Notes for YFAG talk

$\ell^1(\mathbb{Z}) = \{ (a_n)_{n\in\mathbb{Z}} : \|(a_n)\| = \sum |a_n| < \infty \}$
$c_0(\mathbb{Z}) = \{ (x_n)_{n\in\mathbb{Z}} : \|(x_n)\| = \sup |x_n| < \infty, \ x_n \to 0 \text{ as } |n| \to \infty \}$.

Then $c_0(\mathbb{Z})^* = \ell^1(\mathbb{Z})$.
Means: $\exists$ I.M. $\theta : \ell^1(\mathbb{Z}) \to c_0(\mathbb{Z})^*$
$$\langle \theta(\underline{a}), \underline{x} \rangle = \sum a_n x_n.$$

But $\ell^1(\mathbb{Z})$ is the dual space to lots of other spaces: if X is a compact Hausdorff space which is also countable, then Riesz $\Rightarrow$ $C(X)^* = M(X) \cong \ell^1(X)$ as all measures are countably additive. Pick $\alpha : \mathbb{Z} \to X$ bijection then set
$$\beta : \ell^1(\mathbb{Z}) \to C(X)^*,$$
$$\langle \beta(a), f \rangle = \sum_{n\in\mathbb{Z}} f(\alpha(n))\, a_n.$$

I'm interested in the resulting w^*-topology on $\ell^1(\mathbb{Z})$. E.g. if $X = \mathbb{Z}_\infty$ the one-point compactification,
$$\alpha : 0 \mapsto \infty, \quad 1 \mapsto 0, \quad 2 \mapsto 1, \ \ldots$$
$$-1 \mapsto -1, \quad -2 \mapsto -2, \ \ldots$$
Then $\beta(\delta_n) = \delta_{n-1} \xrightarrow{w^*} \delta_\infty$ in $M(X)$ as $n \to \infty$
$= \beta(\delta_0)$.
Of course, $\theta(\delta_n) \to 0$ w^* as $n \to \infty$.

A <u>concrete predual</u> of $\ell^1(\mathbb{Z})$ is a closed subspace E of $\ell^\infty(\mathbb{Z})$ such that

$$\begin{array}{ccc} \ell^1(\mathbb{Z}) & \hookrightarrow & \ell^\infty(\mathbb{Z})^* \\ & \searrow^{\text{Need I.M.}} & \downarrow \\ & & E^* \cong \ell^\infty(\mathbb{Z})^*/E^\perp \end{array}$$

$$E^\perp = \{ \Phi \in \ell^\infty(\mathbb{Z})^* : \Phi(x) = 0 \ \forall x \in E \}$$

Lemma: If θ: ~~[illegible]~~ $\ell^1(\mathbb{Z}) \to X^*$ is an I.M. for some Banach space X, then $\theta^*(X) = E \subseteq \ell^\infty(\mathbb{Z})$ is a concrete predual.

~~Furthermore, $\ell^1(\mathbb{Z}) \cong X^*$ and two concrete~~

The w^*-top. induced by θ agrees with that induced by $E^* \cong \ell^1(\mathbb{Z})$.

Two concrete preduals E_1, E_2 induce the same w^*-top. $\Leftrightarrow$ $E_1 = E_2$ as subspaces of $\ell^\infty(\mathbb{Z})$ ∎

Question: Is there a predual of $\ell^1(\mathbb{Z})$ which makes the bilateral shift $S: \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$ $\delta_n \mapsto \delta_{n+1}$ weak*-cts.

1 Lemma: $E \subseteq \ell^\infty(\mathbb{Z})$ predual makes S w^*-cts. $\Leftrightarrow$ $S^*(E) \subseteq E$

Proof: ($\Rightarrow$) Have $T \in \mathcal{B}(E)$ with $T^* = S$. So T is the "backwards" shift on $\ell^\infty(\mathbb{Z})$ restricted to E, i.e. $T = S^* \restriction_E$.
So $S^*(E) = T(E) \subseteq E$.
($\Leftarrow$) As $S^*(E) \subseteq E$ $\exists$ $T: E \to E$, $T = S^* \restriction_E$. Then follows that $T^* \cong S$ under IM E^* with $\ell^1(\mathbb{Z})$ ∎

~~Then~~ So $c_0(\mathbb{Z})$ makes the bilateral shift w^*-cts.

Suppose $C(X)$ did, for some IM given by $X \overset{\alpha}{\cong} \mathbb{Z}$.
Then E is an sub-C^*-algebra of $\ell^\infty(\mathbb{Z})$ (unital), and every ~~[illegible]~~ such E arises in this way.

Then $\beta: \ell^1(\mathbb{Z}) \to C(X)^*$ $\quad \delta_n \mapsto \delta_{\alpha(n)}$

So $\exists$ $T \in \mathcal{B}(C(X))$, $\beta^{-1} T^* \beta = S$ $\Leftrightarrow$ $T^* \beta = \beta S$

$\Leftrightarrow$ $T^*(\delta_{\alpha(n)}) = \beta S(\delta_n) = \beta(\delta_{n+1}) = \delta_{\alpha(n+1)}$.

So T is the composition operator with cts. map $f: X \to X$ given by $X \xrightarrow{\alpha^{-1}} \mathbb{Z} \xrightarrow{\text{shift}} \mathbb{Z} \xrightarrow{\alpha^{-1}} X$.

Or: Pull back topology from X to $\mathbb{Z}$; $\mathbb{Z}$ becomes a

compact (Hausdorff) space such that $n \mapsto n+1$ is cts. As
① S^{-1} also w^*-cts., $n \mapsto n-1$ is cts.
Compact countable spaces are Baire, so $\exists\, n$ with $\{n\}$ open.
Then translate forwards + backwards to see that every singleton is open $\Rightarrow$ $\mathbb{Z}$ has the discrete topology (# technically).

Part II: Construction

Fix $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. For $n \geq 0$ let $b(n) =$ # ones in binary expansion of n, so $b(0) = 0$, $b(1) = b(2) = 1$, $b(3) = 2$, $b(7) = 3$ etc. Define $b(n) = -\infty$ for $n < 0$, $\lambda^{-\infty} := 0$. Let $x_0 = (\lambda^{-b(n)})_{n \in \mathbb{Z}} \in l^\infty(\mathbb{Z})$,

$$x_0 = (\ldots 0 \underset{0}{1} \underset{1}{\lambda^{-1}} \underset{2}{\lambda^{-1}} \underset{3}{\lambda^{-2}} \underset{4}{\lambda^{-1}} \underset{5}{\lambda^{-1}} \underset{6}{\lambda^{-2}} \underset{7}{\lambda^{-3}} \underset{8}{\lambda^{-1}} \ldots$$

~~The~~ Let F = closed shift-invariant subspace of $l^\infty(\mathbb{Z})$ generated by x_0
$= \overline{\mathrm{lin}}\, \{ (S^*)^n (x_0) : n \in \mathbb{Z} \}$.

Theorem: $F^* \cong l^1(\mathbb{Z})$ canonically (ie. F a concrete predual).

Obvious that $F \neq c_0(\mathbb{Z})$, so get "new" weak*-topology.

Let $\sigma = (S^*)^{-1}$ on $l^\infty(\mathbb{Z})$, so $\sigma(x)(n) = x(n-1)$. Define $\tau : l^\infty(\mathbb{Z}) \circlearrowleft$,

$$\tau(x)(n) = \begin{cases} x(n/2) & : n \text{ even} \\ 0 & : \text{o/w} \end{cases}$$

e.g. $\tau(x_0) = (\ldots 0, 1, 0, \lambda^{-1}\ 0\ \lambda^{-1}\ 0\ \lambda^{-2}\ 0 \ldots)$

Note that $\tau\sigma = \sigma^2 \tau$.

Lemma: $\tau^k(x_0) \in F$ for all $k \geq 1$
Proof: First, using binary expansion, show that

$$(1-\lambda^{-1}\sigma)(x_0) = (1-\lambda)\sum_{j=1}^{\infty}\lambda^{-j}\tau^{j}(x_0)$$

Then apply $(1-\lambda^{-1}\tau)$: $(1-\lambda^{-1}\tau)(1-\lambda^{-1}\sigma)(x_0) = \frac{\lambda-1}{\lambda}\tau(x_0)$

as get a telescoping sum.

Expand, use $\tau\sigma = \sigma^2\tau$, get $(1-\lambda^{-1}\sigma)(x_0) = (1-\lambda^{-2}\sigma^2)(\tau(x_0))$

As $\|\lambda^{-2}\sigma^2\| = |\lambda^{-2}| < 1$, Neumann Series $\Rightarrow$

$$\tau(x_0) = (1-\lambda^{-2}\sigma^2)^{-1}(1-\lambda^{-1}\sigma)(x_0)$$

$$= \sum_{j\geq 0}\lambda^{-2j}\sigma^{2j}\left((1-\lambda^{-1}\sigma)(x_0)\right) \in F.$$

Then using $\tau\sigma = \sigma^2\tau$, follows that $\tau^2(x_0) \in F$ etc etc ∎

Claim: The map $l^1(\mathbb{Z}) \to E^*$ injective.

Proof: Notice $\tau^k(x_0) = (\dots 0\ 1\ \underbrace{0 \dots 0}_{2^{n}-1}\ \lambda^{-1}\ 0 \dots)$

So $\forall\, \underline{a} \in l^1$, $\langle \tau^k(x_0), \underline{a}\rangle \to a_0$ as $n \to \infty$.

So if $\underline{a}$ annihilates E, then $a_0 = 0$. By shift invariance, $a_n = 0$ $\forall n \Rightarrow \underline{a} = 0$. ∎

On $\ell^\infty(\mathbb{Z})^*$

As $\ell^\infty(\mathbb{Z})$ is a commutative ~~Ban~~ C^*-algebra ~~then~~ the character space of $\ell^\infty(\mathbb{Z})$ is a cmpt Hausdorff space, say $\beta\mathbb{Z}$, (= Stone-Cech compactification). So each point in $\beta\mathbb{Z}$ is a character $\ell^\infty(\mathbb{Z}) \to \mathbb{C}$. In particular, $\mathbb{Z} \subseteq \beta\mathbb{Z}$ is dense; let $\mathbb{Z}^* = \beta\mathbb{Z} \setminus \mathbb{Z}$.

Topology on $\beta\mathbb{Z}$ = Relative w^*-top. So if $A \subseteq \mathbb{Z}$, χ_A = Indicator function of A, in $\ell^\infty(\mathbb{Z})$, then $\chi_A \in C(\beta\mathbb{Z})$ is a $\{0,1\}$-valued cts. function so $v \in \beta\mathbb{Z} \Rightarrow \langle v, \chi_A\rangle = 0$ or 1 and $O_A = \{v : \langle v, \chi_A\rangle = 1\}$ is open (and closed) in $\beta\mathbb{Z}$.

Back to products

Define $X_t^{(k)} = \{v \in \mathbb{Z}^* : \forall m > 0,\ v \in O_A$ where
$A = \{2^{n_1} + \dots + 2^{n_k} + t : m < n_1 < n_2 < \dots < n_k\}\}$.
closed in $\mathbb{Z}^*$.
Let $X^{(\infty)} = \mathbb{Z}^* \setminus \bigcup_{k,t} X_t^{(k)}$.

Lemma: $\mathbb{Z}^*$ disjoint union of $X^{(\infty)}$ and the $X_t^{(k)}$

Proof: Need only show $X_t^{(k)} \cap X_s^{(l)} = \emptyset$ if $(k,t) \neq (l,s)$.
But that can't happen. ■

Theorem: $x \in F \iff$ as a function in $C(\beta\mathbb{Z})$,

$$x(v) = \begin{cases} \lambda^{-k} x(t) & : v \in X_t^{(k)} \\ 0 & : v \in X^{(\infty)} \end{cases}$$

Proof (sketch): Let $G \subseteq \ell^\infty(\mathbb{Z})$ be the (auto. closed) subspace satisfying these conditions.

Note that $\beta\mathbb{Z}$ has a $\mathbb{Z}$ action extending that of $\mathbb{Z}$ on itself. if $v \in \beta\mathbb{Z}$, $t \in \mathbb{Z}$ then $v+t$ is the character

$x \mapsto v[(S^*)^t(x)]$. As $(S^*)^t(x)(n) = x(n+t)$ for $n \in \mathbb{Z}$, this does ~~extend~~ extend action.

Then $(S^*)^t(\chi_A)(n) = \chi_A(n+t) = 1 \iff n \in A - t$

So $\mathcal{O}_A + t = \{ v+t : \langle v, \chi_A \rangle = 1 \}$

$= \{ v : \langle v - t, \chi_A \rangle = 1 \}$

$= \{ v : \langle v, (S^*)^{-t}(\chi_A) \rangle = \langle v, \chi_{A+t} \rangle = 1 \}$

$= \mathcal{O}_{A+t}$

Note $v \in \mathbb{Z}^* \iff v+t \in \mathbb{Z}^*$ $(\forall t)$.

Then $X_t^{(k)} + s = X_{t+s}^{(k)}$.

So if $x \in G$, $v \in X_t^{(k)}$ then $v+1 \in X_{t+1}^{(k)}$, so

$S^*(x)(v) = x(v+1) = \lambda^{-k} x(t+1) = \lambda^{-k} S^*(x)(t)$

(and same if $v \in X^{(\infty)}$) $\Rightarrow$ <u>G shift-invariant</u>

<u>Claim</u>: $x_0 \in G$.

<u>Proof</u>: If $t \geq 0$, then if $m >> 0$, $b(2^{n_1} + \dots + 2^{n_k} + t)$

$= k + b(t)$ if $m < n_1 < \dots$

If $t < 0$ then $b(2^{n_1} + \dots + 2^{n_k} + t) \to \infty$ as $m \to \infty$.

So if $A = \{ 2^{n_1} + \dots + 2^{n_k} + t : m < n_1 < \dots \}$ then restricted to A,

$x_0 \simeq \lambda^{-k} x_0(t)$. So for $v \in \mathcal{O}_A$,

$v(x_0) = v(\chi_A) v(x_0) = v(\chi_A x_0) \simeq \lambda^{-k} x_0(t) v(\chi_A)$

$= \lambda^{-k} x_0(t)$.

~~$\Rightarrow v \in G$.~~ ~~QED~~

Remains to show $v \in X^{(\infty)} \Rightarrow v(x_0) = 0$.

If <u>not</u>, then note that x_0 only takes the values $\{0\} \cup \{\lambda^{-k} : k > 0\}$.

So $\exists k$, $v(x_0) = \lambda^{-k}$

$\Rightarrow$ If $B = \{ n : x_0(n) = \lambda^{-k} \} = \{ n : b(n) = k \} =$

$\{ 2^{n_1} + \dots + 2^{n_k} : n_1 < n_2 < \dots \}$

then $v \in \mathcal{O}_B$ (else $v(\chi_B) = 0 \Rightarrow \#$). But $v \notin X_0^{(k)}$

So $B_m = \{ 2^{n_1} + \dots + 2^{n_k} : n_1 < \dots < n_k, m \leq n_1 \}$ then

$v \in \mathcal{O}_{B_m}$. Finite partition $\Rightarrow$ can fix n_1.

Then $v \notin X_{2^{n_1}}^{(k-1)}$ $\Rightarrow$ fix n_2 etc. $\Rightarrow v \in \mathbb{Z}$ $\#$.

So $F \subseteq G$, so $F^{\perp} \supseteq G^{\perp}$

$$\begin{tikzcd} \ell^1(\mathbb{Z}) \arrow[r, "\theta_G"] \arrow[dr, "\theta_F"'] & G^* = \ell^\infty(\mathbb{Z})^*/G^{\perp} \arrow[d] \\ & \ell^\infty(\mathbb{Z})^*/F^{\perp} = F^* \end{tikzcd}$$

θ_F injective $\Rightarrow$ so is θ_G.

Let $\mu \in G^*$, Hahn-Banach μ to member of $\ell^\infty(\mathbb{Z})^* = M(\beta\mathbb{Z})$. As disjoint sets, for $x \in G$

$$\langle \mu, x \rangle = \int_{\beta\mathbb{Z}} x \, d\mu$$

$$= \int_{X^{(\infty)}} x \, d\mu + \sum_{t\in\mathbb{Z}} \left(x(t)\mu(\{t\}) + \sum_{n\geq 1} \int_{X_t^{(n)}} x \, d\mu \right)$$

$$= \sum x(t)\mu(\{t\}) + \sum \lambda^{-k} x(t) \mu(X_t^{(n)})$$

$$= \sum x(t) \, a(t) \quad \text{say, where } a \in \ell^1(\mathbb{Z}).$$

So θ_G onto $\Rightarrow$ θ_F onto, so Open Mapping $\Rightarrow$ θ_F I.M. (and so is $\theta_G \Rightarrow F = G$). ∎

<u>What is F ?</u>

$F = G$ is defined by a "family of equations"

$$x(v) = \lambda^{-n} x(t) \qquad v \in X_t^{(n)}$$
$$x(v) = 0 \qquad v \in X^{(\infty)}$$

So [Benyamini] "G-spaces" $\Rightarrow$ $F \cong C(L)$ (not isometric) for some compact Hausdorff space L.

Szlenk index invariant for Banach spaces (gives ordinal) and can be computed by knowing F and $F^* \cong \ell^1(\mathbb{Z})$. Classifies $C(L)$ spaces (for "small" L, which exists).

<u>Thm</u>. $F \cong C(\omega+1) \cong c_0$ as a Banach space.

<u>Think more abstractly</u>

<u>Semitopological semigp. compactification</u> of $\mathbb{Z}$ is a compact S, which is a semigp, $\forall s \in S$, $t \mapsto st$, $t \mapsto ts$ are cts. and $\exists \mathbb{Z} \hookrightarrow S$ injective, dense range (normally don't ask for "injective").

$\mathbb{Z} \subseteq S$ dense $\Rightarrow$ S abelian. Then $\ell^1(\mathbb{Z}) \hookrightarrow M(S) = C(S)^*$

$\ell^1(\mathbb{Z})$ is a Banach algebra: $\delta_n \delta_m = \delta_{n+m}$.
$M(S)$ is as well: $\langle \mu * \nu, f \rangle = \int\int f(s+t)\, d\mu(s)\, d\nu(t)$.
$\mu, \nu \in M(S)$, $f \in C(S)$

Then $\ell^1(\mathbb{Z}) \hookrightarrow M(S)$ is a <u>homomorphism</u>.

<u>Theorem</u>: Let $\circledast : M(S) \to \ell^1(\mathbb{Z})$ be a bounded projection which is also a homomorphism. Let

$$F = {}^{\perp}(\ker \circledast) = \{ f \in C(S) : \langle \mu, f \rangle = 0 \ \forall \mu \text{ with } \circledast(\mu) = 0 \}.$$

Then $C(S) \restriction_{\mathbb{Z}}$ identifies $C(S)$ as a closed subspace of $\ell^\infty(\mathbb{Z})$. Under this, F is a shift-invariant predual for $\ell^1(\mathbb{Z})$.

In the example, notice that $E = F$ only "sees" each set $X_t^{(h)}$ as a point. So let

$$S = \mathbb{Z} \cup \{ x_t^{(h)} : t \in \mathbb{Z}, h > 0 \} \cup \{\infty\}.$$

Maybe ② ? Tedious calculation shows

$$u \in X_t^{(h)}, \ v \in X_s^{(l)} \Rightarrow u + v \in X_{s+t}^{(h+l)}.$$

So S becomes a semigroup via $n \in \mathbb{Z} \leftrightarrow x_n^{(0)}$,

$$x_t^{(h)} + x_s^{(l)} = x_{t+s}^{(h+l)}$$

$$x + \infty = \infty \qquad (\forall x).$$

Use $\beta\mathbb{Z} \to S$, $\quad u \in X_t^{(k)} \mapsto x_t^{(k)}$

$\quad u \in X^{(\infty)} \mapsto \infty$

to induce the quotient topology on S. Then S is the one-point compactification of $S \setminus \{\infty\}$, and $S \setminus \{\infty\}$ is loc. cpt Hausdorff with a basic open n'hood of $x_t^{(k)}$ being

$$\{ x_s^{(l)} : l < k, \ t = s + 2^{n_1} + \dots + 2^{n_{k-l}}, \ n < n_1 < \dots < n_{k-l} \} \quad \text{for some } n \geq 0.$$

"Picture": $\mathbb{Z}$ all isolated points.

$2^n \to x_0^{(1)}$ as $n \to \infty$

$\Rightarrow 2^n + t \to x_t^{(1)}$ as $n \to \infty$

$2^n + 2^m \to x_0^{(2)}$ as $n \to \infty, m \to \infty, n < m$

$\Rightarrow x_0^{(1)} + 2^n \to x_0^{(2)}$ etc.

$(H) : M(S) \to \ell^1(\mathbb{Z}) \qquad \delta_\infty \mapsto 0$

$\delta_{x_t^{(k)}} \mapsto \lambda^{-k} \delta_t$

Non-$\mathscr{G}$ example

Define the same S, but with $(H)(\delta_{x_t^{(k)}}) = a^k * \delta_t$ where $a \in \ell^1(\mathbb{Z})$ is power bounded.

E.g. $a = \frac{1}{2}(\delta_0 + \delta_1)$. Some general theory shows that as $\|a^k\|_\infty \to 0$ as $k \to \infty$, then $\ker (H)$ is w^*-closed in $M(S)$, which is what we need for ${}^\perp(\ker (H))$ to be a predual.

However, Szlenk index calculations show that the resulting predual is not IM to c_0.

Addenda

(1) Hahn-Banach argument $\Rightarrow$ If T is w^*-cts. and invertible then its pre-adjoint is invertible, i.e. T^{-1} is w^*-cts.
So $S^*(E) \subseteq E \Rightarrow S^*(E) = E$.

(2) More on $\beta\mathbb{Z}$? Let X be cmpt, Hausdorff. ~~Then~~ and let $\phi : \mathbb{Z} \to X$ be a (cts.) map. ~~Then~~ with dense range.
Then $\Phi : C(X) \to l^\infty(\mathbb{Z})$, $f \mapsto (f(\phi(n)))_{n\in\mathbb{Z}}$ is an isometric $*$-homomorphism. So $\forall\, v \in \beta\mathbb{Z}$ = characters on $l^\infty(\mathbb{Z})$ $C(X) \to \mathbb{C}$, $f \mapsto v(\Phi(f))$ is a non-zero character, so ~~then~~ $\exists\, x = \check{\phi}(v) \in X$ with

$$\Phi(f)(v) = f(\check{\phi}(v)) \qquad \forall f \in C(X).$$

Then $\check{\phi}$ is cts. (think about w^*-topologies) and $\check{\phi}(n) = \phi(n)$ for $n \in \mathbb{Z}$. So get.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\phi"] \arrow[dr, leftrightarrow] & X \\ & \beta\mathbb{Z} \arrow[u, "\check{\phi}"'] \end{tikzcd}$$

In particular, for $v \in \beta\mathbb{Z}$ get map $\mathbb{Z} \to \beta\mathbb{Z}$, $n \mapsto v+n$.
Extend to $\beta\mathbb{Z} \to \beta\mathbb{Z}$, $u \mapsto v+u$. Then for $A \subseteq \mathbb{Z}$,

$$\langle v+u, \chi_A \rangle = \Phi(\chi_A)(u)$$

where $F = \Phi(\chi_A) \in l^\infty(\mathbb{Z})$ is $F(n) = \langle v+n, \chi_A \rangle = \langle v, \chi_{A-n} \rangle$
so $F = \chi_B$ where $n \in B \Leftrightarrow \langle v, \chi_{A-n} \rangle = 1 \Leftrightarrow v \in C_{A-n}$

[Or: If (n_α) net tending to u then $v+u = \lim_\alpha v+n_\alpha$ etc.