Operator Spaces and the Sz-Nagy Similarity Problem

Matthew Daws

22 November, 2006

Talk Plan

- Contractions on a Hilbert space.
- Models and functional calculus.
- Similarity problem and conjectures.
- Operator spaces.
- A positive result and conclusion.

Contractions on a Hilbert space

Throughout $\mathcal{H}$ will be a Hilbert space. An operator T on $\mathcal{H}$ is a *contraction* if

$$\|T(x)\| \leq \|x\| \qquad (x \in \mathcal{H}).$$

The Sz.-Nagy dilation theorem states that if T is a contraction, then we can find a bigger Hilbert space $\mathcal{H}_0$ with $\mathcal{H} \subseteq \mathcal{H}_0$, and an isometry U on H_0 such that

$$T = P_{\mathcal{H}} U|_{\mathcal{H}},$$

where $P_{\mathcal{H}} : \mathcal{H}_0 \to \mathcal{H}$ is the orthogonal projection, and $U|_{\mathcal{H}}$ is the restriction of U to $\mathcal{H}$.

Contractions on a Hilbert space

Throughout $\mathcal{H}$ will be a Hilbert space. An operator T on $\mathcal{H}$ is a *contraction* if

$$\|T(x)\| \leq \|x\| \qquad (x \in \mathcal{H}).$$

The Sz.-Nagy dilation theorem states that if T is a contraction, then we can find a bigger Hilbert space $\mathcal{H}_0$ with $\mathcal{H} \subseteq \mathcal{H}_0$, and an isometry U on H_0 such that

$$T = P_{\mathcal{H}} U|_{\mathcal{H}},$$

where $P_{\mathcal{H}} : \mathcal{H}_0 \to \mathcal{H}$ is the orthogonal projection, and $U|_{\mathcal{H}}$ is the restriction of U to $\mathcal{H}$.

Dilation Theorem

In fact, we can choose $\mathcal{H}_0$ and U such that

$$T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}, \quad \overline{\text{lin}}\{U^n(\mathcal{H}) : n \in \mathbb{Z}\} = \mathcal{H}_0.$$

Let $\mathcal{H}_1$ be such that $\mathcal{H}_0 = \mathcal{H} \oplus \mathcal{H}_1$, so with respect to this direct sum,

$$U = \begin{pmatrix} T & 0 \\ ? & ? \end{pmatrix}.$$

For many contractions T, we can even choose U to be a suitable generalisation of the bilateral shift.

Dilation Theorem

In fact, we can choose $\mathcal{H}_0$ and U such that

$$T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}, \quad \overline{\text{lin}}\{U^n(\mathcal{H}) : n \in \mathbb{Z}\} = \mathcal{H}_0.$$

Let $\mathcal{H}_1$ be such that $\mathcal{H}_0 = \mathcal{H} \oplus \mathcal{H}_1$, so with respect to this direct sum,

$$U = \begin{pmatrix} T & 0 \\ ? & ? \end{pmatrix}.$$

For many contractions T, we can even choose U to be a suitable generalisation of the bilateral shift.

Dilation Theorem

In fact, we can choose $\mathcal{H}_0$ and U such that

$$T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}, \quad \overline{\text{lin}}\{U^n(\mathcal{H}) : n \in \mathbb{Z}\} = \mathcal{H}_0.$$

Let $\mathcal{H}_1$ be such that $\mathcal{H}_0 = \mathcal{H} \oplus \mathcal{H}_1$, so with respect to this direct sum,

$$U = \begin{pmatrix} T & 0 \\ ? & ? \end{pmatrix}.$$

For many contractions T, we can even choose U to be a suitable generalisation of the bilateral shift.

Functional Calculus: Hardy Spaces

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, and let $L^p(\mathbb{T})$ be the usual Lebesgue space.

For $1 \le p \le \infty$, let $H^p \subseteq L^p(\mathbb{T})$ be the *Hardy Space* of index p, defined as follows. We let $f \in H^p$ if and only if the negative Fourier coefficients of f are zero, that is,

$$\int_0^{2\pi} f(e^{i\theta}) e^{in\theta} \, \frac{d\theta}{2\pi} = 0 \qquad (n < 0).$$

Equivalently, H^p consists of those functions f analytic on the unit disc, and such that

$$\sup_{0<r<1} \left(\int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

Functional Calculus: Hardy Spaces

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, and let $L^p(\mathbb{T})$ be the usual Lebesgue space.
For $1 \leq p \leq \infty$, let $H^p \subseteq L^p(\mathbb{T})$ be the *Hardy Space* of index p, defined as follows. We let $f \in H^p$ if and only if the negative Fourier coefficients of f are zero, that is,

$$\int_0^{2\pi} f(e^{i\theta}) e^{in\theta} \, \frac{d\theta}{2\pi} = 0 \qquad (n < 0).$$

Equivalently, H^p consists of those functions f analytic on the unit disc, and such that

$$\sup_{0<r<1} \left(\int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

Functional Calculus: Hardy Spaces

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, and let $L^p(\mathbb{T})$ be the usual Lebesgue space.
For $1 \leq p \leq \infty$, let $H^p \subseteq L^p(\mathbb{T})$ be the *Hardy Space* of index p, defined as follows. We let $f \in H^p$ if and only if the negative Fourier coefficients of f are zero, that is,

$$\int_0^{2\pi} f(e^{i\theta}) e^{in\theta} \, \frac{d\theta}{2\pi} = 0 \qquad (n < 0).$$

Equivalently, H^p consists of those functions f analytic on the unit disc, and such that

$$\sup_{0<r<1} \left(\int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

A simple-minded definition

Let f be analytic on $\mathbb{D}$, with power-series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in \mathbb{D}).$$

Suppose that $\sum_{n=0}^{\infty} |a_n| < \infty$.
For any contraction T on $\mathcal{H}$, we can define

$$f(T) = \sum_{n=0}^{\infty} a_n T^n,$$

as the sum is absolutely convergent.
Of course, not all analytic functions have such an absolutely convergent power series.

A simple-minded definition

Let f be analytic on $\mathbb{D}$, with power-series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in \mathbb{D}).$$

Suppose that $\sum_{n=0}^{\infty} |a_n| < \infty$.
For any contraction T on $\mathcal{H}$, we can define

$$f(T) = \sum_{n=0}^{\infty} a_n T^n,$$

as the sum is absolutely convergent.
Of course, not all analytic functions have such an absolutely convergent power series.

A simple-minded definition

Let f be analytic on $\mathbb{D}$, with power-series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in \mathbb{D}).$$

Suppose that $\sum_{n=0}^{\infty} |a_n| < \infty$.
For any contraction T on $\mathcal{H}$, we can define

$$f(T) = \sum_{n=0}^{\infty} a_n T^n,$$

as the sum is absolutely convergent.
Of course, not all analytic functions have such an absolutely convergent power series.

The H^∞ functional calculus

In fact, working somewhat harder, we can prove that for each $f \in H^\infty$, we can define a bounded operator $f(T)$ on $\mathcal{H}$.

- The map $H^\infty \to \mathcal{B}(\mathcal{H}); f \mapsto f(T)$ is a norm-decreasing algebra homomorphism;
- For $f \in H^\infty$, define $\tilde{f} \in H^\infty$ by

$$\tilde{f}(z) = \overline{f(\overline{z})} \qquad (z \in \mathbb{D}).$$

  Then $f(T)^* = \tilde{f}(T^*)$.
- If (f_n) is a bounded sequence in H^∞ which converges pointwise to $f \in H^\infty$, then $f_n(T) \to f(T)$ in the strong topology.

The H^∞ functional calculus

In fact, working somewhat harder, we can prove that for each $f \in H^\infty$, we can define a bounded operator $f(T)$ on $\mathcal{H}$.

- The map $H^\infty \to \mathcal{B}(\mathcal{H}); f \mapsto f(T)$ is a norm-decreasing algebra homomorphism;
- For $f \in H^\infty$, define $\tilde{f} \in H^\infty$ by

$$\tilde{f}(z) = \overline{f(\overline{z})} \qquad (z \in \mathbb{D}).$$

  Then $f(T)^* = \tilde{f}(T^*)$.
- If (f_n) is a bounded sequence in H^∞ which converges pointwise to $f \in H^\infty$, then $f_n(T) \to f(T)$ in the strong topology.

The H^∞ functional calculus

In fact, working somewhat harder, we can prove that for each $f \in H^\infty$, we can define a bounded operator $f(T)$ on $\mathcal{H}$.

- The map $H^\infty \to \mathcal{B}(\mathcal{H}); f \mapsto f(T)$ is a norm-decreasing algebra homomorphism;
- For $f \in H^\infty$, define $\tilde{f} \in H^\infty$ by

$$\tilde{f}(z) = \overline{f(\overline{z})} \qquad (z \in \mathbb{D}).$$

  Then $f(T)^* = \tilde{f}(T^*)$.
- If (f_n) is a bounded sequence in H^∞ which converges pointwise to $f \in H^\infty$, then $f_n(T) \to f(T)$ in the strong topology.

The H^∞ functional calculus

In fact, working somewhat harder, we can prove that for each $f \in H^\infty$, we can define a bounded operator $f(T)$ on $\mathcal{H}$.

- The map $H^\infty \to \mathcal{B}(\mathcal{H}); f \mapsto f(T)$ is a norm-decreasing algebra homomorphism;
- For $f \in H^\infty$, define $\tilde{f} \in H^\infty$ by

$$\tilde{f}(z) = \overline{f(\overline{z})} \qquad (z \in \mathbb{D}).$$

  Then $f(T)^* = \tilde{f}(T^*)$.
- If (f_n) is a bounded sequence in H^∞ which converges pointwise to $f \in H^\infty$, then $f_n(T) \to f(T)$ in the strong topology.

Similarity

- We get the impression that contractions are rather nicely behaved objects.
- We define $T \in \mathcal{B}(\mathcal{H})$ to be *similar to a contraction* if there exists an *invertible* map $S \in \mathcal{B}(\mathcal{H})$ such that $S^{-1}TS$ is a contraction.
- All of the previous explained properties can easily be seen to hold for maps similar to a contraction.
- For example, we define a functional calculus by

$$f \mapsto Sf(S^{-1}TS)S^{-1}.$$

- But how can we recognise an operator which is similar to a contraction?

Similarity

- We get the impression that contractions are rather nicely behaved objects.
- We define $T \in \mathcal{B}(\mathcal{H})$ to be *similar to a contraction* if there exists an *invertible* map $S \in \mathcal{B}(\mathcal{H})$ such that $S^{-1}TS$ is a contraction.
- All of the previous explained properties can easily be seen to hold for maps similar to a contraction.
- For example, we define a functional calculus by

$$f \mapsto Sf(S^{-1}TS)S^{-1}.$$

- But how can we recognise an operator which is similar to a contraction?

Similarity

- We get the impression that contractions are rather nicely behaved objects.
- We define $T \in \mathcal{B}(\mathcal{H})$ to be *similar to a contraction* if there exists an *invertible* map $S \in \mathcal{B}(\mathcal{H})$ such that $S^{-1}TS$ is a contraction.
- All of the previous explained properties can easily be seen to hold for maps similar to a contraction.
- For example, we define a functional calculus by

$$f \mapsto Sf(S^{-1}TS)S^{-1}.$$

- But how can we recognise an operator which is similar to a contraction?

Similarity

- We get the impression that contractions are rather nicely behaved objects.
- We define $T \in \mathcal{B}(\mathcal{H})$ to be *similar to a contraction* if there exists an *invertible* map $S \in \mathcal{B}(\mathcal{H})$ such that $S^{-1}TS$ is a contraction.
- All of the previous explained properties can easily be seen to hold for maps similar to a contraction.
- For example, we define a functional calculus by

$$f \mapsto Sf(S^{-1}TS)S^{-1}.$$

- But how can we recognise an operator which is similar to a contraction?

Similarity

- We get the impression that contractions are rather nicely behaved objects.
- We define $T \in \mathcal{B}(\mathcal{H})$ to be *similar to a contraction* if there exists an *invertible* map $S \in \mathcal{B}(\mathcal{H})$ such that $S^{-1}TS$ is a contraction.
- All of the previous explained properties can easily be seen to hold for maps similar to a contraction.
- For example, we define a functional calculus by

$$f \mapsto Sf(S^{-1}TS)S^{-1}.$$

- But how can we recognise an operator which is similar to a contraction?

Sz.-Nagy Conjecture

- If $\|S^{-1}TS\| \leq 1$, then clearly we have that

$$\sup_{n\geq 0} \|T^n\| = \sup_{n\geq 0} \|S(S^{-1}TS)^n S^{-1}\| \leq \|S\|\|S^{-1}\|,$$

so that T is *power-bounded*.

- Sz.-Nagy proved (1959) that if T is compact and power-bounded, then T is similar to a contraction.
- So he conjectured that this was true for general power-bounded operators.

Sz.-Nagy Conjecture

- If $\|S^{-1}TS\| \leq 1$, then clearly we have that

$$\sup_{n\geq 0} \|T^n\| = \sup_{n\geq 0} \|S(S^{-1}TS)^n S^{-1}\| \leq \|S\|\|S^{-1}\|,$$

  so that T is *power-bounded*.
- Sz.-Nagy proved (1959) that if T is compact and power-bounded, then T is similar to a contraction.
- So he conjectured that this was true for general power-bounded operators.

Sz.-Nagy Conjecture

- If $\|S^{-1}TS\| \leq 1$, then clearly we have that

$$\sup_{n\geq 0} \|T^n\| = \sup_{n\geq 0} \|S(S^{-1}TS)^n S^{-1}\| \leq \|S\| \|S^{-1}\|,$$

  so that T is *power-bounded*.
- Sz.-Nagy proved (1959) that if T is compact and power-bounded, then T is similar to a contraction.
- So he conjectured that this was true for general power-bounded operators.

Counter-example

Foguel (1964) found the following counter-example.

- Let ℓ^2 be the usual Hilbert space indexed by $\mathbb{N}$, with standard orthonormal basis $(e_n)_{n\in\mathbb{N}}$.
- Let S be the right shift, $S(e_n) = e_{n+1}$.
- Let Q be the projection onto the lacunary sequence $\{e_{3^k}\}$.
- Foguel's example is $R(Q)$ acting on $\ell^2 \oplus \ell^2$,

$$R(G) = \begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix}.$$

Counter-example

Foguel (1964) found the following counter-example.

- Let ℓ^2 be the usual Hilbert space indexed by $\mathbb{N}$, with standard orthonormal basis $(e_n)_{n \in \mathbb{N}}$.
- Let S be the right shift, $S(e_n) = e_{n+1}$.
- Let Q be the projection onto the lacunary sequence $\{e_{3^k}\}$.
- Foguel's example is $R(Q)$ acting on $\ell^2 \oplus \ell^2$,

$$R(G) = \begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix}.$$

Counter-example

Foguel (1964) found the following counter-example.

- Let ℓ^2 be the usual Hilbert space indexed by $\mathbb{N}$, with standard orthonormal basis $(e_n)_{n\in\mathbb{N}}$.
- Let S be the right shift, $S(e_n) = e_{n+1}$.
- Let Q be the projection onto the lacunary sequence $\{e_{3^k}\}$.
- Foguel's example is $R(Q)$ acting on $\ell^2 \oplus \ell^2$,

$$R(G) = \begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix}.$$

Counter-example

Foguel (1964) found the following counter-example.

- Let ℓ^2 be the usual Hilbert space indexed by $\mathbb{N}$, with standard orthonormal basis $(e_n)_{n\in\mathbb{N}}$.
- Let S be the right shift, $S(e_n) = e_{n+1}$.
- Let Q be the projection onto the lacunary sequence $\{e_{3^k}\}$.
- Foguel's example is $R(Q)$ acting on $\ell^2 \oplus \ell^2$,

$$R(G) = \begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix}.$$

von Neumann inequality

- Let p be a polynomial, so $p \in H^\infty$, and hence by the H^∞-calculus, for a contraction R, $\|p(R)\| \leq \|p\|_\infty$. Hence, if $S^{-1}TS$ is a contraction,

$$\|p(T)\| = \|Sp(S^{-1}TS)S^{-1}\| \leq \|S\|\|S^{-1}\|\|p\|_\infty.$$

- Actually, there is a more elementary proof of this, due to von Neumann.
- So we have a new conjecture: T is similar to a contraction if and only if, for some constant K, we have

$$\|p(T)\| \leq K\|p\|_\infty \qquad (p \text{ a polynomial}).$$

  That is, T is *polynomially bounded*.
- Lebow (1968) showed that Foguel's example is *not* a counter-example to this new conjecture.

von Neumann inequality

- Let p be a polynomial, so $p \in H^\infty$, and hence by the H^∞-calculus, for a contraction R, $\|p(R)\| \leq \|p\|_\infty$. Hence, if $S^{-1}TS$ is a contraction,

$$\|p(T)\| = \|Sp(S^{-1}TS)S^{-1}\| \leq \|S\|\|S^{-1}\|\|p\|_\infty.$$

- Actually, there is a more elementary proof of this, due to von Neumann.
- So we have a new conjecture: T is similar to a contraction if and only if, for some constant K, we have

$$\|p(T)\| \leq K\|p\|_\infty \qquad (p \text{ a polynomial}).$$

  That is, T is *polynomially bounded*.
- Lebow (1968) showed that Foguel's example is *not* a counter-example to this new conjecture.

von Neumann inequality

- Let p be a polynomial, so $p \in H^\infty$, and hence by the H^∞-calculus, for a contraction R, $\|p(R)\| \leq \|p\|_\infty$. Hence, if $S^{-1}TS$ is a contraction,

$$\|p(T)\| = \|Sp(S^{-1}TS)S^{-1}\| \leq \|S\|\|S^{-1}\|\|p\|_\infty.$$

- Actually, there is a more elementary proof of this, due to von Neumann.
- So we have a new conjecture: T is similar to a contraction if and only if, for some constant K, we have

$$\|p(T)\| \leq K\|p\|_\infty \qquad (p \text{ a polynomial}).$$

  That is, T is *polynomially bounded*.
- Lebow (1968) showed that Foguel's example is *not* a counter-example to this new conjecture.

von Neumann inequality

- Let p be a polynomial, so $p \in H^\infty$, and hence by the H^∞-calculus, for a contraction R, $\|p(R)\| \le \|p\|_\infty$. Hence, if $S^{-1}TS$ is a contraction,

$$\|p(T)\| = \|Sp(S^{-1}TS)S^{-1}\| \le \|S\|\|S^{-1}\|\|p\|_\infty.$$

- Actually, there is a more elementary proof of this, due to von Neumann.
- So we have a new conjecture: T is similar to a contraction if and only if, for some constant K, we have

$$\|p(T)\| \le K\|p\|_\infty \qquad (p \text{ a polynomial}).$$

  That is, T is *polynomially bounded*.
- Lebow (1968) showed that Foguel's example is *not* a counter-example to this new conjecture.

Pisier's Counter-example

- Similar counter-examples to Foguel's have been considered, with much more complicated operators Q,

$$R(G) = \begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix}.$$

However, work of Bourgain, Aleksandrov and Peller has shown that this approach is fairly hopeless.

- Pisier instead uses amplifications (Blackboard). He then found a counter-example of the form

$$R(\Gamma_F) = \begin{pmatrix} S^{*(\infty)} & \Gamma_F \\ 0 & S^{(\infty)} \end{pmatrix}.$$

Here Γ_F is an "operator-valued Hankel operator".

Pisier's Counter-example

- Similar counter-examples to Foguel's have been considered, with much more complicated operators Q,

$$R(G) = \begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix}.$$

  However, work of Bourgain, Aleksandrov and Peller has shown that this approach is fairly hopeless.

- Pisier instead uses amplifications (Blackboard). He then found a counter-example of the form

$$R(\Gamma_F) = \begin{pmatrix} S^{*(\infty)} & \Gamma_F \\ 0 & S^{(\infty)} \end{pmatrix}.$$

  Here Γ_F is an "operator-valued Hankel operator".

Operator spaces

- An *operator space* is just a closed subspace of $\mathcal{B}(\mathcal{H})$.
- Obviously, thanks to the GNS construction, we can replace $\mathcal{B}(\mathcal{H})$ be any C*-algebra $\mathcal{A}$.
- So every Banach space is an operator space!
- The difference, however, is the maps which we consider. We replace *bounded* maps by *completely bounded* maps.

Operator spaces

- An *operator space* is just a closed subspace of $\mathcal{B}(\mathcal{H})$.
- Obviously, thanks to the GNS construction, we can replace $\mathcal{B}(\mathcal{H})$ be any C*-algebra $\mathcal{A}$.
- So every Banach space is an operator space!
- The difference, however, is the maps which we consider. We replace *bounded* maps by *completely bounded* maps.

Operator spaces

- An *operator space* is just a closed subspace of $\mathcal{B}(\mathcal{H})$.
- Obviously, thanks to the GNS construction, we can replace $\mathcal{B}(\mathcal{H})$ be any C*-algebra $\mathcal{A}$.
- So every Banach space is an operator space!
- The difference, however, is the maps which we consider. We replace *bounded* maps by *completely bounded* maps.

Operator spaces

- An *operator space* is just a closed subspace of $\mathcal{B}(\mathcal{H})$.
- Obviously, thanks to the GNS construction, we can replace $\mathcal{B}(\mathcal{H})$ be any C*-algebra $\mathcal{A}$.
- So every Banach space is an operator space!
- The difference, however, is the maps which we consider. We replace *bounded* maps by *completely bounded* maps.

Completely bounded maps

Let $E \subseteq \mathcal{B}(\mathcal{H})$. Write $\mathbb{M}_n(E)$ for the set of $n \times n$ matricies with entries in E.

We have the identification

$$\mathbb{M}_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H}).$$

Which induces a norm on $\mathbb{M}_n(\mathcal{B}(\mathcal{H}))$.
As $\mathbb{M}_n(E) \subseteq \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$, we get a norm on $\mathbb{M}_n(E)$.

For $T \in \mathcal{B}(E)$, we let $(T)_n \in \mathcal{B}(\mathbb{M}_n(E))$ be defined by

$$(T)_n : \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \mapsto \begin{pmatrix} T(x_{11}) & \cdots & T(x_{1n}) \\ \vdots & \ddots & \vdots \\ T(x_{n1}) & \cdots & T(x_{nn}) \end{pmatrix}$$

Then T is *completely bounded* if and only if

$$\|T\|_{cb} := \sup_{n \geq 1} \|(T)_n\| < \infty.$$

Completely bounded maps

Let $E \subseteq \mathcal{B}(\mathcal{H})$. Write $\mathbb{M}_n(E)$ for the set of $n \times n$ matricies with entries in E.
We have the identification

$$\mathbb{M}_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H}).$$

Which induces a norm on $\mathbb{M}_n(\mathcal{B}(\mathcal{H}))$.
As $\mathbb{M}_n(E) \subseteq \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$, we get a norm on $\mathbb{M}_n(E)$.
For $T \in \mathcal{B}(E)$, we let $(T)_n \in \mathcal{B}(\mathbb{M}_n(E))$ be defined by

$$(T)_n : \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \mapsto \begin{pmatrix} T(x_{11}) & \cdots & T(x_{1n}) \\ \vdots & \ddots & \vdots \\ T(x_{n1}) & \cdots & T(x_{nn}) \end{pmatrix}$$

Then T is *completely bounded* if and only if

$$\|T\|_{cb} := \sup_{n \geq 1} \|(T)_n\| < \infty.$$

Completely bounded maps

Let $E \subseteq \mathcal{B}(\mathcal{H})$. Write $\mathbb{M}_n(E)$ for the set of $n \times n$ matricies with entries in E.

We have the identification

$$\mathbb{M}_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H}).$$

Which induces a norm on $\mathbb{M}_n(\mathcal{B}(\mathcal{H}))$.

As $\mathbb{M}_n(E) \subseteq \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$, we get a norm on $\mathbb{M}_n(E)$.

For $T \in \mathcal{B}(E)$, we let $(T)_n \in \mathcal{B}(\mathbb{M}_n(E))$ be defined by

$$(T)_n : \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \mapsto \begin{pmatrix} T(x_{11}) & \cdots & T(x_{1n}) \\ \vdots & \ddots & \vdots \\ T(x_{n1}) & \cdots & T(x_{nn}) \end{pmatrix}$$

Then T is *completely bounded* if and only if

$$\|T\|_{cb} := \sup_{n \geq 1} \|(T)_n\| < \infty.$$

Completely bounded maps

Let $E \subseteq \mathcal{B}(\mathcal{H})$. Write $\mathbb{M}_n(E)$ for the set of $n \times n$ matricies with entries in E.

We have the identification

$$\mathbb{M}_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H}).$$

Which induces a norm on $\mathbb{M}_n(\mathcal{B}(\mathcal{H}))$.

As $\mathbb{M}_n(E) \subseteq \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$, we get a norm on $\mathbb{M}_n(E)$.

For $T \in \mathcal{B}(E)$, we let $(T)_n \in \mathcal{B}(\mathbb{M}_n(E))$ be defined by

$$(T)_n : \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \mapsto \begin{pmatrix} T(x_{11}) & \cdots & T(x_{1n}) \\ \vdots & \ddots & \vdots \\ T(x_{n1}) & \cdots & T(x_{nn}) \end{pmatrix}$$

Then T is *completely bounded* if and only if

$$\|T\|_{cb} := \sup_{n \geq 1} \|(T)_n\| < \infty.$$

Completely bounded maps

Let $E \subseteq \mathcal{B}(\mathcal{H})$. Write $\mathbb{M}_n(E)$ for the set of $n \times n$ matricies with entries in E.

We have the identification

$$\mathbb{M}_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H}).$$

Which induces a norm on $\mathbb{M}_n(\mathcal{B}(\mathcal{H}))$.

As $\mathbb{M}_n(E) \subseteq \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$, we get a norm on $\mathbb{M}_n(E)$.

For $T \in \mathcal{B}(E)$, we let $(T)_n \in \mathcal{B}(\mathbb{M}_n(E))$ be defined by

$$(T)_n : \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \mapsto \begin{pmatrix} T(x_{11}) & \cdots & T(x_{1n}) \\ \vdots & \ddots & \vdots \\ T(x_{n1}) & \cdots & T(x_{nn}) \end{pmatrix}$$

Then T is *completely bounded* if and only if

$$\|T\|_{cb} := \sup_{n \geq 1} \|(T)_n\| < \infty.$$

Example

- Let $\mathcal{H} = \mathbb{C}^2$ be a two-dimensional Hilbert space, so we can idenfity $\mathcal{B}(\mathcal{H})$ with $\mathbb{M}_2$.
- Let $T \in \mathcal{B}(\mathbb{M}_2)$ be transposition:

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

- For example, we identify $\mathbb{M}_2(\mathbb{M}_2)$ with $\mathbb{M}_4$, and then we have

$$(T)_2\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{21} & x_{13} & x_{23} \\ x_{12} & x_{22} & x_{14} & x_{24} \\ x_{31} & x_{41} & x_{33} & x_{43} \\ x_{32} & x_{42} & x_{34} & x_{44} \end{pmatrix}$$

Example

- Let $\mathcal{H} = \mathbb{C}^2$ be a two-dimensional Hilbert space, so we can idenfity $\mathcal{B}(\mathcal{H})$ with $\mathbb{M}_2$.
- Let $T \in \mathcal{B}(\mathbb{M}_2)$ be transposition:

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

- For example, we identify $\mathbb{M}_2(\mathbb{M}_2)$ with $\mathbb{M}_4$, and then we have

$$(T)_2 \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{21} & x_{13} & x_{23} \\ x_{12} & x_{22} & x_{14} & x_{24} \\ x_{31} & x_{41} & x_{33} & x_{43} \\ x_{32} & x_{42} & x_{34} & x_{44} \end{pmatrix}$$

Example

- Let $\mathcal{H} = \mathbb{C}^2$ be a two-dimensional Hilbert space, so we can idenfity $\mathcal{B}(\mathcal{H})$ with $\mathbb{M}_2$.
- Let $T \in \mathcal{B}(\mathbb{M}_2)$ be transposition:

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

- For example, we identify $\mathbb{M}_2(\mathbb{M}_2)$ with $\mathbb{M}_4$, and then we have

$$(T)_2 \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{21} & x_{13} & x_{23} \\ x_{12} & x_{22} & x_{14} & x_{24} \\ x_{31} & x_{41} & x_{33} & x_{43} \\ x_{32} & x_{42} & x_{34} & x_{44} \end{pmatrix}$$

Example continued

- So we see that

$$(T)_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

- Call the left matrix A and the right on B. Then A is just a permutation operator, so that $\|A\| = 1$, while

$$B(2^{-1/2}, 0, 0, 2^{-1/2}) = (2^{1/2}, 0, 0, 2^{1/2}),$$

  so that $\|B\| \geq 2$.
- This example can be extended to construct operators which are bounded, but not completely bounded.

Example continued

- So we see that

$$(T)_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

- Call the left matrix A and the right on B. Then A is just a permutation operator, so that $\|A\| = 1$, while

$$B(2^{-1/2}, 0, 0, 2^{-1/2}) = (2^{1/2}, 0, 0, 2^{1/2}),$$

  so that $\|B\| \geq 2$.

- This example can be extended to construct operators which are bounded, but not completely bounded.

Example continued

- So we see that

$$(T)_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

- Call the left matrix A and the right on B. Then A is just a permutation operator, so that $\|A\| = 1$, while

$$B(2^{-1/2}, 0, 0, 2^{-1/2}) = (2^{1/2}, 0, 0, 2^{1/2}),$$

  so that $\|B\| \geq 2$.

- This example can be extended to construct operators which are bounded, but not completely bounded.

The Disc Algebra

- The *disc algebra* $A(\mathbb{D})$ is the closure of the space of polynomials in $C(\overline{\mathbb{D}})$.
- Alternatively, $A(\mathbb{D})$ is the space of functions $f : \mathbb{D} \to \mathbb{C}$ which are analytic and have a continuous extension to $\mathbb{T}$.
- We turn $A(\mathbb{D})$ into an operator space by embedding $A(\mathbb{D})$ into the C*-algebra $C(\overline{\mathbb{D}})$.
- Let T be a *polynomially bounded* operator. Then, by continuity, $f(T)$ is defined for each $f \in A(\mathbb{D})$, and $\|f(T)\| \leq C\|f\|_\infty$.

The Disc Algebra

- The *disc algebra* $A(\mathbb{D})$ is the closure of the space of polynomials in $C(\overline{\mathbb{D}})$.
- Alternatively, $A(\mathbb{D})$ is the space of functions $f : \mathbb{D} \to \mathbb{C}$ which are analytic and have a continuous extension to $\mathbb{T}$.
- We turn $A(\mathbb{D})$ into an operator space by embedding $A(\mathbb{D})$ into the C*-algebra $C(\overline{\mathbb{D}})$.
- Let T be a *polynomially bounded* operator. Then, by continuity, $f(T)$ is defined for each $f \in A(\mathbb{D})$, and $\|f(T)\| \leq C\|f\|_\infty$.

The Disc Algebra

- The *disc algebra* $A(\mathbb{D})$ is the closure of the space of polynomials in $C(\overline{\mathbb{D}})$.
- Alternatively, $A(\mathbb{D})$ is the space of functions $f : \mathbb{D} \to \mathbb{C}$ which are analytic and have a continuous extension to $\mathbb{T}$.
- We turn $A(\mathbb{D})$ into an operator space by embedding $A(\mathbb{D})$ into the C*-algebra $C(\overline{\mathbb{D}})$.
- Let T be a *polynomially bounded* operator. Then, by continuity, $f(T)$ is defined for each $f \in A(\mathbb{D})$, and $\|f(T)\| \leq C\|f\|_\infty$.

The Disc Algebra

- The *disc algebra* $A(\mathbb{D})$ is the closure of the space of polynomials in $C(\overline{\mathbb{D}})$.
- Alternatively, $A(\mathbb{D})$ is the space of functions $f : \mathbb{D} \to \mathbb{C}$ which are analytic and have a continuous extension to $\mathbb{T}$.
- We turn $A(\mathbb{D})$ into an operator space by embedding $A(\mathbb{D})$ into the C*-algebra $C(\overline{\mathbb{D}})$.
- Let T be a *polynomially bounded* operator. Then, by continuity, $f(T)$ is defined for each $f \in A(\mathbb{D})$, and $\|f(T)\| \leq C\|f\|_\infty$.

Paulsen's characterisation

Paulsen (1984) proved that the following are equivalent for $T \in \mathcal{B}(\mathcal{H})$ for $C \geq 0$

- There exists an invertible $S \in \mathcal{B}(\mathcal{H})$ with $\|S\|\|S^{-1}\| \leq C$ and $\|S^{-1}TS\| \leq 1$;
- T is polynomially bounded, so there is a bounded map $u_T : A(\mathbb{D}) \to \mathcal{B}(\mathcal{H}); f \mapsto f(T)$, and furthermore, $\|u_T\|_{cb} \leq C$;
- For each $n \geq 1$, and each $n \times n$ matrix with polynomial entries $(p_{ij})_{1\leq i,j\leq n}$, we have that

$$\big\| [p_{ij}(T)] \big\|_{\mathbb{M}_n(\mathcal{B}(\mathcal{H}))} \leq C \sup_{|z|<1} \big\| [p_{ij}(z)] \big\|_{\mathbb{M}_n},$$

  where we give $\mathbb{M}_n$ the norm coming from $\mathbb{M}_n$ acting on n-dimensional Hilbert space.

Paulsen's characterisation

Paulsen (1984) proved that the following are equivalent for $T \in \mathcal{B}(\mathcal{H})$ for $C \geq 0$

- There exists an invertible $S \in \mathcal{B}(\mathcal{H})$ with $\|S\|\|S^{-1}\| \leq C$ and $\|S^{-1}TS\| \leq 1$;
- T is polynomially bounded, so there is a bounded map $u_T : A(\mathbb{D}) \to \mathcal{B}(\mathcal{H}); f \mapsto f(T)$, and furthermore, $\|u_T\|_{cb} \leq C$;
- For each $n \geq 1$, and each $n \times n$ matrix with polynomial entries $(p_{ij})_{1 \leq i,j \leq n}$, we have that

$$\big\| [p_{ij}(T)] \big\|_{\mathbb{M}_n(\mathcal{B}(\mathcal{H}))} \leq C \sup_{|z|<1} \big\| [p_{ij}(z)] \big\|_{\mathbb{M}_n},$$

  where we give $\mathbb{M}_n$ the norm coming from $\mathbb{M}_n$ acting on n-dimensional Hilbert space.

Paulsen's characterisation

Paulsen (1984) proved that the following are equivalent for $T \in \mathcal{B}(\mathcal{H})$ for $C \geq 0$

- There exists an invertible $S \in \mathcal{B}(\mathcal{H})$ with $\|S\|\|S^{-1}\| \leq C$ and $\|S^{-1}TS\| \leq 1$;
- T is polynomially bounded, so there is a bounded map $u_T : A(\mathbb{D}) \to \mathcal{B}(\mathcal{H}); f \mapsto f(T)$, and furthermore, $\|u_T\|_{cb} \leq C$;
- For each $n \geq 1$, and each $n \times n$ matrix with polynomial entries $(p_{ij})_{1 \leq i,j \leq n}$, we have that

$$\left\| \left[p_{ij}(T) \right] \right\|_{\mathbb{M}_n(\mathcal{B}(\mathcal{H}))} \leq C \sup_{|z|<1} \left\| \left[p_{ij}(z) \right] \right\|_{\mathbb{M}_n},$$

  where we give $\mathbb{M}_n$ the norm coming from $\mathbb{M}_n$ acting on n-dimensional Hilbert space.

Paulsen's characterisation

Paulsen (1984) proved that the following are equivalent for $T \in \mathcal{B}(\mathcal{H})$ for $C \geq 0$

- There exists an invertible $S \in \mathcal{B}(\mathcal{H})$ with $\|S\|\|S^{-1}\| \leq C$ and $\|S^{-1}TS\| \leq 1$;
- T is polynomially bounded, so there is a bounded map $u_T : A(\mathbb{D}) \to \mathcal{B}(\mathcal{H}); f \mapsto f(T)$, and furthermore, $\|u_T\|_{cb} \leq C$;
- For each $n \geq 1$, and each $n \times n$ matrix with polynomial entries $(p_{ij})_{1 \leq i,j \leq n}$, we have that

$$\left\| \left[p_{ij}(T) \right] \right\|_{\mathbb{M}_n(\mathcal{B}(\mathcal{H}))} \leq C \sup_{|z|<1} \left\| \left[p_{ij}(z) \right] \right\|_{\mathbb{M}_n},$$

  where we give $\mathbb{M}_n$ the norm coming from $\mathbb{M}_n$ acting on n-dimensional Hilbert space.