Involutions on algebras of operators

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Involutions on $\mathcal{B}(E)$

Let E be a Banach space, and let $\mathcal{B}(E)$ be the algebra of operators on E.
We asked the question: when does $\mathcal{B}(E)$ admit an *involution*:

- $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$;
- $(a^*)^* = a$.

The Hilbert space, with the standard involution, is the obvious example.

Before continuing, note that Johnson's uniqueness of norm theorem shows that any involution on $\mathcal{B}(E)$ is automatically continuous. We shall hence assume that involutions are *continuous*, but maybe not *isometric*.

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Proper involutions and Hilbert spaces

This subject seems full of repeated discoveries and forgotten results. Niels and I hope we have a fairly accurate history of events.

An involution is *proper* if $a^* a = 0$ only when $a = 0$.

Theorem (Kakutani-Mackey-Kawada)

Let E be a Banach space such that $\mathcal{B}(E)$ has a proper involution. Then there is an inner-product $[\cdot, \cdot]$ on E such that:

1. $[T(x), y] = [x, T^*(y)]$;
2. *the norm given by $x \mapsto [x, x]^{1/2}$ is equivalent to the norm on E.*

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Involutions and Banach spaces

Theorem (Bognar)

Let E be a Banach space such that $\mathcal{B}(E)$ has an involution. There is a bounded sesquilinear form $[\cdot,\cdot]$ on E such that:

1. $[T(x), y] = [x, T^*(y)]$;
2. $[x, y] = \overline{[y, x]}$;
3. *for each $x \neq 0$, there exists y with $[x, y] \neq 0$.*

In particular, we need not have that $[x, x] \geq 0$.

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In particular, we need not have that $[x, x] \geq 0$.

Involution inducing maps

Let E be a Banach space, and $[\cdot,\cdot]$ be a sesquilinear form as in Bognar's Theorem. As the form is bounded, there exists a conjugate-linear map $J : E \to E'$ such that

$$[x, y] = \langle x, J(y)\rangle = J(y)(x) \qquad (x, y \in E).$$

Then the involution associated with the form satisfies

$$JT^* = T'J \qquad (T \in \mathcal{B}(E)),$$

where $T' \in \mathcal{B}(E')$ is the linear *adjoint* or *transpose* of E,

$$\langle x, T'(\mu)\rangle = \langle T(x), \mu\rangle \qquad (\mu \in E', x \in E).$$

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Surprisingly, Bognar did not see the following result. One proof has recently been found by Becerra Guerrero, Burgos, Kaidi, and Rodríguez-Palacios.

Theorem

Let E be a Banach space such that $\mathcal{B}(E)$ has an involution. Let $J : E \to E'$ be the conjugate-linear map given by Bognar's Theorem. Then J is a homeomorphism (that is, J has a bounded inverse) and so the involution is given by

$$T^* = J^{-1} T' J \qquad (T \in \mathcal{B}(E)).$$

This new condition on J is equivalent to the statement that for each $\mu \in E'$, there exists $y \in E$ with

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Reflexivity

The proof shows that any involution on $\mathcal{B}(E)$ restricts to $\mathcal{F}(E)$, the *finite-rank* operators, and is completely determined by this restriction.

One can easily show that if E admits such a map $J : E \to E'$, then E must be *reflexive*. That is, the canonical map from E to its bidual is surjective.

Call such J *involution-inducing*.

So, does every reflexive E admit an involution on $\mathcal{B}(E)$?

Infact, it is simple to see that $\mathcal{B}(\ell^p)$, for $1 < p < \infty$, admits an involution if and only if $p = 2$.

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Flip example

This example goes back to Aronszajn.
Let E be reflexive, and suppose that there is a bounded, invertible, conjugate-linear map $\Gamma : E \to E$. An example of a *twisted Hilbert space* due to Kalton and Peck gives a reflexive Banach space Z for which no such map Γ can exist. However your favourite reflexive Banach space surely will (for example, all L^p spaces do).
We can define an involution on $E \oplus E'$, termed the *flip*, by defining a sesquilinear form as follows:

$$\big[(x,\mu),(y,\lambda)\big] = \overline{\langle \Gamma(x),\lambda\rangle} + \langle \Gamma(y),\mu\rangle \qquad ((x,\mu),(y,\lambda) \in E \oplus E').$$

If one starts with a Hilbert space H, then $H' \cong H$, and hence $H \oplus H' \cong H$. However, the flip involution is not the same as the usual involution.

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Involutions on Hilbert spaces

Let H be a Hilbert space, let $J : H \to H'$ be involution-inducing, and let $[\cdot,\cdot]$ be the *usual* inner-product on H.

We may define $S \in \mathcal{B}(H)$ by

$$\langle x, J(y)\rangle = [x, U(y)] \qquad (x, y \in H).$$

Then U is invertible, as J is, and U is self-adjoint, with respect to the usual involution.

By the Spectral Theory for normal operators, there exists a measure space (X, μ) such that H is unitarily equivalent to $L^2(X, \mu)$, and such that under this identification, U is given by multiplication by a function $f \in L^\infty(X, \mu)$. As U is self-adjoint and invertible, we see that f is real-valued and bounded above and below.

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Krein spaces

Now identify H with $L^2(X, \mu)$. Define $g : X \to \pm 1$ by setting $g(w) = 1$ when $f(w) > 0$, and $g(w) = -1$ when $f(w) < 0$. Let $V \in \mathcal{B}(H)$ be given by multiplication by g, so as f bounded above and below, there exists an invertible, positive map W such that $U = VW$.

We can define an involution-inducing map $K : H \to H'$ by

$$\langle x, K(y) \rangle = [x, V(y)] \qquad (x, y \in H).$$

Then H, with the sesquilinear form induced by K, is a *Krein space* (actually, Krein spaces are more general than this). Let the involutions induced by J and K be written as $\sharp$ and $\flat$ respectively. It then follows that as W is positive, the algebras $(\mathcal{B}(H), \sharp)$ and $(\mathcal{B}(H), \flat)$ are $*$-isomorphic.

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Decomposition of Krein spaces

With notation as above, let H_+ be the functions in $L^2(X, \mu)$ supported on the set $\{w : g(w) = 1\}$, and let H_- be the functions in $L^2(x, \mu)$ supported on $\{w : g(w) = -1\}$. Then $L^2(X, \mu) = H_+ \oplus H_-$ is an orthogonal decomposition, and the involution-inducing map K is given by

$$\langle x_+ + x_-, J(y_+ + y_-)\rangle = [x_+, y_+] - [x_-, y_-]$$

for $x_+, y_+ \in H_+$ and $x_-, y_- \in H_-$.

If you think hard enough about this, you'll see that this is, roughly, the infinite-dimensional version of Sylvester's Inertia Law.

We've hence seen that, essentially, any involution on $\mathcal{B}(H)$ arises in this way. Of course, the picture for general Banach spaces seems much more complicated.

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Renormings

We now come to some work by Chris Lance, done at the tail end of interest in representing Banach $*$-algebras, before such study settled on C*-algebras as the “correct” axiomisation.

Lance studied the case when $\mathcal{B}(E)$ admits a partially defined involution, again defined using a sesquilinear form. He gives a renorming result which, starting from a fairly general, bounded, sesquilinear form $[\cdot,\cdot]$ on a Banach space E, gives a norm $\|\cdot\|$ on E such that

$$\|x\| = \sup\{|[x,y]| : \|y\| \leq 1\} \qquad (x \in E).$$

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Renormings in our case

However, if we apply this result to an involution-inducing map, then the new norm will be *equivalent* to the old norm:

Theorem

Let E be a (reflexive) Banach space with an involution-inducing map $J : E \to E'$. Then there is an equivalent norm on E making J an isometry. This is equivalent to the involution induced by J being an isometry.

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Representing Banach $*$-algebras

Lance was interested in representing certain Banach $*$-algebras which are not C*-algebras. We can use our ideas to a similar end.

Let $\mathcal{A}$ be a Banach algebra, and let $\mu \in \mathcal{A}'$ be a functional. We say that μ is *weakly almost periodic* if the map $L_\mu : \mathcal{A} \to \mathcal{A}'$ defined by

$$\langle a, L_\mu(b) \rangle = \langle ab, \mu \rangle \quad (a, b \in \mathcal{A})$$

is *weakly-compact*.

By a clever use of interpolation spaces, Davis, Figiel, Johnson and Pelczynski showed that a map $T : E \to F$ between Banach spaces is weakly-compact if and only if T factors through a *reflexive* Banach space.

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By a clever use of interpolation spaces, Davis, Figiel, Johnson and Pelczynski showed that a map $T : E \to F$ between Banach spaces is weakly-compact if and only if T factors through a *reflexive* Banach space.

Representations on reflexive spaces

N. Young showed how to use the proof of this result to show that μ is weakly almost periodic if and only if there is a reflexive Banach space E, a representation $\pi : \mathcal{A} \to \mathcal{B}(E)$, and $x \in E, \lambda \in E'$ such that

$$\langle a, \mu \rangle = \langle \pi(a)(x), \lambda \rangle \qquad (a \in \mathcal{A}),$$

with $\|\mu\| = \|x\|\|\lambda\|$.

Compare this to the Gelfand-Naimark-Segal construction for a state on a C*-algebra.

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Continued

Theorem (Young)

Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:

1. *there is a faithful (bounded below) representation $\pi : \mathcal{A} \to \mathcal{B}(E)$ with E reflexive;*
2. *the weakly almost periodic functionals on $\mathcal{A}$ separate the point of $\mathcal{A}$ (form a quasi-norming set for $\mathcal{A}$).*

Here, $X \subseteq \mathcal{A}'$ is *quasi-norming* if for some $\delta > 0$, we have that

$$\sup\{|\langle a, \mu\rangle| : \mu \in X, \|\mu\| \leq 1\} \geq \delta\|a\| \qquad (a \in \mathcal{A}).$$

Representing Banach $*$-algebras

By using interpolation spaces in a more complicated way than Young, we can prove the following result. For a Banach $*$-algebra $\mathcal{A}$, a functional $\mu \in \mathcal{A}'$ is *self-adjoint* if

$$\overline{\langle a^*, \mu \rangle} = \langle a, \mu \rangle \qquad (a \in \mathcal{A}).$$

Theorem

The following are equivalent:

1. $\mu \in \mathcal{A}'$ *is self-adjoint;*
2. *there is a reflexive Banach space E such that $\mathcal{B}(E)$ admits an involution, and a $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(E)$ such that*

   $$\langle a, \mu \rangle = \langle \pi(a)(x), \lambda \rangle \qquad (a \in \mathcal{A}),$$

   for some $x \in E, \lambda \in E'$ with $\|x\|\|\lambda\| = \|\mu\|$.

Representing Banach $*$-algebras (cont.)

Theorem

Let $\mathcal{A}$ be a Banach $*$-algebra. Then the following are equivalent:

1. *there is a reflexive Banach space F and a bounded-below representation $\pi : \mathcal{A} \to \mathcal{B}(F)$;*
2. *there is a reflexive Banach space E such that $\mathcal{B}(E)$ admits an involution, and a bounded-below $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(E)$.*

We can use Lance's renorning result to ensure that the involution on $\mathcal{B}(E)$ in (2) is isometric (because of the use of interpolation spaces, which are of an isomorphic character, it seems to be necessary to use Lance's result here).

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Application

Let G be a discrete group. We form the group algebra $\mathbb{C}[G]$, which is formal linear combinations of "point-masses" δ_g, for $g \in G$, with multiplication given by convolution

$$\delta_g \delta_h = \delta_{gh} \qquad (g, h \in G),$$

and an involution by $\delta_g^* = \delta_{g^{-1}}$. We norm $\mathbb{C}[G]$ by taking the sum of absolute values of the coefficients: the completion is denoted $\ell^1(G)$.

From classical results on weakly almost periodic functionals on $\ell^1(G)$, Young's theorem tells us that $\ell^1(G)$ is isometric to a subalgebra of $\mathcal{B}(F)$ for a suitable reflexive space F.

Hence $\ell^1(G)$ is certainly isomorphic to a closed $*$-subalgebra of $\mathcal{B}(E)$ for a suitable E, with $\mathcal{B}(E)$ having an involution.

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Application continued

The interesting point about $\ell^1(G)$ is that $\ell^1(G)$ cannot be isomorphic to a closed subalgebra of $\mathcal{B}(H)$ for a Hilbert space H (indeed, for any uniformly-convex Banach space). This follows by looking at Arens products, and does not involve the involution at all.

So, the space E we get, such that $\ell^1(G)$ embeds into $\mathcal{B}(E)$, cannot be a Hilbert space as a Banach space (that is, E is not a Krein space).

Can we choose E to be a “flip” space?

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