duality.

- Write $\mathbb{G}$ for the "abstract quantum group" and $L^\infty(\mathbb{G})$ and $C_0(\mathbb{G})$ for the associated algebras.
- The correct notion of the "group inverse" here is the *antipode* S, which in interesting examples turns out to be unbounded.
- Can "polar decompose" $S = R\tau_{-i/2}$ where R is the *unitary antipode* (and anti-$*$-automorphism), and...
- (τ_t) is the *scaling group*, a 1-parameter group of $*$-automorphisms of $L^\infty(\mathbb{G})$.
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Furthermore, S, R and (τ_t) all drop to $C_0(\mathbb{G})$ which is weak*-dense in $L^\infty(\mathbb{G})$.

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Von Neumann setting

Each α_t is normal, and for $x \in M$, the orbit map $R \to M; \quad t \mapsto \alpha_t(x)$ is weak*-continuous.

- Form α_z in the same way, but we only require a weak*-regular extension.
- (But weak*-holomorphic implies norm analytic. The extension to the boundary is only weak*-continuous).
- Then $\mathcal{G}(\alpha_z)$ is weak*-closed.
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Setup

We will suppose we have:

- a C^*-algebra A which is weak*-dense in a von Neumann algebra M;
- A (strongly-continuous) 1-parameter $*$-automorphism group (α_t^A) on A, which extends to a (weak*-continuous) 1-parameter $*$-automorphism group (α_t^M) on M.

So we can consider:

$$\alpha_{-i}^A \text{ a norm-closed, norm-densely defined operator on } A,$$
$$\alpha_{-i}^M \text{ a weak}^*\text{-closed, weak}^*\text{-densely defined operator on } M.$$

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One can show that actually

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Kaplansky

Theorem

The unit ball of $\mathcal{G}(\alpha^A_{-i})$ is weak-dense in the unit ball of $\mathcal{G}(\alpha^M_{-i})$.*

To be concrete, this means that given $x \in D(\alpha^M_{-i})$ with

$$\|x\| \leq 1 \text{ and } \|\alpha^M_{-i}(x)\| \leq 1,$$

there is a net (a_j) in $D(\alpha^A_{-i})$ with $a_j \to x$ and $\alpha^A_{-i}(a_j) \to \alpha^M_{-i}(x)$ weak*, and with

$$\|a_j\| \leq 1 \text{ and } \|\alpha^A_{-i}(a_j)\| \leq 1.$$

Sketch of proof

The key idea is von Neumann algebraic:

- Using Kaplansky density for $A \subseteq M$ we see that A norms the predual M_*.
- Equivalently, the induced map $M_* \to A^*$ (given by restricting functions in M_* to $A \subseteq M$) is an isometry.
- The resulting subspace of A^* is an A-bimodule, and so there is a central projection $z \in A^{**}$ with $A^*z = M_*$.
- Thus $A^{**}z \cong M$.

We now consider $\mathcal{G}(\alpha^A_{-i})^{**} \subseteq A^{**} \oplus A^{**}$. One can carefully show that

$$\mathcal{G}(\alpha^M_{-i}) \cong \mathcal{G}(\alpha^A_{-i})^{**}(z \oplus z) \text{ and } \mathcal{G}(\alpha^M_{-i}) \subseteq \mathcal{G}(\alpha^A_{-i})^{**}.$$

Sketch of proof

The key idea is von Neumann algebraic:

- Using Kaplansky density for $A \subseteq M$ we see that A norms the predual M_*.
- Equivalently, the induced map $M_* \to A^*$ (given by restricting functions in M_* to $A \subseteq M$) is an isometry.
- The resulting subspace of A^* is an A-bimodule, and so there is a central projection $z \in A^{**}$ with $A^* z = M_*$.
- Thus $A^{**} z \cong M$.

We now consider $\mathcal{G}(\alpha^A_{-i})^{**} \subseteq A^{**} \oplus A^{**}$. One can carefully show that

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Open question

Swap things about:

- The adjoints of (α_t^A) give rise to a weak*-continuous 1-parameter isometry group on A^*.
- The pre-adjoints of (α_t^M) give rise to a norm-continuous 1-parameter isometry group on M_*.

We have the isometric inclusion $M_* \to A^*$ which leads to

$$\mathcal{G}(\alpha_{-i}^{M_*}) \subseteq \mathcal{G}(\alpha_{-i}^{A^*}),$$

which is weak*-dense.

Does Kaplansky Density hold for this?

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