The Haagerup approximation property for discrete quantum groups

Matthew Daws

Leeds

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Amenable groups

Definition

A discrete group Γ is amenable ($C_r^*(\Gamma)$ is nuclear) if and only if there is a net of *finitely supported* positive definite functions f_i on Γ such that (f_i) forms an approximate identity for $c_0(\Gamma)$.

Proof.

($\Rightarrow$) Følner net.
($\Leftarrow$) A finitely supported positive definite function is in the Fourier Algebra $A(\Gamma)$ (the ultraweakly continuous functionals on $VN(\Gamma)$). So we obtain a bounded net in $A(\Gamma)$ converging pointwise to the constant function. Hence this is a bounded approximate identity for $A(\Gamma)$, and so Γ is amenable (Leinert). □

(All works for locally compact, with "finite" replaced by "compact".)

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A discrete group Γ has the Haagerup approximation property if and only if there is a net of positive definite functions f_i on Γ such that (f_i) forms an approximate identity for $c_0(\Gamma)$.

So "finitely-supported" becomes "vanishes at infinity; i.e. in $c_0(\Gamma)$".

- [Haagerup] F_n has HAP.
- Groups acting on trees have HAP.
- Stable under (amalgamated over a finite subgroup) free products.

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Applications to operator algebras

Let (M, τ) be a finite von Neumann algebra, with GNS space $L^2(M, \tau)$ and cyclic vector ξ_0. If $\Phi : VN(\Gamma) \to VN(\Gamma)$ is positive, $\tau \circ \Phi \leq \tau$, and $\Phi(x)^*\Phi(x) \leq \Phi(x^*x)$, then there is a bounded map T on $L^2(M, \tau)$ with

$$T(x\xi_0) = \Phi(x)\xi_0 \qquad (x \in M).$$

Theorem (Choda, 83)

Γ has the Haagerup approximation property if and only if $VN(\Gamma)$ has the HAP, defined as: there is a net (Φ_i) of normal UCP maps on $VN(\Gamma)$, approximating the identity point-σ-weakly, and preserving the trace, such that the induced maps on $\ell^2(\Gamma)$ are compact.

This leads to the HAP for finite von Neumann algebras. [Jolissaint, '02] showed this is independent of the choice of trace.

K-amenability etc.

Theorem (Tu, 99)

If Γ has HAP then Γ is K-amenable.

"Morally", this means that the left-regular representation $\lambda : C^*(\Gamma) \to C^*_r(\Gamma)$ induces isomorphisms in K-theory, $(\lambda)_* : K_i(C^*(\Gamma)) \to K_i(C^*_r(\Gamma))$. Actually definition involves KK-theory.

Theorem (Higson, Kasparov, 97, Tu, 99)

If Γ has HAP then the Baum-Connes conjecture holds for Γ.

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Quantum groups

Definition (Woronowicz)

A compact quantum group is (A, Δ) where A is a unital C^*-algebra, $\Delta : A \to A \otimes A$ is a $*$-homomorphism which is "coassociative": $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$; and such that "quantum cancellation" holds:

$$\mathrm{lin}\{\Delta(a)(b \otimes 1) : a, b \in A\}, \quad \mathrm{lin}\{\Delta(a)(1 \otimes b) : a, b \in A\}$$

are dense in $A \otimes A$.

Motivation: Let G be a compact semigroup, set $A = C(G)$, and define

$$\Delta : C(G) \to C(G \times G); \quad \Delta(f)(s,t) = f(st) \quad (f \in C(G), s, t \in G).$$

Then Δ is coassociative as the product in G is associative, and quantum cancellation holds if and only if

$$st = st' \implies t = t', \quad ts = t's \implies t = t' \qquad (s, t, t' \in G).$$

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Compact groups to quantum groups

Lemma

A compact semigroup with cancellation is a group.

The Haar (probability) measure (the unique invariant Borel measure on G) induces a state φ in $C(G)$ such that

$$(\varphi \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \varphi)\Delta(a) = \varphi(a)1_G \qquad (a \in C(G)).$$

(Remember that $\Delta(f)(s,t) = f(st)$, so $\varphi \otimes \mathrm{id})\Delta$ is integrating out the first variable.)

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Discrete groups to quantum groups

Let Γ be a discrete group, and form $C_r^*(\Gamma)$ acting on $\ell^2(\Gamma)$, generated by the left translation operators $(\lambda_t)_{t\in\Gamma}$. We claim that there is a $*$-homomorphism

$$\Delta : C_r^*(\Gamma) \to C_r^*(\Gamma) \otimes C_r^*(\Gamma); \quad \lambda_t \mapsto \lambda_t \otimes \lambda_t.$$

Proof: Fell absorption principle, or observe that

$$\Delta(x) = W^*(1 \otimes x)W \quad \text{for} \quad W(\delta_s \otimes \delta_t) = \delta_{t^{-1}s} \otimes \delta_t.$$

Then Δ obviously coassociative and satisfies quantum cancellation:

$$\begin{aligned} (\Delta \otimes \mathrm{id})\Delta &= (\mathrm{id} \otimes \Delta)\Delta, \\ \overline{\mathrm{lin}}\{\Delta(a)(b \otimes 1) : a, b \in C_r^*(\Gamma)\} &= \overline{\mathrm{lin}}\{\Delta(a)(1 \otimes b) : a, b \in C_r^*(\Gamma)\} \\ &= C_r^*(\Gamma) \otimes C_r^*(\Gamma). \end{aligned}$$

Universal case

If φ is the canonical trace on $C_r^*(\Gamma)$ then

$$(\varphi \otimes \text{id})\Delta(a) = (\text{id} \otimes \varphi)\Delta(a) = \varphi(a)1 \qquad (a \in C(G)).$$

Can also do all this with $C^*(\Gamma)$:

- here the existence of $\Delta(\lambda_t) = \lambda_t \otimes \lambda_t$ follows by universality— the map $t \mapsto \lambda_t \otimes \lambda_t$ is a unitary representation of Γ.

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General case

In general, starting from (A, Δ) can prove the existence of a "Haar state" φ on A with

$$(\varphi \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \varphi)\Delta(a) = \varphi(a)1 \qquad (a \in A).$$

Important that φ may fail to be a trace.

- Maybe φ won't be faithful, but can always quotient to obtain (A_r, Δ_r).
- On the GNS space $L^2(\varphi)$ set $M = A_r''$.
- Then Δ extends to M (because we can always construct a suitable unitary W with $\Delta(\cdot) = W^*(1 \otimes \cdot)W$).
- Can always form a "universal" version of A, say A_u.
- Generalises the passage between $C_r^*(\Gamma)$, $C^*(\Gamma)$ and $VN(\Gamma)$.

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Representation Theory

Dualising the notion of a group representation, we obtain:

Definition

A finite-dimensional unitary corepresentation is $U = (U_{ij}) \in M_n(A)$ with $\Delta(U_{ij}) = \sum_{k=1}^n U_{ik} \otimes U_{kj}$.

- The collection of all elements U_{ij} forms a dense $*$-subalgebra of A, say A_0, such that Δ gives a map $A_0 \to A_0 \odot A_0$.
- In fact, (A_0, Δ) is a Hopf $*$-algebra.
- For $C(G)$ get the "polynomials" on G; for $C_r^*(\Gamma)$ get $\mathbb{C}[\Gamma]$.
- The enveloping C^*-algebra of A_0 is the "full" or "universal" version of A, which we denoted A_u.
- Pick representatives of the irreducibles, say $\{U^\alpha : \alpha \in \Lambda\}$.
- Peter-Weyl theory: $L^2(\varphi) \cong \bigoplus L^2(M_{n_\alpha}, \varphi_\alpha)$. (Classically each φ_α a trace, but maybe not here).

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Dual groups

- The "dual group" $\widehat{A} = \bigoplus M_{n_\alpha}$ carries a coassociative map $\widehat{\Delta}$ (whose definition involves multiplier algebras).
- For $C(G)$ we obtain $C_r^*(G)$; for $C_r^*(\Gamma)$ we obtain $c_0(\Gamma)$
- Can axiomatise such objects: "discrete quantum groups" (van Daele).
- Can dualise again, and get back to the compact quantum group we started with.
- I'll adopt the notation that $\mathbb{G}$ is a discrete quantum group, $C_r^*(\mathbb{G}) = A$, $C^*(\mathbb{G}) = A_u$, $VN(\mathbb{G}) = M$.
- Similarly, $c_0(\mathbb{G}) = \widehat{A}$ the C*-algebra representing the discrete quantum group.
- The "left-regular representation" or "Fourier transform" is the map $\lambda : C^*(\mathbb{G})^* \to \ell^\infty(\mathbb{G}) = M(c_0(\mathbb{G}))$; $\mu \mapsto \big(\langle \mu, U_{ij}^\alpha \rangle\big) \in \prod M_{n_\alpha}$.

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HAP

Work with Fima, Skalski, White.

Definition

A "positive definite function" on $\mathbb{G}$ is any element $\lambda(\mu) \in \ell^\infty(\mathbb{G})$ with $\mu \in C^*(\mathbb{G})^*_+$. (I.e. the Fourier transform of a positive functional.)

[D., Salmi, 13] give various intrinsic characterizations involving, mainly, certain completely positive maps.

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$\mathbb{G}$ has HAP if and only if there is a net (a_i) of positive definite functions, such that (a_i) forms an approximate identity for $c_0(\mathbb{G})$.

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Various other characterisations?

As in the classical setting, we have various other characterisations:

- $\mathbb{G}$ has HAP if and only if there is a "mixing" representation of $\mathbb{G}$ weakly containing the trivial representation.
- $\mathbb{G}$ has HAP if and only if mixing representations are dense.

Classically G has HAP iff it admits a proper, conditionally negative definite function.

- Working with the Hopf $*$-algebra, can get a notion of this for $\mathbb{G}$.
- Quantum probability ideas allow one to form semigroups (under the convolution product) which give our approximate identities.
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For von Neumann algebras

We shall say that $\mathbb{G}$ is of "Kac type" if φ is a trace on $VN(\mathbb{G})$. Equivalent to the antipode of $c_0(\mathbb{G})$ or $C_r^*(\mathbb{G})$ being bounded.

Theorem (DFSW)

If $\mathbb{G}$ is of Kac type then $\mathbb{G}$ has HAP if and only if $VN(\mathbb{G})$ has HAP.

Proof.

($\Rightarrow$)As in the classical case, states μ on $C^*(\mathbb{G})$ induce *multipliers* on $VN(\mathbb{G})$ which are normal, UCP, and preserve φ. A calculation shows that the induced maps on $L^2(\varphi)$ agree with $\lambda(\mu)$; so if $\lambda(\mu) \in c_0(\mathbb{G})$ they are compact.

($\Leftarrow$)We use a (vaguely complicated) "averaging" argument to turn arbitrary normal UCP maps Φ on $VN(\mathbb{G})$ into multipliers. Then [D. 12] shows that CP multipliers come from states on $C^*(\mathbb{G})$. □

The UCP maps we construct restrict to $C_r^*(\mathbb{G})$ cf [Dong], [Suzuki].

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Examples

Let $U_N^+ = (A_u(N), \Delta)$ be the free unitary quantum group: $A_u(N)$ is the universal C*-algebra generated by elements $\{u_{ij} : 1 \leq i, j \leq N\}$ such that:

- $U = [u_{ij}]$ is unitary and $\overline{U} = [u_{ij}^*]$ is unitary.
- need latter condition for quantum cancellation laws.
- U_N^+ is of Kac type, so has a trace.
- the dual of U_N^+ is not amenable.

Theorem (Brannan, 12)

$L^\infty(U_N^+)$ has the HAP.

Corollary

The discrete dual of U_N^+ has HAP.

[Freslon, 13?] has a “transportation” procedure using monodial equivalence: gives HAP for various non-Kac examples.

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Question

Theorem (DFSW)

For all $\mathbb{G}$, if $\mathbb{G}$ has HAP then VN$(\mathbb{G})$ has HAP.

Of course φ is not a trace anymore...

- There is a tight relation between quantum group theory and KMS states: φ is KMS on $C_r^*(\mathbb{G})$ and $C^*(\mathbb{G})$.
- It's been suggested that maybe HAP for a state should include the condition that each map Φ "commute" with the modular automoprhism group.
- Not particularly clear for what values of "commute" this would be true, for the multipliers constructed above...
- Not clear what uses this definition might have...