The Fourier Algebra

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Colloquium talk

- In this month's Notices of the AMS, we have articles by Peller, suggesting that one should not give a computer presentation in a talk; and an article by Kra suggesting one should spend about half the talk on your own work.
- I shall break both these rules!
- I'm going to try just to give a survey talk about a particular area at the interface between algebra and analysis.
- Please ask questions!

Circle group

Let's look at the group $\mathbb{T}$, which can be thought of as:

- The interval $[0, 1)$ with addition modulo 1;
- The quotient group $\mathbb{R}/\mathbb{Z}$;
- The complex numbers $\{z \in \mathbb{C} : |z| = 1\}$ with multiplication.

We can think of functions on $\mathbb{T}$ as being the same as functions on $\mathbb{R}$ which are *periodic*.

A *character* on $\mathbb{T}$ is a group homomorphism from $\mathbb{T}$ to $\mathbb{T}$. There are lots of these!

The *continuous* characters are precisely the maps

$$\hat{n} : e^{i\theta} \mapsto e^{in\theta}$$

where $n \in \mathbb{Z}$.

Fourier series

Given a periodic function $f : \mathbb{R} \to \mathbb{C}$ the Fourier series of f is $(\hat{f}(n))_{n \in \mathbb{Z}}$ where

$$\hat{f}(n) = \int_0^1 f(\theta) e^{-2\pi i n\theta}\, d\theta.$$

We have the well-known “reconstruction”:

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n\theta}.$$

Of course, a great deal of classical analysis is concerned with the question of in what sense does this sum actually converge?

Convergence

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n\theta}.$$

- If f is twice continuously differentiable, then the sum converges uniformly to f (that is, $\lim_{N\to\infty} \sum_{n=-N}^{N}$).
- (Kolmogorov) There is a (Lebesgue integrable) function f such that the sum diverges everywhere.
- (Carleson) If f is continuous then the sum converges almost everywhere.

If $f \in L^2(\mathbb{T})$ (so $\int_0^1 |f|^2 < \infty$) then the sum always converges in the Banach space $L^2(\mathbb{T})$.

Banach spaces

So thinking more abstractly, Parseval's Theorem,

$$\int_0^1 |f|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2,$$

implies that the *Fourier transform* $\mathcal{F} : f \mapsto \hat{f}$ is a linear, isometric bijection

$$\mathcal{F} : L^2([0,1]) \to \ell^2(\mathbb{Z}).$$

The Riemann-Lebesgue Lemma shows that

$$\mathcal{F} : L^1([0,1]) \to c_0(\mathbb{Z})$$

is a linear contraction. That is, if $\int_0^1 |f| < \infty$, then

$$\lim_{n\to\pm\infty} \hat{f}(n) = 0, \qquad \max |\hat{f}(n)| \le \int_0^1 |f|.$$

Banach algebras

We turn $L^1([0,1])$ into a Banach algebra for the convolution product

$$(f * g)(t) = \int_0^1 f(s)g(t-s)\,ds,$$

where all addition is modulo 1. Turn $c_0(\mathbb{Z})$ into a Banach algebra for the pointwise product. Then

$$\mathcal{F} : L^1([0,1]) \to c_0(\mathbb{Z})$$

is an algebra homomorphism.

- The image is denoted $A(\mathbb{Z})$, the *Fourier algebra* of $\mathbb{Z}$. We give $A(\mathbb{Z})$ the norm coming from $L^1([0,1])$.
- So in one sense $A(\mathbb{Z})$ is just a different way to view $L^1([0,1])$.
- But $A(\mathbb{Z})$ is an interesting algebra of functions on $\mathbb{Z}$: if $K \subseteq \mathbb{Z}$ finite and $F \subseteq \mathbb{Z}$ disjoint from K then there is $a \in A(\mathbb{Z})$ with $a \equiv 1$ on K and $a \equiv 0$ on F.

Generalisation one

Given a locally compact abelian group G, let $\hat{G}$ be the collection of continuous *characters* on G, that is, group homomorphisms $\phi : G \to \mathbb{T}$.

- We turn $\hat{G}$ into a group by pointwise multiplication: $\phi\psi : G \to \mathbb{T}; g \mapsto \phi(g)\psi(g)$.
- We turn $\hat{G}$ into a locally compact space for the topology of uniform convergence on compact sets.
- Then $\hat{G}$ is a locally compact abelian group.
- We have that $\hat{\hat{G}} \cong G$ in a canonical way: $g \in G$ induces $\hat{\hat{g}} \in \hat{\hat{G}}$ by
$$\hat{\hat{g}} : \phi \mapsto \phi(g).$$
Then $g \mapsto \hat{\hat{g}}$ is a homeomorphism.
- This is reminiscent of the fact that $V \cong V^{**}$ for a finite-dimensional vector space V.

Generalisation one cont.

The other key fact about locally compact groups is that they admit a Haar measure: a Radon measure μ such that

$$\mu(A) = \mu(tA) \text{ where } tA = \{ts : s \in A\}$$

for any measurable set A.

- On $\mathbb{R}$ this is the Lebesgue measure;
- On $\mathbb{Z}$ this is just the counting measure.

For a suitably normalised Haar measure $\hat{\mu}$ on $\hat{G}$ we have a Fourier transform

$$L^1(G) \to C_0(\hat{G}); \quad f \mapsto \hat{f}, \quad \hat{f}(\phi) = \int_G f(s)\overline{\phi(s)}\, d\mu(s).$$

Again, this induces an isometry $L^2(G) \to L^2(\hat{G})$.

Generalisation two

If G is abelian, we define the *Fourier algebra* on G to be the image of $\mathcal{F} : L^1(\hat{G}) \to C_0(G)$ (recalling that $\hat{\hat{G}} = G$). Denote this $A(G)$:

- so this is some collection of functions on G, which vanish at infinity;
- given the norm from $L^1(\hat{G})$, we get a Banach algebra;
- it's "regular": given disjoint K, F with K compact and F closed, there is $a \in A(G)$ with $a \equiv 1$ on K and $a \equiv 0$ on F;
- functions of compact support are dense in $A(G)$;
- a *character* on $A(G)$, a non-zero multiplicative continuous map $A(G) \to \mathbb{C}$, is always given by "evaluation at a point of G".

It turns out that for any G we can find an algebra $A(G) \subseteq C_0(G)$ which behaves "as if" it is $\mathcal{F}(L^1(\hat{G}))$, even though $\hat{G}$ might not exist.

Restart– think about algebras

For now, let G be a finite (not assumed abelian) group. Let V be the $\mathbb{C}$-vector space which has G as a basis– so V is formal linear combinations of the elements of G.

- We can think of V as being functions from G to $\mathbb{C}$, written $\mathbb{C}^G$, turned into an algebra for the pointwise operations:

$$(f \cdot g)(s) = f(s)g(s) \qquad (f, g \in \mathbb{C}^G).$$

- We can think of V as being the $\mathbb{C}$ group ring of G, written $\mathbb{C}[G]$, with multiplication now given by "convolution":

$$\Big(\sum_{g \in G} a_g\, g\Big)\Big(\sum_{h \in G} b_h\, h\Big) = \sum_{g,h \in G} a_g b_h\, gh = \sum_{g \in G} \Big(\sum_{h \in G} a_h b_{h^{-1}g}\Big) g.$$

Hopf algebras

I can't resist pointing out that these algebras have extra structure.

On $\mathbb{C}^G$ we define a "coproduct"

$$\Delta : \mathbb{C}^G \to \mathbb{C}^G \otimes \mathbb{C}^G = \mathbb{C}^{G \times G}; \quad \Delta(f)(g, h) = f(gh).$$

Also define a "counit", and "coinverse"

$$\epsilon : \mathbb{C}^G \to \mathbb{C}; \epsilon(f) = f(e), \qquad S : \mathbb{C}^G \to \mathbb{C}^G; S(f)(g) = f(g^{-1}).$$

These interact in a "dual way" to how multiplication, identity and inverse work for algebras. For example, Δ is "coassociative" meaning that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

Notice that Δ and ϵ are homomorphisms, and S is an anti-homomorphism.

Duality

Similarly such maps exist on $\mathbb{C}[G]$; for $g \in G$ define

$$\Delta : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G] = \mathbb{C}[G \times G]; \quad \Delta(g) = g \otimes g,$$
$$\epsilon : \mathbb{C}[G] \to \mathbb{C}; \epsilon(g) = 1, \quad S : \mathbb{C}[G] \to \mathbb{C}[G]; S(g) = g^{-1}.$$

and extend by linearity. Again, these are all algebra homomorphisms.

Then $\mathbb{C}[G]$ and $\mathbb{C}^G$ are naturally dual vector spaces to each other, for the dual pairing

$$\langle f, \sum_g a_g\, g \rangle = \sum_g a_g f(g) \qquad (f \in \mathbb{C}^G, \sum_g a_g\, g \in \mathbb{C}[G]).$$

This is “canonical” as then the counit for one algebra becomes the unit for the other, and the coproduct gives the product. (The coinverse doesn’t quite fit here– I’m really talking about bialgebras at this point).

Some analysis: norms

Recall that V was our underlying vector space– turn this into a Euclidean space (a Hilbert space) for the canonical inner-product:

$$\big(\sum_g a_g\, g \big| \sum_h b_h\, h\big) = \sum_g a_g \overline{b_h}.$$

Then $\mathbb{C}[G]$ acts on V by the (left) regular representation. Then we can give $\mathbb{C}[G]$ the induced operator norm:

$$\|a\| = \max\left\{\|av\|_V : v \in V, \|v\|_V = (v|v)^{1/2} \leq 1\right\}$$

Then this is an algebra norm: $\|ab\| \leq \|a\|\|b\|$.

- For example, if $a = \sum_g a_g\, g$ with $a_g \geq 0$ for all g, then $\|a\| = \sum_g a_g$.
- Proof: "$\leq$" is easy inequality; let $v = |G|^{-1/2} \sum_g g \in V$ so $\|v\|_V = 1$ and $av = \left(\sum_g a_g\right) v$.
- Other cases are (much) harder to calculate!

The coinverse appears!

Recall the coinverse of $\mathbb{C}^G$: $S(f)(t) = f(t^{-1})$.

- As V is a Hilbert space, for each $a \in \mathbb{C}[G]$ there is a linear map $a^* : V \to V$ given by

$$(a^* v|u) = (v|au) \qquad (u, v \in V).$$

- A calculation shows that

$$\Big(\sum_g a_g\, g\Big)^* = \sum_g \overline{a_g}\, g^{-1}.$$

- The converse plays a role here; for $f \in \mathbb{C}^G$,

$$\langle f, a^* \rangle = \sum_g f(g^{-1})\overline{a_g} = \overline{\langle S(f)^*, a \rangle}.$$

- So again, we can argue that this *-structure is canonical.

C*-algebras and duality again

As V is a Hilbert space, and we have now identified $\mathbb{C}[G]$ as a $*$-subalgebra of $\mathrm{Hom}(V)$, we find that $\mathbb{C}[G]$ is a C*-algebra.

- Such algebras can be characterised as those Banach algebras which satisfy the C*-condition: $\|a^*a\| = \|a\|^2$.

We now use the duality between $\mathbb{C}[G]$ and $\mathbb{C}^G$ to induce the dual norm on $\mathbb{C}^G$:

$$\|f\| = \sup\left\{|\langle f, a\rangle| : a \in \mathbb{C}[G], \|a\| \leq 1\right\}.$$

- This is an algebra norm on $\mathbb{C}^G$.
- Actually the coproduct on $\mathbb{C}[G]$, being an injective $*$-homomorphism, is an isometry.
- Then $|\langle fg, a\rangle| = |\langle f \otimes g, \Delta(a)\rangle| \leq \|f\|\|g\|\|a\|$ so $\|fg\| \leq \|f\|\|g\|$.

The Fourier algebra

If G is abelian, then $\mathbb{C}^G$, with this norm, is precisely $A(G) \cong L^1(\hat{G})$, the Fourier algebra.

- The Fourier transform converts convolution to pointwise actions, and a bit of calculation shows that it establishes an isomorphism between $\mathbb{C}[G]$ and $C(\hat{G})$, the continuous functions on $\hat{G}$ with the max norm.
- Routine calculations show that $C(\hat{G})^* = L^1(\hat{G})$. And so $A(G) = \mathbb{C}[G]^* \cong L^1(\hat{G})$.

This then forms our definition of $A(G)$ for non-abelian G.

Infinite groups

Given a locally compact group G with the left invariant Haar measure, our analogue of V is $L^2(G)$, the square-integrable functions on G:

$$(f|h) = \int_G f(g)\overline{h(g)}\, dg.$$

- Let $C_c(G)$ be the space of compactly supported, continuous functions. This acts on $L^2(G)$ by left convolution, and forms a $*$-subalgebra of $\mathrm{Hom}(L^2(G))$.
- The closure is $C_r^*(G)$, the reduced group C*-algebra. This is our analogue of $\mathbb{C}[G]$.
- Reduced because we could have taken the supremum over all C*-norms. This gives $C^*(G)$, which if G is not *amenable* is larger.
- Sadly we're not quite done...

Infinite groups continued

The dual of $C_r^*(G)$ is somewhat "too large". Instead we use a different topology on $C_r^*(G)$:

- The *strong operator topology* on $\mathrm{Hom}(L^2(G))$ is such that a net (T_i) converges to T if and only if $\|T_i(f) - T(f)\| \to 0$ for all $f \in L^2(G)$.
- This is "locally convex" and metrisable if G is separable, but is not given by a norm unless G is finite.
- What you gain: the unit ball is now compact.
- We then define $A(G)$ to be the collection of linear functionals $C_r^*(G) \to \mathbb{C}$ which are continuous for the strong operator topology.
- Functional Analysis arguments show that $A(G)$ is then a closed subspace of the dual of $C_r^*(G)$ and hence a Banach space.

Why an algebra?

If we take the strong operator closure of $C^*_r(G)$ we get another, larger C*-algebra, $VN(G)$.

- This is a von Neumann algebra (as it's strongly closed!)
- von Neumann bicommutant theorem: $VN(G) = C^*_r(G)''$ where

$$X' = \{T \in \mathrm{Hom}(L^2(G)) : TR = RT \ (R \in X)\}.$$

- Then $A(G)^* = VN(G)$.
- $VN(G)$ is also equal to the von Neumann algebra generated by the left translation operators given by group elements $g \in G$.
- As before, the coproduct $\Delta(g) = g \otimes g$ is well-defined and strongly continuous as a map $VN(G) \to VN(G) \overline{\otimes} VN(G)$.
- This then shows that $A(G)$ is an algebra, just as in the finite-group case.

Philosophy

A large number of constructions which one can do in pure algebra for a "finite structure" can be carried out for an infinite structure which has a topology by use of operator-algebraic (C^* or von Neumann algebra techniques).

When G is an infinite discrete group, then G might not have any topology, but the "infinite" nature makes operator-algebraic methods useful (e.g. the "abstract" side of geometric group theory).

As a function algebra

For a finite group G, we saw that $A(G)$ is just $\mathbb{C}^G$ but with a different norm. In general:

- We use that $A(G)^* = VN(G)$.
- The map $G \to VN(G); g \mapsto g$ is continuous, and so for $a \in A(G)$ the map $g \mapsto \langle g, a \rangle$ is continuous.
- If $\langle g, a \rangle = 0$ for all g, then as G generates $VN(G)$, we see that $a = 0$.
- So we identify $a \in A(G)$ with a continuous function $G \to \mathbb{C}$.
- By getting your hands dirty, you can show that functions of the form $f_1 * f_2$, for $f_i \in C_c(G)$, are dense in $A(G)$.
- So $A(G)$ is a dense subalgebra of $C_0(G)$.
- The character space of $A(G)$ is G.

What you get

If G is finite, then the isomorphism class of $\mathbb{C}^G$ just depends on $|G|$.

- [Walter] If $A(G)$ is *isometrically* isomorphic to $A(H)$, then G is isomorphic to either H or H^{op} (and one can describe very concretely the isomorphism $A(G) \cong A(H)$).
- That is, with the norm, $\mathbb{C}^G$ completely determines G or G^{op}.
- Philosophical point: I don't know of any "algorithm" that takes $A(G)$ and gives the group structure on G and/or G^{op}.

Approximation properties

Recall that G is amenable if and only if $C_r^*(G) = C^*(G)$.

- G is amenable if it has certain "averaging" properties: compact groups are amenable, but $\mathbb{F}_2$ is not. This is related to "paradoxical decompositions".
- $A(G)$ has an identity (is unital) if and only if G is compact.
- G is amenable if and only if $A(G)$ has a bounded approximate identity: a bounded net (a_i) with $a_i a \to a$ for all $a \in A(G)$.
- There are various weaker notions of "amenability" which can be defined using weaker forms of "bounded".
- Related properties (e.g. the Haagerup approximation property) have links to e.g. the Baum–Connes conjecture in K-Theory.

Example: compact groups

Let G be a compact group (in particular, finite!)

- Let Γ be a set of representatives for the classes of irreducible representations of G. For $\pi \in \Gamma$ let n_π be the dimension.
- Then the Peter-Weyl theorem tells us that as a left G-module,
$$L^2(G) \cong \bigoplus_{\pi\in\Gamma} n_\pi \mathbb{C}^{n_\pi}.$$
- Then $C_r^*(G)$ decomposes as
$$C_r^*(G) \cong \bigoplus_{\pi\in\Gamma} \mathbb{M}_{n_\pi}.$$
- We get $VN(G)$ by taking the ℓ^∞ direct sum, instead of the c_0 direct sum. Then
$$A(G) \cong \ell^1 - \bigoplus_{\pi\in\Gamma} \mathbb{M}_{n_\pi}^*.$$

Example: compact groups cont.

$$A(G) \cong \ell^1 - \bigoplus_{\pi \in \Gamma} \mathbb{M}^*_{n_\pi}, \qquad \Delta(g) = g \otimes g.$$

- Let $\pi_1, \pi_2 \in \Gamma$ and consider the irreducible decomposition:

$$\pi_1 \otimes \pi_2 \cong \pi_{k_1} \oplus \pi_{k_2} \oplus \cdots \oplus \pi_{k_n}.$$

- So if $a = (a_\pi), b = (b_\pi), c = (c_\pi) \in A(G)$ with $ab = c$, then

$$c_\pi = \sum_{\pi_1, \pi_2} \{\text{The } \pi \text{ component of } a_{\pi_1} \otimes b_{\pi_2}\}.$$

- If G is abelian then each π is a character and Γ is a (discrete) group, and the above becomes

$$c_\pi = \sum_{\phi,\psi} \{a_\phi b_\psi : \phi\psi = \pi\}.$$

That is, convolution, as we expect given $A(G) \cong L^1(\Gamma)$.

Takesaki operator

Consider the operator W on $L^2(G \times G)$

$$Wf(g,h) = f(h^{-1}g,h) \qquad (g,h \in G).$$

As the Haar measure is left invariant, W is an isometry, with obvious inverse, so W is unitary.

We can "slice" W: given $\xi, \eta \in L^2(G)$, consider the operator

$$(\mathrm{id} \otimes \omega_{\xi,\eta})(W) = T \quad \Leftrightarrow \quad (Tf|g) = (W(f \otimes \xi)|g \otimes \eta).$$

If we do the calculation, then

$$T = \text{left convolution by } \xi\overline{\eta} \in L^1(G).$$

So $C_r^*(G)$ ($VN(G)$) is the norm (strong) closure of such slices. We can recover Δ via the map

$$\Delta(x) = W(1 \otimes x)W^* \qquad (x \in VN(G) \subseteq \mathrm{Hom}(L^2(G))).$$

Towards (locally compact) quantum groups

So the single operator W allows us to reconstruct $C_r^*(G), VN(G), \Delta$ and hence $A(G)$.

- By taking slices of the other side, we can reconstruct $C_0(G), L^\infty(G), L^1(G)$.
- That Δ is coassociative is reflected in the "Pentagonal equation":

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

- Slicing W might not give a reasonable algebra. But under various extra conditions:
  - W is (semi)-regular, [Baaj-Skandalis]
  - W is manageable [Woronowicz-Sołtan]

  we get a C*-algebra A, a von Neumann algebra M, a coproduct $\Delta : M \to M\overline{\otimes}M$ and (perhaps unbounded) counit and coinverse.
- By starting with algebras, one can write down some axioms to get the notion of a "locally compact quantum group" [Kustermans-Vaes].

One result

So we get a quantum group $\mathbb{G}$:

- An "abstract" object represented by various operator algebras: $C_r^*(\mathbb{G})$ and $VN(\mathbb{G})$.
- (This is "non-commutative topology".)
- Can get a Banach algebra $A(\mathbb{G})$: if $\mathbb{G}$ is a genuine group (or dual) then we get $A(G)$ (or $L^1(G)$).
- [D., Le Pham] If $A(\mathbb{G})$ and $A(\mathbb{H})$ are *isometrically* isomorphic, then $\mathbb{G}$ is isomorphic to $\mathbb{H}$ (or its opposite).