Uniqueness of preduals

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9th July 2007

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Dual Banach algebras

A *dual Banach algebras* is a Banach algebra $\mathcal{A}$ which is a dual space, $\mathcal{A} = E'$, such that the product is separately weak*-continuous.

- Recall that a C*-algebra which is a dual Banach space is called a W*-algebra or a von Neumann algebra.
- Then the product, and involution, are weak*-continuous.

If $E' = \mathcal{A}$ is a dual Banach algebra, then $E \hookrightarrow E'' = \mathcal{A}'$. Then the product is weak*-continuous if and only if E is a *submodule* of $\mathcal{A}'$.

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It is common knowledge that "A von Neumann algebra has a unique predual".

What, exactly, do we mean by this?

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However, Pełczyński showed that ℓ^∞ and $L^\infty[0,1]$ are *isomorphic* as Banach spaces. So the "isometric" condition before is essential.

Theorem (D. & White)

Let $\mathcal{M}$ be a von Neumann algebra, let $\mathcal{A}$ be a dual Banach algebra, and let $\theta : \mathcal{M} \rightarrow \mathcal{A}$ be a Banach algebra isomorphism. Then θ is weak-continuous.*

Thus, if we ignore the involution, and the isometric structure, of a von Neumann algebra, but not its algebra structure, then we still have the unique predual property.

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Discrete groups

Let G be a discrete group, and form the convolution Banach algebra $\ell^1(G)$. Every $a \in \ell^1(G)$ admits a representation of the form

$$a = \sum_{g \in G} a_g \delta_g, \quad \|a\| = \sum_{g \in G} |a_g|.$$

This is a dual Banach algebra with predual $c_0(G)$.

Is this the only predual which makes the product weak*-continuous?

If G is countable, and K is locally compact and countable, then

$$C_0(K)' = M(K) = \ell^1(K) \cong \ell^1(G).$$

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C*-preduals

Notice that the canonical predual $c_0(G)$, and the dual $\ell^\infty(G)$, are C*-algebras.

Theorem (D. & White)

Let $E \subseteq \ell^\infty(G)$ be a predual for $\ell^1(G)$. If E is also a C^*-subalgebra of $\ell^\infty(G)$, then $E = c_0(G)$.

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Coassociative products

Let E be a predual for $\ell^1(G)$. Then

$$(E\check{\otimes}E)' = E'\widehat{\otimes}E' \cong \ell^1(G \times G).$$

Consider the map

$$\Delta : \ell^1(G) \to \ell^1(G \times G), \quad \delta_g \mapsto \delta_g \otimes \delta_g = \delta_{(g,g)}.$$

This is a coassociative product; compare with work of Effros and Ruan, "Operator space tensor products and Hopf convolution algebras."

A simple calculation shows that Δ is weak*-continuous if and only if E is a subalgebra of $\ell^1(G)' = \ell^\infty(G)$.

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Duality: Compact groups

Dual to $\ell^1(G)$ for a discrete group G is $A(H)$, the *Fourier algebra* of a *compact* group H. This is the dual of $C^*(H)$, and the predual of $VN(H)$.

Example: $SU(2)$.

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Semigroups

We now turn to finding general preduals of $\ell^1(G)$. This is joint work with Haydon, Schlumprecht and White.

Let $E \subseteq \ell^\infty(G)$ be a predual. Let $\mathcal{A}$ be the unital C*-algebra generated by E in $\ell^\infty(G)$. Thus $\mathcal{A} \cong C(K)$ and so $\mathcal{A}' \cong M(K)$. We get an injection $G \to K$ which we can use to extend the product on G to K. Then K becomes a *semitopological semigroup* containing G densely.

Theorem

There is a projection $P : M(K) \to \ell^1(G)$ which is an algebra homomorphism, and such that

$$E = {}^\perp(\ker P) = \{f \in C(K) : f(a) = 0 \ (P(a) = 0)\}.$$

Furthermore, ker P *is weak*-closed.*

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Constructions

We simply reverse the above argument. That is, we find a compact semitopological semigroup K which contains G densely, so we can identify $\ell^1(G) \subset M(K)$. Suppose we have defined a projection $P : M(K) \to \ell^1(G)$ which is a homomorphism.

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Let ${}^{\perp}(\ker P)$ *induce a space of functionals on* $\ell^1(G)$*, say* $E \subseteq \ell^\infty(G)$*. Then* E *is a predual for* $\ell^1(G)$ *if and only if* $\ker P$ *is weak*-closed.*

What is E though?

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Let (a_α) *be a bounded net in* $\ell^1(G)$ *which tends weak* to* $b \in C(K)' = M(K)$*. Then* (a_α) *tends to* $a = P(b)$ *in the weak*-topology on* $\ell^1(G)$ *induced by* E*.*

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Case study: $G = \mathbb{Z}$

Pick the easiest compacitification of $\mathbb{Z}$. Let z be some extra generator, and form the free abelian semigroup generated by $\mathbb{Z}$ and z, together with ∞. So everything in K is of the form

$$nz + k \qquad (n \geq 0, k \in \mathbb{Z}).$$

We give K some complicated topology.
The projection $P : M(K) = \ell^1(K) \to \ell^1(\mathbb{Z})$ is uniquely determined by

$$P(\delta_k) = \delta_k, \quad P(\delta_z) = a \qquad (k \in \mathbb{Z}),$$

for some $a \in \ell^1(\mathbb{Z})$.
Given a suitable topology on K, we can prove some abstract results about when (ker P) is weak*-closed. In particular, this will hold if $\sum_n \|a^n\| < \infty$.

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Conclusions

Theorem

Let $J \subseteq \mathbb{Z}$ be a "sparse" set, let $a \in \ell^1(\mathbb{Z})$ with $\|a\| < 1$. There exists a predual for $\ell^1(\mathbb{Z})$ such that $\delta_n \to a$, as n tends through J, in the weak*-topology.

- Let $a = \lambda\delta_0$ for some $|\lambda| < 1$, and let $J = \{2^n\}$. Then the predual we construct is isomorphic to a $C(K)$ space. Furthermore, we can calculate the *Szlenk index*, showing that $E \cong c_0$.
- Of course, such an isomorphism does not respect duality.
- For this example, the involution on $\ell^1(\mathbb{Z})$ is not weak*-continuous.
- Let $a = \delta_0$; then if we could find a predual as above, this predual would have uncountable Szlenk index, showing that the predual were itself uncountable!
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- Let $a = \lambda\delta_0$ for some $|\lambda| < 1$, and let $J = \{2^n\}$. Then the predual we construct is isomorphic to a $C(K)$ space. Furthermore, we can calculate the *Szlenk index*, showing that $E \cong c_0$.
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Future work

Runde defined the concept of *Connes-amenability* for dual Banach algebras: simply take the usual notion of amenability, and make everything in sight weak*-continuous.

- Then $\ell^1(G)$ is Connes-amenable if and only if G is amenable.
- Of course, this is with respect to the predual $c_0(G)$.
- If G is amenable, then $\ell^1(G)$ is amenable, so $\ell^1(G)$ is Connes-amenable for any predual.
- However, could we find a predual of $\ell^1(\mathbb{F}_2)$, say, making this algebra Connes-amenable?

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