Some operator algebra problems from quantum computing
Or. . . What are some nice maps between matrices?

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Matrices

Throughout, $\mathbb{M}_n$ means $\mathbb{M}_n(\mathbb{C})$, that is, $n \times n$ complex matrices.
I will consider $\mathbb{M}_n$ as acting on $\mathbb{C}^n$, the latter coming equipped with the usual Euclidean inner product:

$$(\xi|\eta) = \sum_{j=1}^{n} \overline{\xi_j}\eta_j = \overline{\xi}^t \eta$$

if we think of ξ and η as column vectors. (Here I use Physics notation).
Then $\mathbb{M}_n$ has the *operator norm*:

$$\|x\| = \sup\left\{\|x\xi\| : \|\xi\| \leq 1\right\} = \sup\left\{(\overline{\xi}^t x^* x\xi)^{1/2} : \|\xi\| \leq 1\right\}.$$

Here $x^* = \overline{x}^t$ and $\|\xi\|^2 = \overline{\xi}^t \xi$.
As x^*x is hermitian, we can find a new orthonormal basis such that x^*x becomes diagonal: the entries being the eigenvalues. A little thought then shows that

$$\|x\|^2 = \|x^*x\| = \max\left\{|\lambda| : \lambda \text{ an eigenvalue of } x^*x\right\}.$$

Remember that $\|xy\| \leq \|x\|\|y\|$ for any $x, y \in \mathbb{M}_n$.

Maps between matrices

Let $\{u_1, \cdots, u_k\}$ and $\{w_1, \cdots, w_k\}$ be finite sets in $\mathbb{M}_n$. We can then define a linear map $\varphi : \mathbb{M}_n \to \mathbb{M}_n$ by

$$\varphi(x) = \sum_j u_j x w_j.$$

What is the norm of φ? That is, compute $\|\varphi\| = \sup\big\{\|\varphi(x)\| : \|x\| \leq 1\big\}$. The triangle inequality shows trivially that

$$\|\varphi\| \leq \sum_j \|u_j\|\|w_j\|.$$

Identify $\mathbb{C}^n \otimes \mathbb{C}^k$ with $\mathbb{C}^{nk}$. In $\mathbb{C}^k$, let $\{\delta_1, \cdots, \delta_k\}$ be the standard basis. Then we can define maps

$$u, w : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^k; \quad u(\xi) = \sum_j u_j^*(\xi) \otimes \delta_j, \quad w(\xi) = \sum_j w_j(\xi) \otimes \delta_j.$$

Then

$$\varphi(x) = u^*(x \otimes I)w \implies \|\varphi\| \leq \|u^*\|\|w\| = \Big\| \sum_j u_j u_j^* \Big\|^{1/2} \Big\| \sum_j w_j^* w_j \Big\|^{1/2}.$$

Norms

This is a better estimate, but not tight. Indeed, identify $\mathbb{M}_m \otimes \mathbb{M}_n$ with $\mathbb{M}_{mn}$. Then we can consider

$$\iota_m \otimes \varphi : \mathbb{M}_m \otimes \mathbb{M}_n \to \mathbb{M}_m \otimes \mathbb{M}_n.$$

However, now we have

$$(\iota_m \otimes \varphi)(x) = (I \otimes u)^*(x \otimes I)(I \otimes w) \quad \text{for } x \in \mathbb{M}_{mn}.$$

Thus also

$$\|\iota_m \otimes \varphi\| \leq \|I \otimes u\| \|I \otimes w\| \leq \Big\| \sum_j u_j u_j^* \Big\|^{1/2} \Big\| \sum_j w_j^* w_j \Big\|^{1/2}.$$

It turns out that

$$\sup_m \|\iota_m \otimes \varphi\| = \|\iota_n \otimes \varphi\| = \inf \Big\| \sum_j u_j u_j^* \Big\|^{1/2} \Big\| \sum_j w_j^* w_j \Big\|^{1/2}.$$

We call this quantity the *completely bounded norm* of φ.

Positivity

For me, a *positive* matrix means a semi-definite positive matrix, that is

$$\overline{\xi}^t x \xi \geq 0 \quad \text{for all } \xi \in \mathbb{C}^n.$$

- A matrix x is positive if and only if $x = y^* y$ for some matrix y.
- For $x \in \mathbb{M}_n$, we write $x \geq 0$ to mean that x is positive.
- A matrix a is hermitian, $a^* = a$, if and only if $a = x - y$ for $x, y \geq 0$.
- Hence we can define a partial order on the hermitians by $a \geq b$ if and only if $a - b \geq 0$.
- This order is tightly linked to the norm structure: for a hermitian matrix a, we have that $\|a\| \leq 1$ if and only if $-I \leq a \leq I$.
- All this can be proved easily by diagonalisation.

Maps which respect positivity

- A map $\varphi : \mathbb{M}_n \to \mathbb{M}_n$ is *positive* if it send positive matrices to positive matrices.
- Notice that then $\varphi(x^*) = \varphi(x)^*$.
- We say that φ is *m-positive* if $\iota_m \otimes \varphi : \mathbb{M}_{mn} \to \mathbb{M}_{mn}$ is positive.
- Finally, φ is *completely positive* if φ is m-positive for all m. Again, enough to check the case $m = n$.
- The canonical example of a positive, not completely positive map is the transpose map $\varphi(x) = x^t$:

$$\iota_2 \otimes \varphi \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \varphi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \varphi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \varphi\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \varphi\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- Not obvious what the bound of a completely positive map is.

Maps and functionals

There is a bijection between linear maps $\varphi : \mathbb{M}_n \to \mathbb{M}_n$ and linear functionals $\hat{\varphi} : \mathbb{M}_{n^2} = \mathbb{M}_n \otimes \mathbb{M}_n \to \mathbb{C}$ given by

$$\hat{\varphi}\big(a \otimes e_{ij}\big) = \varphi(a)_{ij}.$$

Here $e_{ij} \in \mathbb{M}_n$ is the obvious elementary matrix which has 1 in the (i, j) position, and 0 elsewhere.

Theorem (Choi, 1975)

The map φ is completely positive if and only if $\hat{\varphi}$ is positive.

Proof.

$(\Rightarrow)$ Let $(\delta_i)_{i=1}^n$ be the canonical basis of $\mathbb{C}^n$, and let $\xi_0 = \sum_i \delta_i \otimes \delta_i \in \mathbb{C}^{n^2}$. Then

$$\big(\xi_0 \big| (\iota_n \otimes \varphi)(e_{ij} \otimes a)\xi_0\big) = \sum_{s,t} \big(\delta_s \otimes \delta_s \big| (e_{ij} \otimes \varphi(a))\delta_t \otimes \delta_t\big) = \big(\delta_i \big| \varphi(a)\delta_j\big) = \varphi(a)_{ij}.$$

Thus $\hat{\varphi}(x) = (\xi_0 | (\iota_n \otimes \varphi)(x)\xi_0)$ for any $x \in \mathbb{M}_{n^2}$, so $\hat{\varphi}$ is positive. □

More on positive functionals on matrices

We identify the dual space of $\mathbb{M}_m$ with $\mathbb{M}_m$ via *trace duality*:

$$\langle x, y \rangle = \mathrm{Tr}(xy).$$

Here I write $\langle \cdot, \cdot \rangle$ for a *bilinear* pairing between vector spaces.

- If we give $\mathbb{M}_m$ the operator norm, then the dual space gets the *trace class norm*

$$\|y\|_1 = \sup \big\{ |\mathrm{Tr}(xy)| : \|x\| \leq 1 \big\} = \sum \big\{ \lambda : \lambda^2 \text{ an eigenvalue of } y^* y \big\}.$$

- If $y \in \mathbb{M}_m$ is positive, then the functional $x \mapsto \mathrm{Tr}(xy)$ is positive. Indeed, we can write $y = u^* u$, and then

$$\mathrm{Tr}(z^* z y) = \mathrm{Tr}(z^* z u^* u) = \mathrm{Tr}(u z^* z u^*) = \mathrm{Tr}\left((z u^*)^* z u^* \right) \geq 0.$$

- If the functional $x \mapsto \mathrm{Tr}(xy)$ is positive, then for $\xi \in \mathbb{C}^m$, let $x = \xi \overline{\xi}^t \in \mathbb{M}_m$, so that $x \geq 0$, and hence

$$0 \leq \mathrm{Tr}(xy) = \mathrm{Tr}(\xi \overline{\xi}^t y) = \mathrm{Tr}(\overline{\xi}^t y \xi) = \overline{\xi}^t y \xi.$$

  Thus y is positive.

Towards the converse

Recall: $\varphi : \mathbb{M}_n \to \mathbb{M}_n$ and $\hat\varphi : \mathbb{M}_{n^2} = \mathbb{M}_n \otimes \mathbb{M}_n \to \mathbb{C}$ linked by

$$\hat\varphi\big(a \otimes e_{ij}\big) = \varphi(a)_{ij}.$$

So $\hat\varphi$ is identified with $x \in \mathbb{M}_{n^2}$. If this is positive, then diagonalise, and pick the unique positive square-root, say $y \in \mathbb{M}_{n^2}$. Let

$$y = \sum_{s,t} y_{st} \otimes e_{st}.$$

As y is positive, also $y^* = y$, so $y_{st} = y_{ts}^*$ for all s, t.
Then, for $a \in \mathbb{M}_n$,

$$\begin{aligned}\big(\delta_j \big| \varphi(a)\delta_i\big) &= \varphi(a)_{ji} = \hat\varphi(a \otimes e_{ij}) = \operatorname{Tr}\big(x(a \otimes e_{ji})\big) = \operatorname{Tr}\big(y(a \otimes e_{ji})y\big) \\ &= \sum_{s,t,r,u} \operatorname{Tr}(y_{st}ay_{ru})\operatorname{Tr}(e_{st}e_{ji}e_{ru}) = \sum_s \operatorname{Tr}(y_{sj}ay_{is}) \\ &= \sum_s \operatorname{Tr}(y_{js}^*ay_{is}).\end{aligned}$$

Link with completely bounded norms

So we have

$$\left(\delta_j \middle| \varphi(a)\delta_i\right) = \sum_s \mathrm{Tr}(y_{js}^* a y_{is}).$$

Let the kth row of y_{is} be $\xi_{i,s,k}$, and define

$$v : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n; \quad \delta_i \mapsto \sum_{s,k} \delta_k \otimes \delta_s \otimes \xi_{i,s,k}.$$

Then

$$\begin{aligned}\left(\delta_j \middle| v^*(e_{ab} \otimes I) v \delta_i\right) &= \sum_{s,r,k,l} \left(\delta_k \otimes \delta_s \otimes \xi_{j,s,k} \middle| e_{ab}\delta_l \otimes \delta_r \otimes \xi_{i,r,l}\right) \\ &= \sum_{s,r} \left(\delta_s \otimes \xi_{j,s,a} \middle| \delta_r \otimes \xi_{i,r,b}\right) = \sum_s \left(\xi_{j,s,a} \middle| \xi_{i,s,b}\right) \\ &= \sum_s \xi_{i,s,b} \overline{\xi_{j,s,a}^t} = \sum_s (y_{is} y_{js}^*)_{ba} = \sum_s \mathrm{Tr}(e_{ab} y_{is} y_{js}^*) = \left(\delta_j \middle| \varphi(e_{ab})\delta_i\right).\end{aligned}$$

Thus $\varphi(x) = v^*(x \otimes I)v$ for any $x \in \mathbb{M}_n$. Thus φ is certainly completely positive.

Stinespring Theorem and links to completely bounded norms

Theorem (Stinespring, 1955)

Let $\varphi : \mathbb{M}_n \to \mathbb{M}_n$ *be a completely positive map. There exists an inner-product space* K*, of dimension at most* n^2*, and a linear map* $v : \mathbb{C}^n \to \mathbb{C}^n \otimes K$*, such that* $\varphi(x) = v^*(x \otimes I)v$*.*

If we pick an orthonormal basis (e_j) for K, then we can find matrices (v_j) with $v(\xi) = \sum_j v_j(\xi) \otimes e_j$, and so

$$\varphi(x) = \sum_j v_j^* x v_j.$$

This result is also attributed to Choi and Kraus. Hence

$$\|\varphi\|_{cb} \leq \Big\| \sum_j v_j^* v_j \Big\| = \|\varphi(I)\|.$$

Actually we have equality throughout.

Spans of completely positive maps

Remember from before that to compute the completely bounded norm of $\varphi : \mathbb{M}_n \to \mathbb{M}_n$, we looked at maps $u, w : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^k$ with

$$\varphi(x) = u^*(x \otimes I)w \qquad (x \in \mathbb{M}_n).$$

Now we know that φ is completely positive if and only if we can choose $u = w$. However, polarisation gives

$$\varphi(x) = \frac{1}{4} \sum_{k=0}^{3} i^k (u + i^k w)^* (x \otimes I)(u + i^k w).$$

Thus any linear map $\mathbb{M}_n \to \mathbb{M}_n$ is a linear combination of 4 completely positive maps.

Quantum channels

I am far from an expert here!

- In quantum information theory, a *quantum channel* is a mathematical model of the evolution of an "open" quantum system.
- This is a trace preserving, completely positive map $\varphi : \mathbb{M}_n \rightarrow \mathbb{M}_n$.
- The trace is used to evaluate the probability of quantum states occurring, and so trace preservation reflects conservation of probability.
- Complete positivity is required to allow tensoring with other quantum systems without losing positivity.
- Given $\varphi : \mathbb{M}_n \rightarrow \mathbb{M}_n$ we can define $\varphi^\dagger : \mathbb{M}_n \rightarrow \mathbb{M}_n$ by using trace-duality: $\mathrm{Tr}(\varphi^\dagger(x)y) = \mathrm{Tr}(x\varphi(y))$. This operation preserves complete positivity, but φ is trace preserving if and only if $\varphi^\dagger$ is unital: $\varphi^\dagger(I) = I$.
- This swaps between the Schrödinger and Heisenberg pictures.

Some open problems

We focus on completely positive unital maps $\varphi : \mathbb{M}_n \to \mathbb{M}_n$, say with

$$\varphi(x) = \sum_i v_i^* x v_i.$$

- The collection of such maps, say UCP_n, is a bounded, convex subset of the collection of linear maps $\mathbb{M}_n \to \mathbb{M}_n$. So we might ask what the extreme points are (recall that a theorem of Minkowski shows that then UCP_n is the convex hull of its extreme points).
- Choi showed that φ is extreme if and only if we can choose the matrices (v_i) with $\{v_i^* v_j\}$ a linearly independent set.
- The closure of the set of extreme points in UCP_n is those φ which admit a representation as above, with at most n matrices v_i.
- There seems to be considerable interest in "characterising" or "classifying" the closure of the extreme points in UCP_n.

For example, when $n = 2$

Ruskai, Szarek, Wener showed that a ucp map $\varphi : \mathbb{M}_2 \to \mathbb{M}_2$ which is in the closure of the extreme points is of the following form:

$$\varphi(x) = u_1^* x u_1 + u_2^* x u_2,$$

where

$$u_1 = \sum_{j=1}^{2} \alpha_j \xi_j \overline{\eta_j^t}, \quad u_2 = \sum_{j=1}^{2} \sqrt{1-\alpha_j^2} \rho_j \overline{\eta_j^t}$$

where $\{\xi_1, \xi_2\}, \{\eta_1, \eta_2\}, \{\rho_1, \rho_2\}$ are three orthonormal bases of $\mathbb{C}^2$, and $0 \leq \alpha_j \leq 1$.
Apparently (caveat emptor!) there is nothing known for $\mathbb{M}_n \to \mathbb{M}_n$ for $n > 2$.

More on convexity

- Any $\varphi \in UCP_n$ can be written as a convex combination of extreme points. But how many?
- As before, we allow ourselves also to work with the closure of the extreme points. This is the maps of the form $\phi(x) = \sum_{i=1}^n u_i^* x u_i$.
- Conjecture: Any $\varphi \in UCP_n$ can be written as

$$\frac{1}{n} \sum_{j=1}^{n} \phi_j,$$

where each ϕ_j is in the closure of the extreme points.

- You can restate this in terms of matrices: suppose $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{M}_n)$ is positive. Conjecture: there are n matrices $B_k = (b_{ij}^{(k)})$, each of rank at most n, with

$$A = \frac{1}{n} \sum_k B_k, \quad \sum_j b_{jj}^{(k)} = \sum_j a_{jj} \text{ for each } k.$$

Entropy problem

- A *density matrix* is a positive matrix $x \in \mathbb{M}_n$ with $\mathrm{Tr}(x) = 1$.
- The *von Neumann entropy* of a density matrix x is $S(x) = -\mathrm{Tr}(x \log(x))$.
- Let $\varphi : \mathbb{M}_n \to \mathbb{M}_n$ be a completely positive, trace-preserving map. The *minimal entropy* of φ is

$$S_{\min}(\varphi) = \inf \left\{ S(\varphi(x)) : x \text{ a density matrix} \right\}.$$

- Is the following additivity conjecture true?

$$S_{\min}(\varphi \otimes \phi) = S_{\min}(\varphi) + S_{\min}(\phi).$$

Back to algebra

Let $A, B \subseteq \mathbb{M}_n$ be (unital) algebras. Then $\sum_i a_i \otimes b_i \in A \otimes B$ induces $\varphi : \mathbb{M}_n \to \mathbb{M}_n$ by

$$\varphi(x) = \sum_i a_i x b_i .$$

The completely bounded norm is estimated by

$$\|\varphi\|_{cb} \leq \Big\| \sum_i a_i a_i^* \Big\|^{1/2} \Big\| \sum_i b_i^* b_i \Big\|^{1/2} .$$

The infimum of the RHS is the *Haagerup tensor norm* on $A \otimes B$ (and is $\|\varphi\|_{cb}$). Conversely, suppose that $\varphi : \mathbb{M}_n \to \mathbb{M}_n$ is a linear map which is a left A'-module homomorphism, and a right B'-module homomorphism. Here $A' = \{x \in \mathbb{M}_n : xa = ax \ (a \in A)\}$ the *commutant* of A, and similarly for B'. Then we have that

$$\varphi(x) = \sum_i a_i x b_i \qquad \text{where } (a_i) \subseteq A'', (b_i) \subseteq B'' .$$

We can still compute $\|\varphi\|_{cb}$ be just considering these special forms.

Application to Hopf algebras

Recall that a Hopf $*$-algebra is, for me, a hermitian algebra $A \subseteq \mathbb{M}_m$ (so $a \in A \implies a^* \in A$) which admits a *coproduct* $\Delta : A \to A \otimes A$: that is, Δ is an algebra homomorphism, and $(\iota \otimes \Delta)\Delta = (\Delta \otimes \iota)\Delta$.

It's usual to either specify a counit and antipode, or to require some generalised cancellation rule which implies the existence of a counit and an antipode. But for me, a coalgebra is enough.

The dual space $A^\dagger = \hom(A, \mathbb{C})$ becomes a hermitian algebra for the product

$$\langle \mu\lambda, a\rangle = \langle \mu \otimes \lambda, \Delta(a)\rangle \qquad (a \in A, \mu, \lambda \in A^\dagger),$$

and the $*$-operation

$$\langle \mu^*, a\rangle = \overline{\langle \mu, a^*\rangle} \qquad (a \in A, \mu \in A^\dagger).$$

A *representation* of $A^\dagger$ is an algebra homomorphism $\pi : A^\dagger \to \mathbb{M}_n$ which preserves the $*$ operation.

Coefficients

Fix a representation $\pi : A^\dagger \to \mathbb{M}_n$, and pick $\xi, \eta \in \mathbb{C}^n$. These induce the *coefficient* $a_{\xi,\eta} \in A$, which satisfies

$$\langle \mu, a_{\xi,\eta} \rangle = \overline{\xi^t} \pi(\mu) \eta \qquad (\mu \in A^\dagger).$$

With the usual orthonormal basis (δ_i) for $\mathbb{C}^n$, we have that

$$\begin{aligned} \langle \mu \otimes \lambda, \Delta(a_{\xi,\eta}) \rangle &= \overline{\xi^t} \pi(\mu\lambda) \eta = \overline{\xi^t} \pi(\mu) \pi(\lambda) \eta \\ &= \sum_j \overline{\xi^t} \pi(\mu) \delta_j \overline{\delta_j^t} \pi(\lambda) \eta = \sum_j \langle \mu \otimes \lambda, a_{\xi,\delta_j} \otimes a_{\delta_j,\eta} \rangle. \end{aligned}$$

Thus

$$\Delta(a_{\xi,\eta}) = \sum_j a_{\xi,\delta_j} \otimes a_{\delta_j,\eta}.$$

Where the Haagerup norm comes in

$$\Delta(a_{\xi,\eta}) = \sum_j a_{\xi,\delta_j} \otimes a_{\delta_j,\eta}.$$

We can check that $a^*_{\xi,\eta} = a_{\eta,\xi}$. It's a bit tedious to show, but there is an absolute constant K depending only on π such that

$$\Big\| \sum_j a^*_{\delta_j,\eta} a_{\delta_j,\eta} \Big\| = \Big\| \sum_j a_{\eta,\delta_j} a_{\delta_j,\eta} \Big\| \leq K \|\eta\|^2.$$

Hence we see that

$$\|\Delta(a_{\xi,\eta})\|_{\text{Haagerup}} \leq K \|\xi\| \|\eta\|.$$

Consequently, studying these coefficients will have something to do with studying completely bounded, $A = A''$ bimodule maps $\mathbb{M}_m \to \mathbb{M}_m$. This is what I'm interested in.

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