Compactifications and the Fourier Algebra

Matthew Daws

29th February 2008

Outline

Locally compact groups

Let G be a locally compact group; then G has a Haar measure: a left-invariant Radon measure on G.

It is often interesting just to consider a discrete group G. Then the Haar measure is just the counting measure.

If G is a compact group, we normalise the Haar measure to be a probability measure.

The Haar measure on $\mathbb{R}$ is just the Lebesgue measure.

Let $L^1(G)$ be the usual space of integrable functions, with respect to Haar measure. We turn $L^1(G)$ into a Banach algebra with the convolution product.

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von Neumann considerations

$L^1(G)$ is the predual of the commutative von Neumann algebra $L^\infty(G)$. Here we treat $L^\infty(G)$ as an algebra acting on $L^2(G)$. Define a unitary $W : L^2(G \times G) \to L^2(G \times G)$ by

$$WF(s,t) = F(s, s^{-1}t) \qquad (F \in L^2(G \times G), s,t \in G).$$

Then we define $\Delta : L^\infty(G) \to L^\infty(G) \overline{\otimes} L^\infty(G) = L^\infty(G \times G)$ by

$$\Delta(f)(s,t) = f(st) \qquad (f \in L^\infty(G), s,t \in G).$$

Notice that

$$\Delta(f) = W^*(\mathrm{id} \otimes f)W \qquad (f \in L^\infty(G)).$$

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C*-algebras

$L^1(G)$ has a natural involution: for a discrete group, it is just the map sending $s \mapsto s^{-1}$. However, $L^1(G)$ is not a C*-algebra.

$L^1(G)$ acts on $L^2(G)$ by convolution on the left; the closure of $L^1(G)$ in $\mathcal{B}(L^2(G))$ is $C_r^*(G)$, the *reduced group C*-algebra*.

Alternatively, we can give $L^1(G)$ the maximal C*-algebra norm, leading to $C^*(G)$, the *group C*-algebra*.

Recall that $C_r^*(G) = C^*(G)$ if and only if G is *amenable*.

Question: Find a non-amenable group G such that $C_r^*(G)$ and $C^*(G)$ are not the only C*-completions of $L^1(G)$.

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Hopf von Neumann algebras

Let $VN(G)$ be the weak-operator topology closure of $L^1(G)$ in $\mathcal{B}(L^2(G))$, the *group von Neumann algebra*.

Let $\lambda : G \to \mathcal{B}(L^2(G))$ be the left-regular representation

$$(\lambda(s)f)(t) = f(s^{-1}t) \qquad (f \in L^2(G), s, t \in G).$$

Then $VN(G)$ is generated by $\{\lambda(s) : s \in G\}$.

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The Fourier algebra

Let $A(G)$ be the predual of $VN(G)$, the *Fourier algebra of G*.

As $\Delta : VN(G) \to VN(G)\overline{\otimes}VN(G)$ is normal, it has a pre-adjoint, a *completely-contractive* map $\Delta_* : A(G)\widehat{\otimes}A(G) \to A(G)$. We can check that this gives an associative product on $A(G)$.

If G is an *abelian* group, then we have the Pontryagin dual $\hat{G}$, and the Fourier transform $L^1(G) \to C_0(\hat{G})$. The image of $L^1(G)$ is $A(\hat{G})$.

For example, let $G = \mathbb{Z}$, so $\hat{G} = \mathbb{T}$. Hence

$$L^1(\mathbb{Z}) \cong A(\mathbb{T}), L^\infty(G) \cong VN(\mathbb{T}), c_0(\mathbb{Z}) \cong C_r^*(\mathbb{T}),$$
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Hopf-von Neumann algebras

Informally, we think of $A(G)$ as being the L^1-algebra of the dual group of G, even when this strictly doesn't make sense.

A *Hopf-von Neumann algebra* $\mathcal{M}$ is a von Neumann algebra equipped with a *co-associative* $*$-homomorphism $\Delta : \mathcal{M} \to \mathcal{M}\overline{\otimes}\mathcal{M}$; that is

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A *locally compact quantum group* is a Hopf-von Neumann algebra equipped with further structure (a replacement for the Haar measure). In this setting, one can formulate an abstract duality theory: a Hopf-von Neumann algebra $\hat{\mathcal{M}}$, such that $\hat{\hat{\mathcal{M}}} = \mathcal{M}$.

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Compact quantum groups

There is a C*-algebra counterpart to the von Neumann flavour of this theory. This is much more technical, except in the "compact" case.

A *compact quantum group* is a unital C*-algebra $\mathcal{A}$ with a co-associative product $\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\text{min}} \mathcal{A}$, such that the sets $(\mathcal{A} \otimes 1)\Delta(\mathcal{A})$ and $(1 \otimes \mathcal{A})\Delta(\mathcal{A})$ are linearly dense in $\mathcal{A} \otimes \mathcal{A}$.

For example, let G be a compact space, and let $\mathcal{A} = C(G)$. A $*$-homomorphism $\Delta : C(G) \to C(G \times G)$ is equivalent to a continuous map $G \times G \to G$; Δ is co-associative if and only if this product is associative.

A bit of group theory, combined with Stone-Weierstrass, shows that G is a group if and only if the density conditions hold.

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Bohr compactifications

Let G be a topological group. The *Bohr compactification* of G is a compact group $\mathfrak{b}G$:

- there is a continuous homomorphism $\iota : G \to \mathfrak{b}G$ with dense range;
- for all compact groups H,

In contrast to, say, the Stone-Cech Compactification, ι need not be injective. In fact, there exist groups G such that $\mathfrak{b}G = \{1\}$. (The Lorentz group, for example).

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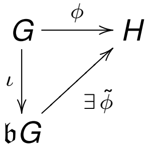

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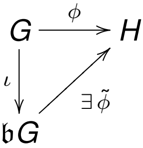

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Existance

To show that $\mathfrak{b}G$ exists, we can simply take the collection of all compact groups H which contain a dense, homomorphic image of G, and then "glue" them together in some sense.

We understand the representation theory of compact groups very well: every irreducible representation is finite-dimensional, and may be assumed to be on a Hilbert space.

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A more concrete approach

Consider the C*-algebra $C^b(G)$. Let $\mathrm{ap}(G)$ be the collection of those $f \in C^b(G)$ such that the family of left translates of f forms a relatively compact subset of $C^b(G)$.

We can show that $\mathrm{ap}(G)$ is a unital C*-subalgebra of $C^b(G)$. So $\mathrm{ap}(G)$ has character space G^{ap}, hence $\mathrm{ap}(G) \cong C(G^{\mathrm{ap}})$.

Clearly G maps into G^{ap}; we can extend the group product from G to G^{ap}, turning G^{ap} into a semigroup.

The topology on G^{ap} is such that this semigroup product is jointly continuous. It follows that we can also extend the inverse operation from G to G^{ap} turning G^{ap} into a compact group.

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Sołtan's Quantum approach

A *quantum semigroup* is a C*-algebra $\mathcal{A}$ equipped with a co-associative *morphism* $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$.

Example: Let S be a topological semigroup, let $\mathcal{A} = C_0(S)$, and define $\Delta : \mathcal{A} \to C^b(S \times S)$ by

$$\Delta(f)(s,t) = f(st) \qquad (f \in C_0(S), s,t \in S).$$

There are notions of *unitary representation* and so forth for quantum groups. In particular, the representation theory of compact quantum groups parallels that for compact groups.

Using the abstract "gluing" idea, Sołtan found that for any quantum semigroup $\mathbb{S} = (\mathcal{A}, \Delta)$, one can find a compact quantum group $\mathfrak{b}\mathbb{S}$ which satisfies the expected universal property.

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Example calculation

Let G be a locally compact group, and consider $\mathbb{G} = C_r^*(G)$.

This has a co-associative product, but this is a little hard to describe.

Then $\mathfrak{b}\mathbb{G}$ is always a unital C*-subalgebra of the *multiplier* algebra of $C_r^*(G)$. We can regard this as the subalgebra

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Then $\mathfrak{b}\mathbb{G} = C_\rho^*(G)$, which is the C*-algebra generated by $\lambda(G) = \{\lambda(s) : s \in G\}$ in $VN(G)$. Recall that $VN(G)$ is the von Neumann algebra generated by $\lambda(G)$.

Let G_d be the group G with the discrete topology. If G_d is amenable, then as $C^*(G_d) = C_r^*(G_d)$, it follows that $C_\rho^*(G) \cong C_r^*(G_d)$.

In general, it seems that $C_\rho^*(G)$ could be the C*-completion of $\ell^1(G_d)$ in some norm such that $C_\rho^*(G)$ is *not* $C^*(G_d)$ or $C_r^*(G_d)$. $G = SO(3)$??

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Let G be a locally compact group, and consider $\mathbb{G} = C_r^*(G)$. This has a co-associative product, but this is a little hard to describe.

Then $\mathfrak{b}\mathbb{G}$ is always a unital C*-subalgebra of the *multiplier* algebra of $C_r^*(G)$. We can regard this as the subalgebra

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More about the Fourier algebra

The setting: G is a locally compact group; $\lambda : G \to \mathcal{B}(L^2(G))$ is the left-regular representation; $VN(G)$ is the group von Neumann algebra generated by $\lambda(G)$; $A(G)$ is the predual, turned into an algebra by the co-assocative product Δ.

$A(G)$ is a regular commutative Banach algebra, which has character space G. More explicitly, given $a \in A(G)$, we regard a as an element in $C_0(G)$ by

$$a(s) = \langle \lambda(s), a \rangle \qquad (a \in A(G), s \in G).$$

Of course, $A(G)$ is not *closed* in $C_0(G)$.

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Almost periodic elements in Banach algebras

Let $\mathcal{A}$ be a Banach algebra. We turn $\mathcal{A}^*$ into a left $\mathcal{A}$-module by

$$\langle a \cdot \mu, b \rangle = \langle \mu, ba \rangle \qquad (\mu \in \mathcal{A}^*, a, b \in \mathcal{A}).$$

We define ap($\mathcal{A}$) to be the collection of $\mu \in \mathcal{A}^*$ such that the map

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is compact.

Example: Let G be a locally compact group, and let $\mathcal{A} = L^1(G)$, so that $\mathcal{A}^* = L^\infty(G) \supseteq C^b(G)$. Then ap($\mathcal{A}$) $=$ ap(G).

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We define $\mathrm{ap}(\hat{G}) = \mathrm{ap}(A(G))$.

Then $\mathrm{ap}(\hat{G})$ behaves *vaguely* like $\mathrm{ap}(G)$.

It is not known if $\mathrm{ap}(\hat{G})$ is always a C^*-algebra, however.

Work of Chou and Rindler shows that there exists *compact* groups G such that $\mathrm{ap}(\hat{G}) \neq C^*_\rho(G)$.

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A way forward: approximation!

The Banach space $L^\infty(G)$ has the *approximation property*; in particular, for $f \in L^\infty(G)$, the map

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is compact if and only if it can be norm approximated by finite-rank maps; is it *approximable*.

In contrast, if $VN(G)$ has the approximation property, then G is abelian-by-finite.

So it is possible that, for some $x \in VN(G)$, the map

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The other ingredient: operator spaces

An *operator space* E is a subspace of $\mathcal{B}(H)$; this induces a norm on $\mathbb{M}_n(E)$, by

$$\mathbb{M}_n(E) \subseteq \mathbb{M}_n(\mathcal{B}(H)) = \mathcal{B}(H \oplus \cdots \oplus H).$$

A map $\phi : E \to F$ between two operator spaces is *completely bounded* if the map $\phi_n : \mathbb{M}_n(E) \to \mathbb{M}_n(F)$,

$$\phi_n : (a_{ij})_{i,j=1}^n \mapsto \big(\phi(a_{ij})\big)_{i,j=1}^n,$$

is uniformly bounded in n.

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In particular, $VN(G)$ carries a natural operator space structure, and so $A(G)$, as the predual of $VN(G)$, also has a natural operator space structure.

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Putting it together

So maybe "compact" is the wrong condition for $A(G)$.

Instead, we ask for "completely bounded approximable". That is, operators which can be approximated by finite-rank maps, in the completely bounded norm.

Then everything works! That is, $C^*_\rho(G)$ is the collection of $x \in VN(G)$ such that the map

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