Fields of algebras

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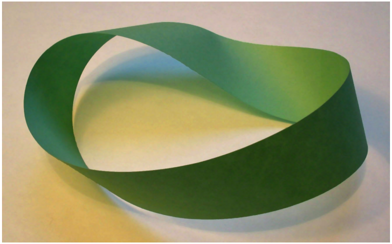
Outline

Bundles

A *bundle* over a topological space X is a topological space B together with $p : B \to X$ a continuous, open, surjective map. The *fibre* at $x \in X$ is $p^{-1}(\{x\}) \subseteq B$.

- The "trivial bundle" with fibre F is $B = X \times F$ with $p(x, f) = x$.
- A Mobius Strip is a bundle over the circle, where each fibre is a copy of the interval, but the global topology is not trivial.
- Typically our fibres will vary, and we do not assume any form of "local triviality", so we are far from the setting of "fibre bundles".



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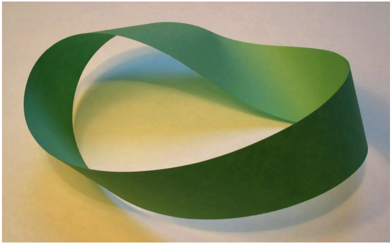

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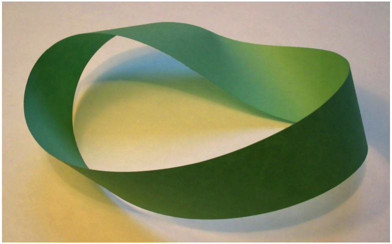

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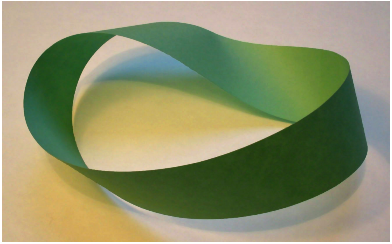

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Banach bundles

A *bundle of Banach spaces* is a bundle where each fibre has the structure of a Banach space, and:

- $B \to [0,\infty); b \mapsto \|b\|$ is continuous (where we use the norm on the fibre at $p(b) \in X$);
- addition is jointly continuous as a map $+ : \{(b_1, b_2) \in B \times B : p(b_1) = p(b_2)\} \to B$;
- for each $\lambda \in \mathbb{C}$ the map $B \to B; b \mapsto \lambda B$ is continuous;
- if (b_i) is a net in B with $p(b_i) \to x$ and $\|b_i\| \to 0$ then $b_i \to 0_x$ (the zero vector in the fibre over x).

Each fibre could be a Hilbert space; the Polarisation Identity shows the inner product is continuous.

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Sections

A *section* is a function $f : X \to B$ with $p(f(x)) = x$ for each $x \in X$. We say that B has *sufficiently many continuous (cross-) sections* if for each $b \in B$ there is a continuous section f with $f(x) = b$ for $x = p(b)$.

- Notice that the axioms imply that the zero section, $x \mapsto 0_x$, is continuous;

Theorem (Douady, Dal Soglio-Hérault)

Let X be locally compact. A bundle of Banach spaces over X has sufficiently many continuous sections.

The topology on B, restricted to the fibre at x, is just the norm topology on the Banach space at x. So the bundle "glues together" the Banach spaces.

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Properties of sections

Some notation: write $E(x)$ for the fibre at $x \in X$, which is a Banach space.

Let Γ be the collection of continuous sections.

- Γ is a closed subspace of $\prod_x E(x)$ for the supremum norm $\|f\| = \sup_x \|f(x)\|_{E(x)}$.
- If $f \in \Gamma$ and $a : X \to \mathbb{C}$ is continuous, then $af : x \mapsto a(x)f(x)$ is also in Γ.
- So Γ is a module over $C^b(X)$;

A bundle of Banach algebras is such that each $E(x)$ is a Banach algebra, and the multiplication map $\{(b_1, b_2) : p(b_1) = p(b_2)\} \to B$ is continuous. Similarly, for a bundle of C^*-algebras, the involution needs to be continuous.

- If each $E(x)$ is a C^*-algebra, then the *C^*-algebra of the bundle* is $\{af : a \in \Gamma, f \in C_0(X)\}$.

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Constructing bundles

Let $(E(x))_{x\in X}$ be a family of Banach spaces and let A be their disjoint union (without topology) together with the obvious map $p : A \to X$.

Suppose we have Γ a set of sections with:

- under pointwise operations, Γ is a vector space;
- for each $f \in \Gamma$, the map $x \mapsto \|f(x)\|$ is continuous;
- for each $x \in X$, the set $\{f(x) : f \in \Gamma\}$ is dense in $E(x)$.

Theorem (Dauns, Hofmann?)

There is a unique topology on A turning it into a Banach bundle such that each $f \in \Gamma$ becomes a continuous section.

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Moral

We can also axiomatise the notion of a section, which is called a "field" in the literature.
Bundles and fields are essentially the same thing.

- A bundle gives rise to continuous sections, which form a field.
- A field defines a topology which allows us to "glue together" the spaces into a bundle.

References:

- Dixmier "Les C^*-algèbres et leurs représentations"
- Fell, Doran, "Representations of $*$-algebras, locally compact groups, and Banach $*$-algebraic bundles".

Outline

Representation theory of C^*-algebras

We setup some notation:

- Let A be a C^*-algebra and $\widehat{A}$ its dual space, the set of unitary equivalence classes of irreducible representations, given the hull-kernel topology.
- Let Prim A be the space of primitive ideals of A, with the hull-kernel topology. For $I \in \text{Prim } A$ let $\pi_I : A \to A/I$ be the quotient map.
- We could then consider the "field" of C^*-algebras given by $(\pi_I(A))_{I\in\text{Prim } A} = (A/I)_{I\in\text{Prim } A}$ and vector fields of the form $\pi \mapsto \pi_I(a) = a + I$ for $a \in A$.
- However, $a \mapsto \|\pi_I(a)\|$ may fail to be continuous. And Prim A is often a "nasty" topological space.

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Hausdorff to the rescue

Theorem (Lee, Tomiyama)

Let $f : \mathrm{Prim}\, A \to X$ *be an open, continuous, surjective map onto a locally compact (Hausdorff)* X*. Let* A_x *be* A *quotiented by* $\bigcap f^{-1}(\{x\})$*. Then* (A_x) *is a continuous field, and the* C^**-algebra of this field is (isomorphic to)* A*.*

Recall that if $I \subseteq A$ is a closed (two-sided) ideal then $\mathrm{Prim}(A/I) = \{P \in \mathrm{Prim}\, A : I \subset P\}$. Thus if $E \subseteq \mathrm{Prim}\, A$ is a subset and $I = \bigcap E = \bigcap_{P \in E} P$, then $\mathrm{Prim}(A/I) = E$.

A detailed example: the Heisenberg group

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Some notes on the structure:

- Can think of as triples (x, y, z) with product $(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + x \cdot y')$.
- The centre is $Z = \{(0, 0, z) : z \in \mathbb{R}\}$ and $\mathbb{H}/Z \cong \mathbb{R}^2$.
- $N = \{(0, y, z)\}$ is a closed normal subgroup, isomorphic to $\mathbb{R}^2$,
- and $A = \{(x, 0, 0)\}$ is a closed group, isomorphic to $\mathbb{R}$,
- with $NA = \mathbb{H}$ and $N \cap A = \{0\}$, so $\mathbb{H} \cong \mathbb{R}^2 \rtimes_\alpha \mathbb{R}$ where $\alpha_x(y, z) = (y, z + x \cdot y)$

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Representation theory

The irreducible representations fall into two classes:

- characters $\{\chi_{\xi,\lambda} : \xi, \lambda \in \mathbb{R}\}$ acting on x, y via $\chi_{\xi,\lambda}(x, y, z) = e^{i(\xi \cdot x + \lambda \cdot y)}$;
- infinite dimensional representations U^λ, for $\lambda \neq 0$, on $L^2(\mathbb{R})$ given by $U^\lambda(x, y, z)f(t) = e^{i\lambda(z - y \cdot t)} f(t - x)$

So as a set, $\widehat{\mathbb{H}} = \mathbb{R}^2 \cup \mathbb{R}^*$ where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

- $\mathbb{R}^2 \to \widehat{\mathbb{H}}$ is a homeomorphism onto a closed set, and $\mathbb{R}^* \to \widehat{\mathbb{H}}$ is a homeomorphism onto an open set.
- $T \subseteq \widehat{\mathbb{H}}$ is closed if and only if:
  - $T \cap \mathbb{R}^2$ and $T \cap \mathbb{R}^*$ are both closed;
  - if $T \cap \mathbb{R}^*$ contains 0 as a limit point, then $\mathbb{R}^2 \subseteq T$.

So $\widehat{\mathbb{T}}$ is far from Hausdorff!

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So $\widehat{\mathbb{T}}$ is far from Hausdorff!

Representation theory

The irreducible representations fall into two classes:

- characters $\{\chi_{\xi,\lambda} : \xi, \lambda \in \mathbb{R}\}$ acting on x, y via $\chi_{\xi,\lambda}(x, y, z) = e^{i(\xi \cdot x + \lambda \cdot y)}$;
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$C^*(\mathbb{H})$ as a bundle

Set $A = C^*(\mathbb{H})$. It is not hard to check that $\widehat{\mathbb{H}} = \widehat{A} = \text{Prim}\, A$.

The map $\theta : \widehat{\mathbb{H}} \to \mathbb{R}$ which is the identity of $\mathbb{R}^*$ and which sends $\mathbb{R}^2$ to 0, is surjective, continuous, and open.

Thus we obtain that A is (isomorphic to) a continuous field over $\mathbb{R}$ with fibres:

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- The bundle over $(0, 1]$ is trivial.

However, what are the vector fields? However, this can be powerful tool, see Elliott, Natsume, Nest, "The Heisenberg Group and K-Theory".

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Outline

Drawbacks

Continuous fields of C^*-algebras are an attractive tool, but it does not interact well with other operations.

Theorem (Kirchberg, Wassermann)

Let B be a C^*-algebra. Consider continuous fields over the base space $\mathbb{N}_\infty$. Tensoring each fibre against B gives a continuous field if and only if B is exact.

There is a similar criteria for nuclearity (using the maximal tensor product). Similar results for crossed-products (characterising exactness of the group) hold.

The Dauns-Hoffman Theorem

For a C^*-algebra A, the *multiplier* algebra of A is the largest C^*-algebra M such that A embeds as a closed ideal of M which is *essential*, that is, $x \in M$ and $xA = \{0\}$ imply $x = 0$. Write $M(A)$ for the multiplier algebra.

- Various well-known constructions: double centralisers, bidual picture, etc.
- If A is unital then $M(A) \cong A$.
- If $A = C_0(X)$ then $M(A) \cong C^b(X) \cong C(\beta X)$

For $P \in \operatorname{Prim} A$ let $\pi_P : A \to A/P$ be the quotient map.

Theorem

There is an isomorphism $\phi : C^b(\operatorname{Prim} A) \to ZM(A)$, the centre of $M(A)$, with

$$\pi_P(\phi(f)a) = f(P)\pi_P(a) \qquad (a \in A, P \in \operatorname{Prim} A, f \in C^b(\operatorname{Prim} A)).$$

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Corollary: $C_0(X)$-algebras

A $C_0(X)$-algebra is a C^*-algebra A together with a $*$-homomorphism $\Phi_A : C_0(X) \to ZM(A)$ which is *non-degenerate*:

$$\Phi_A(C_0(X))A = \operatorname{lin}\{\Phi_A(f)a : f \in C_0(X), a \in A\} \text{ is dense in } A.$$

- A non-degenerate $*$-homomorphism $\Phi : C_0(X) \to C^b(Y)$ is always of the form $\Phi(f) = f \circ \sigma$ with $\sigma : Y \to X$ continuous.
- And any σ gives rise to a Φ.
- We typically assume Φ_A is injective, which is equivalent to σ having dense range.

As $ZM(A) \cong C^b(\operatorname{Prim} A)$ we see that $C_0(X)$-algebras can also be described by continuous maps $\sigma : \operatorname{Prim} A \to X$ (which have dense range).

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$C_0(X)$-algebras as fields/bundles

This setup, $\sigma : \operatorname{Prim} A \to X$, is very close to what we saw before, but we no longer have that σ is *open*.

This corresponds to fields/bundles which are only *upper semicontinuous*

$$\limsup_{x \to x_0} \|x\| \leq \|x_0\|.$$

The benefit is that $C_0(X)$-algebras are better behaved with respect to tensor products and so forth.

References:

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Banach $C_0(X)$-algebras: remove?

A Banach $C_0(X)$-algebra is a Banach algebra A which is an essential (left) $C_0(X)$-module with

$$f \cdot (ab) = a(f \cdot b) = (f \cdot a)b \qquad (f \in C_0(X), a, b \in A).$$

(This is also the same as the definition using the multiplier algebra of A).

- For $x \in X$ let $C_x(X) = C_0(X \setminus \{x\})$ which is identified with $\{f \in C_0(X) : f(x) = 0\}$ a subalgebra of $C_0(X)$.
- Let $N_x = C_x(X) \cdot A$ which by Cohen-Hewitt factorisation is a closed subspace of A. The above conditions show that it is an ideal in A.
- Define $A^x = A/N_x$, the *fibre* at x. Let $\pi^x : A \to A^x$ be the quotient map.

Contrasting to the C^*-case, we no longer have, for example,

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Representations: Hilbert C^*-modules

C^*-algebras are naturally represented on Hilbert spaces. The analogue for a $C_0(X)$-algebra is a *Hilbert C^*-module* over $C_0(X)$.

This is a Banach space E which is a right $C_0(X)$ module, and which has an "inner-product" which is $C_0(X)$-valued.

Such objects behave much like Hilbert spaces. A big difference is that bounded linear maps do not automatically have adjoints; this needs to be an axiom, leading to $\mathcal{L}(E)$.

We can form N_x in the same way as before, and then E^x turns into a genuine Hilbert space.

A representation of a $C_0(X)$-algebra A on E is a $C_0(X)$-module map $\pi : A \to \mathcal{L}(E)$. This fibres to give genuine $*$-homomorphisms $\pi^x : A_x \to \mathcal{B}(E^x)$.

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A representation of a $C_0(X)$-algebra A on E is a $C_0(X)$-module map $\pi : A \to \mathcal{L}(E)$. This fibres to give genuine $*$-homomorphisms $\pi^x : A_x \to \mathcal{B}(E^x)$.

Outline

$SU(2)$

Let $G = SU(2)$ the group of 2×2 complex valued unitary matrices with unit determinant.

Any member of G is of the form

$$s = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$$

with $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$.

Consider $C(G)$ the continuous functions on G. Define

$$\alpha(s) = a, \quad \gamma(s) = b.$$

Then the matrix

$$\begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2(C(G))$$

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$SU(2)$ as a Hopf C^*-algebra

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a unitary in $M_2(A)$. By Stone-Weierstrauss, α, γ generate A.

Of course, A tells us nothing about the group $SU(2)$. We can encode the group product $G \times G \to G$ as a $*$-homomorphism $\Delta : A = C(G) \to C(G \times G) = A \otimes A$.

Then, if $s = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$ and $t = \begin{pmatrix} c & -\overline{d} \\ d & \overline{c} \end{pmatrix}$, then

$$\Delta(\alpha)(s,t) = \alpha(st) = ac - \overline{b}d = \alpha(s)\alpha(t) - \gamma^*(s)\gamma(t).$$

So $\Delta(\alpha) = \alpha \otimes \alpha - \gamma^* \otimes \gamma$. Similarly $\Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$.

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Quantum $SU(2)$

Woronowicz introduced a "deformation" of $SU(2)$ as follows. Let $0 \leq q \leq 1$ and let A be the C^*-algebra generated by elements α, γ such that

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is unitary in $M_2(A)$.

- Such a C^*-algebra does exist. It is non-commutative; for $0 < q < 1$, it is isomorphic to a C^*-algebra related to the Toeplitz algebra.
- If we define $\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma$ and $\Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$ then we obtain a $*$-homomorphism
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Quantum $SU(2)$ continued

$$\Delta : A \to A \otimes A; \quad (\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta.$$

The pair (A, Δ) is a *compact quantum group*, objects which have remarkable similarities to compact groups:

- Subject to "quantum cancellation", that $\mathrm{lin}\{(a \otimes 1)\Delta(b) : a, b \in A\}$ is dense in $A \otimes A$ (and $(1 \otimes a)\Delta(b)$), we get...
- A "Haar measure", a state $\varphi \in A^*$ which is invariant, $(\varphi \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \varphi)\Delta(a) = \varphi(a)1$.
- Have a notion of a "corepresentation", and all unitary corepresentations split into direct sums of finite dimensionals.
- Analogue of the Peter-Weyl theory.
- Lots of operator-algebraic structure appears: for example, φ is a KMS state. So there is automatic interaction with von Neumann algebra theory.

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A field of $SU_q(2)$

Following Bauval and Blanchard, let A be the C^*-algebra generated by α, γ, f such that:

- f commutes with α, γ;
- $f = f^* \geq 0$ has spectrum $[0, 1]$;
- $u = \begin{pmatrix} \alpha & -f\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$ is unitary in $M_2(A)$.

Functional calculus, applied to f, provides a $*$-homomorphism $C([0, 1]) \to A$, so turning A into $C([0, 1])$-algebra.

By restriction, A becomes a $C_0((0, 1])$-algebra.

Theorem

A becomes a continuous field over $(0, 1]$ with fibres A_q where A_q is the C^*-algebra representing $SU_q(2)$. The Haar states vary continuously.

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Questions

If we have a continuous family of Haar states, then there should be a von Neumann algebra picture lurking here. But we somehow want to mix continuity and "measurability": jumping straight to measurable fields of von Neumann algebras seems to lose too much information.

What can one say about *actions*?

What about the locally compact case? Is there a nice source of examples, beyond the "classical" deformation examples?