Multipliers and the Fourier algebra

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Outline

Multipliers of C*-algebras

Let A be a C*-algebra acting non-degenerately on a Hilbert space H. The *multiplier algebra* of A is

$$M(A) = \{T \in \mathcal{B}(H) : Ta, aT \in A \ (a \in A)\}.$$

- If A is unital, then clearly $M(A) = A$.
- Notice that $A \subseteq M(A)$ as an ideal, and $M(A)$ is always unital.
- $M(A)$ is the largest unital algebra containing A as an *essential* ideal: if $I \subseteq M(A)$ is any ideal, then $A \cap I \neq \{0\}$.
- If $A = C_0(X)$ then $M(A) = C^b(X) = C(\beta X)$, so $M(A)$ is a non-commutative Stone-Čech compactification.

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Centralisers

For an algebra $\mathcal{A}$, let $M(\mathcal{A})$ be the space of *double centralisers*, that is, pairs of linear maps (L, R) of $\mathcal{A} \to \mathcal{A}$ with

$$\begin{cases} L(ab) = L(a)b, \quad R(ab) = aR(b), \\ \qquad aL(b) = R(a)b \end{cases} \qquad (a, b \in \mathcal{A}).$$

We always assume that $\mathcal{A}$ is faithful, meaning that if $a \in \mathcal{A}$ with $bac = 0$ for any $b, c \in \mathcal{A}$, then $a = 0$.

For a C*-algebra, this agrees with the notion of a multiplier.

When $\mathcal{A}$ is a Banach algebra, we naturally ask that L and R are linear and bounded. However…

A Closed Graph argument shows that if (L, R) is a pair of maps $\mathcal{A} \to \mathcal{A}$ with

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Centralisers continued

Then $M(\mathcal{A})$ becomes a Banach algebra for the product and norm

$$(L, R)(L', R') = (LL', R'R), \quad \|(L, R)\| = \max(\|L\|, \|R\|).$$

We can identify $\mathcal{A}$ as a subalgebra of $M(\mathcal{A})$ by

$$a \mapsto (L_a, R_a), \qquad L_a(b) = ab, \ R_a(b) = ba \qquad (a, b \in \mathcal{A}).$$

Then $\mathcal{A}$ is an essential ideal in $M(\mathcal{A})$, and $M(\mathcal{A})$ is the largest algebra with this property.

If $\mathcal{A}$ is a Banach algebra with a bounded approximate identity, then most of what we expect from the C*-world works for $M(\mathcal{A})$.

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Dual Banach algebras and multipliers

A *dual Banach algebra* is a Banach algebra $\mathcal{A}$ which is (isomorphic to) the dual of some Banach space $\mathcal{A}_*$, such that the product on $\mathcal{A}$ is separately weak*-continuous.

- Some motivation is the theory of von Neumann algebras. However…
- The multiplier algebra of a C*-algebra is rarely a dual Banach algebra:

$$M(c_0) = \ell^\infty = (\ell^1)^*, \qquad M(C_0(K)) = C^b(K) \cong C(\beta K).$$

- However, for many algebras arising in abstract harmonic analysis, we do have that $M(\mathcal{A})$ is a dual Banach algebra.

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Locally compact groups

Let G be a locally compact group, equipped with a left invariant Haar measure. Examples include:

- Any *discrete* group with the counting measure.
- Any *compact* group, where the Haar measure is normalised to be a probability measure.
- The real line $\mathbb{R}$ with Lebesgue measure.
- Various non-compact Lie groups give interesting examples.

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Group algebras

Turn $L^1(G)$ into a Banach algebra by using the convolution product:

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)\, dt.$$

We can also convolve finite measures. Identify $M(G)$ with $C_0(G)^*$, then

$$\langle \mu * \lambda, F \rangle = \int \int F(st)\, d\mu(s)\, d\lambda(t) \qquad (\mu, \lambda \in M(G), F \in C_0(G)).$$

Then we have that

$$M(L^1(G)) = M(G),$$

where for each $(L, R) \in M(L^1(G))$, there exists $\mu \in M(G)$,

$$L(a) = \mu * a, \quad R(a) = a * \mu \qquad (a \in L^1(G)).$$

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Representations

Building on work of Young and Kaiser, we have

Theorem (Daws, Uygul)

Let $\mathcal{A}$ be a (completely contractive) dual Banach algebra. Then there exists a **reflexive** *Operator / Banach space E and a (completely) isometric, weak*-weak*-continuous homomorphism $\pi : \mathcal{A} \to \mathcal{B}(E)$.*

If we know more about $\mathcal{A}$ (say, $\mathcal{A} = M(L^1(G)) = M(G)$) can we choose E in a "nice" way?

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An idea of Young

Fix a group G. Let $(p_n) \subseteq (1, \infty)$ be a sequence tending to 1, and let

$$E = \ell^2 - \bigoplus_n L^{p_n}(G).$$

- $L^1(G)$ acts by convolution on each $L^{p_n}(G)$, and hence on E.
- Similarly $M(G)$ acts by convolution on E, extending the action of $L^1(G)$.
- Actually, the homomorphism $\pi : M(G) \to \mathcal{B}(E)$ is an *isometry*, and is weak*-weak* continuous.
- The image of $M(G)$ in $\mathcal{B}(E)$ is the *idealiser* of $\pi(L^1(G))$:

$$\pi(M(G)) = \left\{ T \in \mathcal{B}(E) : \begin{matrix} T\pi(a), \pi(a)T \in \pi(L^1(G)) \\ (a \in L^1(G)) \end{matrix} \right\}.$$

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The Fourier Algebra

For a locally compact group G let λ be the left regular representation

$$\big(\lambda(s)\xi\big)(t) = \xi(s^{-1}t) \qquad (s, t \in G, \xi \in L^2(G)).$$

This induces a homomorphism $\lambda : L^1(G) \to \mathcal{B}(L^2(G))$.

Let $C^*_\lambda(G)$ and $VN(G)$ be the norm and σ-weak closures of $\lambda(L^1(G))$, respectively. So $VN(G) = C^*_\lambda(G)''$.

Let $A(G)$ be the predual of $VN(G)$. As $VN(G)$ is in standard position on $L^2(G)$, for each $\omega \in A(G)$, there exist $\xi, \eta \in L^2(G)$ with

$$\omega = \omega_{\xi,\eta} \qquad \langle x, \omega\rangle = \big(x(\xi)|\eta\big) \qquad (x \in VN(G)).$$

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Fourier Algebra: The product

- As $\{\lambda(s) : s \in G\}$ also generates $VN(G)$, we see that $\{\langle\lambda(s), \omega\rangle : s \in G\}$ determines $\omega \in A(G)$.
- So $\omega \in A(G)$ is identified with a function $G \to \mathbb{C}$.
- This function is actually in $C_0(G)$, so we have a map

$$\Phi : A(G) \to C_0(G).$$

- Then $\Phi(A(G))$ is a (not closed!) subalgebra of $C_0(G)$, and $A(G)$ is a Banach algebra.
- If G is abelian with dual group $\hat{G}$, then $A(G)$ is the image, under the Fourier transform, of $L^1(\hat{G})$.

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Multipliers

So we can form $MA(G)$.

- Either abstractly, or…
- As $A(G)$ is a "nice" subalgebra of $C_0(G)$, we have that

$$MA(G) = \{f \in C^b(G) : fa \in A(G) \ (a \in A(G))\}.$$

- $MA(G) = B(G)$, the Fourier-Stieltjes algebra, if and only if G is amenable [Losert].

As the predual of a von Neumann algebra, $A(G)$ is an operator space. Actually a completely contractive Banach algebra. Hence natural to consider the *completely bounded multipliers*, written $M_{cb}(A(G))$.
[De Canniere, Haagerup]: For $f \in MA(G)$, TFAE:

- $f \in M_{cb}A(G)$;
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$MA(G)$ and $M_{cb}A(G)$ are dual

[De Canniere, Haagerup]: Let Q be the completion of $L^1(G)$ under the norm

$$\|f\|_Q = \sup\left\{ \left| \int f(s)a(s)\,ds \right| : a \in MA(G), \|a\| \leq 1 \right\}.$$

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It is well-known that $L^p(G)$ can be realised as the *complex interpolation* space, of parameter $1/p$, between $L^\infty(G)$ and $L^1(G)$.

I won't explain this in detail but observe that:

- We regard $L^\infty = L^\infty(G)$ and $L^1 = L^1(G)$ as spaces of functions on G, so it makes sense to talk about $L^\infty \cap L^1$ and $L^\infty + L^1$.
- We have inclusions $L^\infty \cap L^1 \subseteq L^p \subseteq L^\infty + L^1$ for $p \in (1, \infty)$;
- (Riesz-Thorin) If $T : L^\infty + L^1 \to L^\infty + L^1$ is linear, and restricts to give maps $L^1 \to L^1$ and $L^\infty \to L^\infty$, then

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Convolution action

For $\mu \in M(G)$, we have a convolution action of μ on $L^1(G)$ and $L^\infty(G)$. Interpolating gives the convolution action on $L^p(G)$.

However, from an abstract point of view, this is actually a little odd:

- $M(G)$ acts entirely naturally on $L^1(G)$ as $M(L^1(G)) = M(G)$.
- $L^\infty(G)$ is the dual space of $L^1(G)$.
- So we have the adjoint action of $M(G)$ on $L^\infty(G)$.
- This *is not* the usual convolution action of $M(G)$ on $L^\infty(G)$.

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Analogous ideas for $A(G)$

So for the Fourier algebra, we might proceed as follows:

- Find some way to embed $A(G)$ and $VN(G)$ into a Hausdorff topological space;
- so we can form $VN(G) \cap A(G)$ and $VN(G) + A(G)$.
- Use the complex interpolation method with parameter $1/p$.
- Find some module action of $MA(G)$ on $VN(G)$ which agrees with the standard action of $MA(G)$ on $A(G)$ in $VN(G) \cap A(G)$.
- Then do the same again at the Operator Space level!

Bizarrely, the last point suggests a novel way to get the module actions.

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Non-commutative L^p spaces

Using the complex interpolation method applied to von Neumann algebras is a well established way to construct *non-commutative L^p* spaces, say $L^p(VN(G))$.

- If G is discrete, then $VN(G)$ admits a finite trace: $\varphi : x \mapsto (x\delta_e|\delta_e)$ for $x \in VN(G)$. Then $L^p(VN(G))$ is the completion of $VN(G)$ under the norm $\|x\|_p = \varphi(|x|^p)^{1/p}$, where $|x| = (x^*x)^{1/2}$.
- In general, $VN(G)$ only admits a weight, which satisfies $\varphi(\lambda(f * g)) = (f * g)(e)$ for, say, $f, g \in C_{00}(G)$.
- If G is compact, then

$$VN(G) \cong \prod_i \mathbb{M}_{n_i}, \qquad L^p(VN(G)) \cong \ell^p - \bigoplus_i S^p_{n_i},$$

where S^p_n is $\mathbb{M}_n$ equipped with the pth Schatten-class norm.

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Operator Space Structures

For further details on the complex interpolation approach to non-commutative L^p spaces, see [Kosaki], [Terp] and [Izumi].

Eventually we want a *natural* Operator Space structure on $L^p(VN(G))$:

- Under favourable circumstances, we except that non-commutative L^2 is a Hilbert space;
- A Hilbert space is *self-dual*;
- The unique Operator Space structure on a Hilbert space with this property is Pisier's Operator Hilbert Space;
- To recover this, we need to interpolate between a von Neumann algebra M and the *opposite* predual M_*^{op}, see [Pisier].
- Here M_*^{op} is the predual of M equipped with the *opposite* structure,

$$\|(\omega_{ij})\|_{M_*^{op}} = \|(\omega_{ji})\|_{M_*}. \qquad \big((\omega_{ij}) \in \mathbb{M}_n(M_*)\big).$$

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For further details on the complex interpolation approach to non-commutative L^p spaces, see [Kosaki], [Terp] and [Izumi]. Eventually we want a *natural* Operator Space structure on $L^p(VN(G))$:

- Under favourable circumstances, we except that non-commutative L^2 is a Hilbert space;
- A Hilbert space is *self-dual*;
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For the Fourier algebra

- As $VN(G)$ is in standard position on $L^2(G)$, we can identify $A(G)^{\mathrm{op}}$ with the predual of the *commutant* $VN(G)'$.
- However, $VN(G)'$ is simply $VN_r(G)$, the *right* group von Neumann algebra, which is generated by the right regular representation.
- So if we privilege $A(G)$, it makes sense to interpolate between $VN_r(G)$ and $A(G)$.
- If we follow Terp's interpolation method through, then in $A(G) \cap VN_r(G)$, we find that

$$a = \rho(\nabla^{-1/2} a) \qquad (a \in A(G) \cap C_{00}(G)^2).$$

Here ρ is the right regular representation, and ∇ is the modular function of G.

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The spaces

- So we interpolate between $VN_r(G)$ and $A(G)$, leading to $L^p(\hat{G})$ say. If G is abelian, this *is* the L^p space of the dual group $\hat{G}$.
- As a Banach space, $L^p(\hat{G})$ is just $L^p(VN(G))$.
- It turns out we can find a (rather natural, in the end) action of $MA(G)$ on $VN_r(G)$ which makes sense on $A(G) \cap VN_r(G)$.
- So we interpolate the module actions, and hence $L^p(\hat{G})$ becomes a (completely contractive) $A(G)$ module. A similar argument establishes that $MA(G)$ and $M_{cb}A(G)$ act on $L^p(\hat{G})$, extending the action of $A(G)$.
- Work of Izumi shows that there is a natural dual pairing between $L^p(\hat{G})$ and $L^{p'}(\hat{G})$, where $p^{-1} + p'^{-1} = 1$.
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The theorem

Let (p_n) be a sequence in $(1, \infty)$ tending to 1. Let

$$E = \ell^2 - \bigoplus_n L^{p_n}(\hat{G}).$$

Let $\pi : MA(G) \to \mathcal{B}(E)$ be the diagonal action.

Theorem

The homomorphism π is an isometric, weak-weak*-continuous isomorphism onto its range, which is equal to the idealiser of $\pi(A(G))$ in $\mathcal{B}(E)$,*

$$\pi(MA(G)) = \left\{ T \in \mathcal{B}(E) : \begin{matrix} T\pi(a), \pi(a)T \in \pi(A(G)) \\ (a \in A(G)) \end{matrix} \right\}.$$

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Notice that E, and the $A(G)$ action, is the same in either case. The idealiser in $\mathcal{B}(E)$ is $MA(G)$, while the idealiser in $\mathcal{CB}(E)$ is $M_{cb}A(G)$.

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Preprints

"Multipliers, Self-Induced and Dual Banach Algebras", arXiv:1001.1633v1 [math.FA]

"Representing multipliers of the Fourier algebra on non-commutative L^p spaces", arXiv:0906.5128v2 [math.FA]