Weakly almost periodic functionals on the measure algebra

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Outline

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Weakly almost periodic functionals

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Generalise: f is *weakly almost periodic* if $L_G(f)$ is (relatively) compact, in the *weak* topology on $C^b(G)$.

Links with compactifications

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Replace "compact group" by "compact semitopological semigroup" (that is, separate continuity of the product) and we replace "almost periodic" by "weakly almost periodic".

For Banach algebras

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$\mathrm{wap}(\mathcal{A})$ has interesting links with the Arens products on $\mathcal{A}^{**}$.

In general, little can be said about $\mathrm{wap}(\mathcal{A})$ and $\mathrm{ap}(\mathcal{A})$.

Measure algebras

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To be more precise: the history above was backwards. To show that $\text{wap}(L^1(G))$ is a subalgebra of $L^\infty(G)$ requires the result that $\text{wap}(L^1(G)) = \text{wap}(G)$, and then an application of Grothendieck's repeated limit criterion for weak compactness.

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Bounded approximate identities allows you to build π from $\hat{\pi}$.

"Multiplying" functionals

Given $\pi : G \to \text{iso}(E)$, a *coefficient functional* of π is

$$F \in C^b(G), \quad F(s) = \langle \mu, \pi(s)x \rangle \quad (s \in G),$$

where $\mu \in E^*$ and $x \in E$. Write $F = \omega_{\pi,\mu,x}$.

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This is exactly the proof that the Fourier-Stieltjes algebra is an algebra (all coefficient functionals of unitary representations).

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Theorem

$\mu \in \mathrm{wap}(\mathcal{A}^*)$ *if and only if there exists a reflexive Banach space* E, *a representation* $\pi : \mathcal{A} \to \mathcal{B}(E)$, *and* $x \in E, \mu \in E^*$ *with*

$$\langle \mu, a \rangle = \langle \mu, \pi(a)(x) \rangle \qquad (a \in \mathcal{A}).$$

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So $F \in \text{wap}(L^1(G))$ if and only if F is the coefficient functional of a representation on a *reflexive* Banach space.

Reflexive tensor products

Let E and F be reflexive Banach spaces. There exists a norm on $E \otimes F$ such that:

1. $\|x \otimes y\| = \|x\|\|y\|$ for $x \in E, y \in F$;
2. Given $T \in \mathcal{B}(E)$ and $S \in \mathcal{B}(F)$, the map $T \otimes S$ is bounded, with norm $\|T\|\|S\|$;
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- wap$(L^1(G))$ is the space of coefficient functionals on reflexive spaces;
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Change categories!

Look at Hopf von Neumann algebras and corepresentations.

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Then Γ is co-associative if and only if this product is associative.

Examples

The motivating example is $L^\infty(G)$ with the map

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We can lift the product from $M(G)$ to a co-associative map on $M(G)^*$, turning $M(G)^*$ into a Hopf von Neumann algebra.

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$$(\Gamma \otimes \mathrm{id})W = W_{13}W_{23} \in L^\infty(X \times X)\overline{\otimes}\mathcal{B}(H).$$

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A co-representation of $L^\infty(X)$ on a Hilbert space H is an element $W \in L^\infty(X)\overline{\otimes}\mathcal{B}(H)$ (von Neumann tensor product); with

$$(\Gamma \otimes \mathrm{id})W = W_{13}W_{23} \in L^\infty(X \times X)\overline{\otimes}\mathcal{B}(H).$$

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The von Neumann algebra $L^\infty(X)\overline{\otimes}\mathcal{B}(H)$ has predual

$$L^1(X)\widehat{\otimes}\mathcal{T}(H),$$

the *projective tensor product* of $L^1(X)$ and the trace-class operators on H.

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via the dual pairing

$$\langle T, f\otimes\tau\rangle = \langle T(f),\tau\rangle \quad \begin{pmatrix} T\in\mathcal{B}(L^1(X),\mathcal{B}(H)), \\ f\in L^1(X), \tau\in\mathcal{T}(H)\end{pmatrix}$$

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So $W\in L^\infty(X)\overline{\otimes}\mathcal{B}(H)$ induces $\pi : L^1(X)\to\mathcal{B}(H)$; W is a corepresentation if and only if π is a (Banach algebra) representation.

Tensoring co-representations

Given $\pi_i : L^1(X) \rightarrow \mathcal{B}(H_i)$ representations, the tensored representation

$$\pi = \pi_1 \otimes \pi_2 : L^1(X) \rightarrow \mathcal{B}(H_1 \otimes H_2),$$

is associated to

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where $\omega_{\mu,x} \in \mathcal{T}(H)$ is the normal functional

$$\mathcal{B}(H) \to \mathbb{C}; \quad T \mapsto \langle \mu, T(x) \rangle.$$

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So we need a co-representation theory for reflexive Banach spaces!

Weak*-tensor products

Fix a reflexive space E. We define $L^\infty(X)\overline{\otimes}\mathcal{B}(E)$ to be the weak*-closure of $L^\infty(X) \otimes \mathcal{B}(E)$ inside $\mathcal{B}(L^2(X, E))$.

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Then co-representations all still work, and are compatible with our way of tensoring reflexive spaces.

A result!

Theorem

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The analogous result for $\mathrm{ap}(L^1(X))$ is easy, once you think in terms of Γ (and not just look at $L^1(X)$).

But what is wap($M(G)$)?

For $L^1(G)$, we have that $\text{wap}(L^1(G)) = \text{wap}(G) = C(K)$ where K is some compact semigroup, which we can characterise in terms of G.

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But we only expect *separate* continuity, so we cannot expect something simple, like Γ restricting to a map $C(K) \to C(K \times K)$.

Not clear that co-representations give much insight.

Weakly compact operators

We have that

$$L^\infty(X \times X) = L^\infty(X)\overline{\otimes}L^\infty(X) = \big(L^1(X)\widehat{\otimes}L^1(X)\big)^* = \mathcal{B}(L^1(X), L^\infty(X)).$$

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Again using factorisation results, it is possible to show:

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This immediately implies that $\text{wap}(L^1(X))$ is a subalgebra!

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Theorem

Let $(L^\infty(X), \Gamma)$ *be a commutative Hopf von Neumann algebra. Let* K *be the character space of* $\mathrm{wap}(L^1(X))$. *Then* Γ *naturally induces a semigroup product on* K *turning* K *into a compact semitopological semigroup.*

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But what can we say about K? It would be good to have an abstract characterisation of K in terms of G.

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3. How to tensor two reflexive operator spaces?