L^p-operator algebras in the case $p = 1$

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Group algebras

Consider a locally compact group G and the left-regular representation of G on $L^2(G)$,

$$(\lambda_s \xi)(t) = \xi(s^{-1}t) \qquad (s, t \in G, \xi \in L^2(G)).$$

Integrating gives a contractive algebra homomorphism

$$\lambda\colon L^1(G) \to \mathcal{B}(L^2(G)), \qquad \lambda(f)(\xi) = f * \xi.$$

The norm closure of the image is the (reduced) group C^*-algebra $C_r^*(G)$, and the weak*-closure is the group von Neumann algebra $VN(G)$.

Varying p

There is nothing special about $p = 2$ here. The norm closure of the image of $\lambda_p \colon L^1(G) \to \mathcal{B}(L^p(G))$ has long been studied, and is the algebra of *p-pseudo functions*, $PF_p(G)$ or $F^p_\lambda(G)$.
The weak*-closure is $PM_p(G)$ the algebra of *p-pseudo measures*.

- At least if $1 < p < \infty$ of course, so that $L^p(G)$ is reflexive, with dual space $L^{p'}(G)$ for $1/p + 1/p' = 1$.
- There is a natural pairing between $\mathcal{B}(L^p)$ and $\mathcal{N}(L^p)$ the *nuclear operators* on L^p, which turns $\mathcal{B}(L^p)$ into a dual Banach algebra.
- Indeed, $\mathcal{N}(L^p)$ is the dual of the compact operators $\mathcal{K}(L^p)$.
- Reminder: T is *nuclear* where there are (f_n) in $L^p(G)$ and (g_n) in $L^{p'}(G)$ with

$$\sum_n \|f_n\|_p \|g_n\|_{p'} < \infty, \qquad T(f) = \sum_n \langle g_n, f \rangle f_n \quad (f \in L^p(G)).$$

Commutants

We think of $PM_p(G)$ as being like $VN(G)$.

- We also have the right-regular representation ρ_p of G on $L^p(G)$, which leads to $PM_p^\rho(G)$ say.
- Thinking about von Neumann's bicommutant theorem, we could also define

  $$CV_p(G) = \lambda_p(G)'', \qquad CV_p^\rho(G) = \rho_p(G)'',$$

  the algebras of *p-convolvers*. These are weak*-closed.
- [D.–Spronk; folklore] we have $PM_p(G)' = CV_p^\rho(G)$ and $CV_p(G) = CV_p^\rho(G)'$ (and so forth).
- But...do we have $PM_p(G) = PM_p(G)''$ $(= CV_p(G))$? von Neumann's result uses *projections*.
- [Cowling; D.–Spronk] when G has the approximation property (e.g. G is amenable, weakly-amenable, etc.) then $PM_p(G) = CV_p(G)$. Unknown in general.

Cuntz algebras; Cohn algebras; Leavitt algebras

The *Cohn algebra* C_2 has generators s_1, s_2, t_1, t_2 subject to the relations

$$t_1 s_1 = t_2 s_2 = 1, \qquad t_1 s_2 = t_2 s_1 = 0.$$

If we impose the further relation

$$s_1 t_1 + s_2 t_2 = 1,$$

we obtain the *Leavitt algebra* L_2.

Of course, if we consider the C^*-algebra with $t_i = s_i^*$ then we obtain the *Cuntz algebra* $\mathcal{O}_2$.

As L^p-operator algebras

Consider $\ell^p = \ell^p(\mathbb{N})$, thought of as sequences $x = (x_n)$ with $\|x\|_p^p = \sum_n |x_n|^p < \infty$.
Define $S_i, T_i \colon \ell^p \to \ell^p$ by

$$S_1(x) = (x_1, 0, x_2, 0, \cdots), \qquad S_2(x) = (0, x_1, 0, x_2, 0, \cdots)$$
$$T_1(x) = (x_1, x_3, x_5, x_7, \cdots), \qquad T_2(x) = (x_2, x_4, x_6, x_8, \cdots).$$

Then S_1, S_2 are isometries, T_1, T_2 are contractions, and $S_1 T_1, S_2 T_2$ are contractive idempotents. We check the relations:

$$T_1 S_1 = T_2 S_2 = 1, \quad T_1 S_2 = T_2 S_1 = 0, \quad S_1 T_1 + S_2 T_2 = 1.$$

So we obtain a representation of L_2; this is faithful.
Denote by $\mathcal{O}_2^p$ the closure of L_2 in $\mathcal{B}(\ell^p)$. [Phillips]

Spatial partial isometries

Question

To what extent is $\mathcal{O}_2^p$ unique, for $p \neq 2$?

Fix $p \neq 2$. There are not many isometries on an $L^p(\Omega)$ space:

- We can form a composition operator ("Koopman operator") with a measure-preserving transformation;
- Or more generally allow a change-of-density;
- We can multiply by $f \in L^\infty(\Omega)$ where $|f(\omega)| = 1$ a.e.

Lamperti's Theorem states that this is it.
We obtain a notion of a "spatial partial isometry" $L^p(\Omega) \to L^p(\Omega')$ given by an isometry $L^p(E) \to L^p(E')$ where $E \subseteq \Omega$ and $E' \subseteq \Omega'$.

Notice that each S_i, T_i is a spatial partial isometry on ℓ^p.

Matrix algebras; isometric representations

Notice that $e_{ij} = s_i t_j \in L_2$ form a copy of the matrix units of $\mathbb{M}_2$. We can identify $\mathcal{B}(\ell^p_2)$ with $\mathbb{M}_2$ which gives the L^p-operator algebra M^p_2.

Theorem (Phillips)

Any contractive representation $\rho\colon M^p_2 \to \mathcal{B}(L^p)$ is automatically isometric, and $\rho(e_{ij})$ is a spatial partial isometry for each i, j.

Theorem (Phillips)

Let $\rho\colon L_2 \to \mathcal{B}(L^p)$ be a representation which is contractive on M^p_2 and with $\rho(t_i), \rho(s_i)$ contractions for $i = 1, 2$. Then ρ is "spatial" and the identification of $L_2 \subseteq \mathcal{O}^p_2$ and $\rho\colon L_2 \to \overline{\rho(L_2)}$ extends to an isometric isomorphism $\mathcal{O}^p_2 \to \overline{\rho(L_2)}$.

Thus, $\mathcal{O}^p_2$ is unique, for *contractive* homomorphisms.

Contractive representations

These ideas of "contractive (often) implies isometric" and "there are not many isometries" can be further exploited.

Theorem (Gardella–Thiel)

For $p \neq 2$, and G, H locally compact groups, we have that $PM_p(G)$ is isometrically isomorphic to $PM_p(H)$ if and only if G and H are isomorphic. Similarly for PF_p and CV_p.

This fails when $p = 2$.

Question

Should we be more interested in the isomorphic theory?

Are we looking for our keys under the lamppost, or being guided by (non-selfadjoint) operator algebra theory?

The non-reflexive case

Much of the theory works when $p = 1$, but a little care is needed.

- $\mathcal{B}(L^1)$ is not naturally a dual Banach algebra (unless the L^1 space is finite-dimensional).
- $\mathcal{B}(\ell^1)$ is a one-sided dual Banach algebra, for the predual given by the nuclear operators on c_0.

So it's not clear what $PM_1(G)$ should be, and $CV_1(G)$ need not be weak*-closed, at least from general theory.

Result

For any G, the left-regular representation $\lambda\colon L^1(G) \to \mathcal{B}(L^1(G))$ is an isometry. Thus $PF_1(G) = L^1(G)$ canonically and isometrically.

Result

For discrete G, we have $CV_1(G)' = \rho(\ell^1(G))$ and $CV_1^\rho(G)' = \lambda(\ell^1(G))$. In particular, $PF_1(G) = PM_1(G) = CV_1(G)$.

Idea: what's the ℓ^1-Leavitt/Cuntz algebra?

We can of course consider $\mathcal{O}_2^1 \subseteq \mathcal{B}(\ell^1)$, but this is a "concrete" algebra, which seems very different to $\ell^1(G)$ in construction.

Instead, let's start by looking at the "Cuntz semgrioup" (monoid) which is a semigroup with zero, which we denote $\Diamond$, with presentation

$$Cu_2 = \big\langle s_1, s_2, t_1, t_2 : t_1 s_1 = t_2 s_2 = 1, t_1 s_2 = t_2 s_1 = \Diamond \big\rangle.$$

(This is actually an involutive semigroup, if we define $s_i^* = t_i$. Our representations won't have any star structures however.)

We consider the Banach algebra $\ell^1(Cu_2)$. I will abuse notation, and for $s \in Cu_2$ write $s \in \ell^1(Cu_2)$ for the point-mass at s.

Identifying zeroes

We'd like to identify the semigroup zero and the algebra 0. At present, $\Diamond \neq 0$ as of course $\|\Diamond\| = 1$.

- Notice that $\mathbb{C}\Diamond$ is a (two-sided) ideal in $\ell^1(Cu_2)$.
- So we can quotient by it, leading to $\mathcal{A} = \ell^1(Cu_2)/\mathbb{C}\Diamond$.
- Slightly informally, this is just the same as setting $\Diamond = 0$.
- Nothing strange happens with the norm.

Indeed, we could instead take the Cohn algebra C_2, thought of as $\mathbb{C}$-linear combinations of words in s_i, t_i. Define a norm by taking the ℓ^1-sum of the coefficients of these words. Then $\mathcal{A}$ is the completion.

[Dales, Laustsen, Read, 2003] studied $\mathcal{A}$.

To Leavitt algebras

We now further quotient $\mathcal{A} = \ell^1(Cu_2)/\mathbb{C}\Diamond$ by the relation

$$s_1 t_1 + s_2 t_2 = 1.$$

That is, we consider the closed two-sided ideal $\mathcal{J}$ generated by the element $f_0 = s_1 t_1 + s_2 t_2 - 1$ and quotient by this ideal.

- Both $\mathcal{A}$ and $\mathcal{A}/\mathcal{J}$ are extremely agreeable to combinatorial study.
- We can combinatorially characterise when $f \in \mathcal{J}$.

I want to argue that $\mathcal{A}/\mathcal{J}$ is an "ℓ^1-completion of L_2".

- It seems rare in the Banach algebra world to study quotients of semigroup algebras.

Purely infinite algebras

Theorem (D.–Horvath)

For $f \in \mathcal{A}$, the following are equivalent:

1. $f \in \mathcal{J}$;
2. f *has zero sums at every* $v \in Cu_2 \setminus \{\lozenge\}$ *without symmetric core;*
3. *there are no* $g, h \in \mathcal{A}$ *with* $gfh = 1$.

Corollary

$\mathcal{A}/\mathcal{J}$ *is purely infinite: for each* $x \neq 0$ *there are* y, z *with* $yxz = 1$.

This is perhaps not the usual C^*-algebraic meaning of "purely infinite", but it is equivalent to it.
Phillips showed that $\mathcal{O}_2^p$ is purely infinite.

Our motivation: ultrapowers

Recall the ultrapower construction for a Banach algebra A. We first consider $\ell^\infty(A)$, the algebra of all bounded sequences (x_n) in A. Let $\mathcal{U}$ be a countably-incomplete ultrafilter. Then

$$\Big\{(x_n) \in \ell^\infty(A) : \lim_{n\to\mathcal{U}} \|x_n\| = 0\Big\}$$

is a closed ideal in $\ell^\infty(A)$; we denote the quotient algebra by $(A)_\mathcal{U}$ the *ultra-power* of A.

We were seeking a counter-example to the claim that a purely infinite Banach algebra has purely infinite ultrapowers. (This holds for C^*-algebras.)

Theorem (D.–Horvath)

$(\mathcal{A}/\mathcal{J})_\mathcal{U}$ *is not simple, so certainly not purely infinite.*

(non-)Isomorphisms and questions

The natural representation $L_2 \to \mathcal{B}(\ell^1)$ which defines $\mathcal{O}_2^1$ gives an injective representation $\mathcal{A}/\mathcal{J} \to \mathcal{O}_2^1 \subseteq \mathcal{B}(\ell^1)$ but one can show that it is not bounded below.

Theorem (D.–Horvath)

There are no non-zero bounded traces on $\mathcal{O}_2^1$, but there are on $\mathcal{A}/\mathcal{J}$. Thus $\mathcal{A}/\mathcal{J}$ is not isomorphic to $\mathcal{O}_2^1$ in any way.

Question

Does $\mathcal{O}_2^1$ have purely infinite ultrapowers?

Question

Is $\mathcal{A}/\mathcal{J}$ amenable?

Phillips showed, using ideas from the C^*-world, that $\mathcal{O}_2^1$ is amenable, but the norm estimates cannot work for $\mathcal{A}/\mathcal{J}$.

Coxeter groups

As a second example, I will consider Coxeter groups.

A Coxeter group W has generators $s \in S$ subject to relations of the form

$$s^2 = 1 \quad (s \in S), \qquad (st)^{m_{s,t}} = 1,$$

where $m_{s,t} = m_{t,s} \in \{2, 3, \cdots, \infty\}$ gives the order of the element st for $s \neq t$.

Example

Take $S = \{s, t\}$ with $m_{s,t} = 3$, so

$$W = \langle s, t : s^2 = t^2 = 1, ststst = 1 \rangle.$$

This is isomorphic to S_3, via $s \mapsto (1\,2), t \mapsto (2\,3)$.

Generic algebras

Let $(a_s),(b_s)$ be scalars indexed by the generators S, with the only constraint that if s,t are conjugate in W, then $a_s = a_t$ and $b_s = b_t$.

Definition

The *generic algebra* $\mathbb{C}_{a,b}[W]$ is generated by elements $\{T_s : s \in S\}$ with the relations

$$T_s^2 = a_s T_s + b_s T_1, \qquad \underbrace{T_s T_t T_s \cdots}_{m_{s,t} \text{ times}} = \underbrace{T_t T_s T_t \cdots}_{m_{s,t} \text{ times}}$$

- Although not clear in this presentation, the T_s act like generators of the group algebra, except in the case when cancellation occurs in the combinatorics of W.
- If $a_s = 0, b_s = 1$ then we obtain exactly the group algebra.

Weighted semigroup algebras

Given a reduced word $w = s_1 \cdots s_n$, we (well-)define

$$\omega(w) = \prod_{j=1}^{n} \mathsf{max}(1, |a_{s_j}| + |b_{s_j}|).$$

Let $\mathbb{S}_S$ be the free monoid on generators S, and define ω on $\mathbb{S}_S$ in the same way. Consider the weighted semigroup algebra

$$\ell^1(\mathbb{S}_S, \omega) = \Big\{ (a_u)_{u \in \mathbb{S}_S} : \sum_u \omega(u)|a_u| < \infty \Big\}.$$

Denote by δ_u the point mass at $u \in \mathbb{S}_S$.
Let I be the ideal in $\ell^1(\mathbb{S}_S, \omega)$ generated by the relations in $\mathbb{C}_{a,b}[W]$,

$$\delta_s^2 = a_s\delta_s + b_s\delta_1, \quad \underbrace{\delta_s\delta_t\delta_s \cdots}_{m_{s,t} \text{ times}} = \underbrace{\delta_t\delta_s\delta_t \cdots}_{m_{s,t} \text{ times}}$$

An ℓ^1-generic algebra

There is a natural map $\mathbb{S}_S \to W$: send a word $u \in \mathbb{S}_S$ to the element of W it defines. This induces a Banach space isometric isomorphisms

$$\ell^1(\mathbb{S}_S, \omega)/I \to \ell^1(W, \omega).$$

The LHS is an algebra, so we have effectively defined a "twisted" product on $\ell^1(W, \omega)$. Indeed, the natural map

$$\mathbb{C}_{a,b}[W] \to \ell^1(W, \omega); \quad T_s \mapsto \delta_s,$$

is an injective homomorphism. We think of $\ell^1(W, \omega)$ with this product as being the ℓ^1 version of the Generic algebra.

Again, there is a natural action of $\mathbb{C}_{a,b}[W]$ on $\ell^1(W)$, leading to $F^1_{a,b}(W)$ say. [Ruam–Skalski]

This extends to a contraction $\ell^1(W, \omega) \to F^1_{a,b}(W)$, but again this is rarely bounded below.

Final question

I deliberately ellided the difference between C^*-algebras and non-self-adjoint operator algebras when motivating L^p-operator algebras.

Question

The examples here all seem more "C^*-algebra like" than "non-self-adjoint", although there was no involution.
Is there some sense in which an L^p-operator algebra can be considered "self-adjoint".
I mean to ask this by analogy.

Thanks to the organisers!

And let's hope for a repeat of this conference in the future.