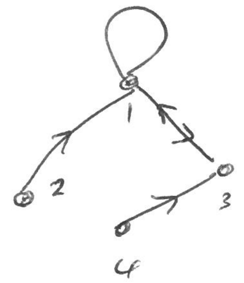
"Non-commutative Mathematics" talk

Consider a finite (directed) graph G



The adjacency matrix is

$$A = A_G = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Loops undirected.

This gives a bijection, so instead of looking a graphs, we could study 0-1 valued matrices.

$\rightarrow$ Some comments about "spectral graph theory"?

We could also look at

$$S_G = \{ \text{All matrices which a "shadow" like } A_G \}.$$

$$= \left\{ \begin{pmatrix} * & 0 & * & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & * & 0 \end{pmatrix} : * \text{ any value in } \mathbb{C} \right\}$$

This is a subspace of $\mathbb{M}_4$. – you can add such matrices and scalar multiply them. Can obviously recover A_G from S_G.

- What properties does S_G have?

Let $D_4 \subseteq \mathbb{M}_4$ be the subspace of diagonal matrices.

This is an algebra: a vector space over $\mathbb{C}$ with a (bilinear, associative) multiplication. Clearly we can multiply diagonal matrices to obtain another diagonal matrix

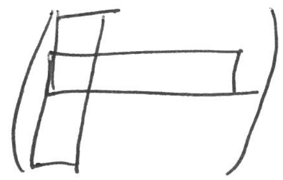
<u>Notation</u>: $e_{ij} = \begin{array}{c|c} & \overrightarrow{\quad j \quad} \\ i\downarrow & \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} \end{array}$ "matrix unit"

$D_n = \text{lin}\{e_{11}, e_{22}, \dots, e_{nn}\}$ and $M_n = \text{lin}\{e_{ij}\}$.

<u>Claim</u>: S_G is a <u>bimodule</u> over D_n. That is, multiplying on the left / right by elements of D_n maps S_G to itself.

<u>Proof</u>: Enough to check using e_{ii} and e_{jj}. E.g.

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{e_{22}} \begin{pmatrix} * & * & * & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & * & 0 \end{pmatrix} e_{11} =$$



$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in S_G.$$

<u>Claim</u>: If $S \subseteq M_n$ is a subspace and a bimodule over D_n, then $S = S_G$ for some graph G.

<u>Proof</u>: $e_{ii}\, x\, e_{jj} = 0$ or non-zero scalar multiple of e_{ij}.

So if $\exists\, x \in S$ with $e_{ii} x e_{jj} \neq 0$ then $e_{ij} \in S$, and conversely. So $S = \text{lin}\{e_{ij} : e_{ij} \in S\} = S_G$ where G is the graph with $i \to j$ an edge $\Leftrightarrow e_{ij} \in S$. $\blacksquare$

So Graphs $=$ D_n-bimodules in M_n.

Undirected graphs : means $i \to j \Rightarrow j \to i$

So $e_{ij} \in S \Rightarrow e_{ji} \in S$

So $x \in S \Rightarrow x^* \in S$ Hermitian Conjugate.

Make "non-commutative". The diagonal matrices are a commutative algebra:

$xy = yx$.

Look at other algebras in M_n, $A \subseteq M_n$.

→ Want these to be "self-adjoint" which means

$x \in A \Rightarrow x^* \in A$.

→ These are the finite-dimensional C^*-algebras.

→ Lots and lots of structure.

E.g. up to unitary equivalence, $A = M_{d_1} \oplus M_{d_2} \oplus \dots \oplus M_{d_k}$

$$\begin{pmatrix} \square & & \\ & \square & \\ & & \square \end{pmatrix}$$

i.e. "diagonal" but the blocks can be larger than 1×1.

A quantum / non-commutative graph is a self-adjoint subspace $S \subseteq M_n$

which is a bimodule over A.

Application: A communication channel sends "tokens" to "tokens", sometimes making mistakes due to noise. E.g. with my hand-writing, maybe a, u, get confused or a and o. But b, a don't.

Make a graph of the tokens, and edges link things that might be confused, then a maximal collection of tokens which can't be confused is an <u>independent set</u> in G. "confusability graph"

A <u>quantum communication channel</u> replaces tokens by vectors: the magic of quantum mechanics, very roughly, arises from the ability to have linear combinations of vectors – so "mixed states". These "quantum graphs" occurred ~~as~~ as an analogue of confusability graphs.

→ Can carry out a programme of "graph theory" for these objects.

<u>Analogue of the adjacency matrix?</u>

Yes, but complicated. E.g. how do we detect a 0-1 valued matrix.

Use the "wrong" matrix product: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \circ \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 5 & 12 \\ 21 & 32 \end{pmatrix}$

Then a matrix x with $x \circ x = x$ must have $0, 1$ in each entry.

"Hadamard product" → $\exists$ way to define this on ~~an algebra A,~~ ~~and then~~ maps $A \to A$ and so speak about an "quan. adjacency matrix" which is really a sort of linear map $A_G : A \to A$.

<u>Outlook</u>: Associate a mathematical object to a commutative algebra. Then noncommutative analogues of these algebras represent "quantum" objects.

E.g. groups → Hopf Algebras.

I'm an analystic, so interested in infinite dimensional versions ~~of~~ these [illegible] where convergence is all important