Non-commutative graphs

Matthew Daws

Lancaster

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Channels

A channel sends an input message (element of a finite set A) to an output message (element of a finite set B) perhaps with *noise* so that there is a probability that $a \in A$ is mapped to various different $b \in B$.

$$p(b|a) = \text{probability that } b \text{ is received given that } a \text{ was sent}$$

Define a (simple, undirected) graph structure on A by

$$(a_1, a_2) \text{ an edge when } p(b|a_1)p(b|a_2) > 0 \text{ for some } b.$$

This is the *confusability graph* of the channel.
If we want to communicate with *zero error* then we seek a maximal *independent set* in A.

Physics notation

I will follow physics notation, so inner products $(\cdot|\cdot)$ are linear on the right.

- Use bra-ket notation: $|\psi\rangle$ is a vector in a Hilbert space H, and $\langle\psi|$ is a member of the dual space, identified with the conjugate $\overline{H}$.
- Then $\langle\psi|\phi\rangle = (\psi|\phi)$ the inner-product...
- and $|\phi\rangle\langle\psi|$ is the rank-one operator $H \to H;\ \alpha \mapsto (\psi|\alpha)\phi$.
- Given an operator T on H we tend to write $\langle\psi|T|\phi\rangle$ which means $(\psi|T(\phi)) = (T^*(\psi)|\phi)$.

Give an index set I, consider $\ell^2(I)$; the canonical orthonormal basis is often denoted by $(e_i)_{i\in I}$ or $(\delta_i)_{i\in I}$. We abuse notation and write $|i\rangle$ for these basis vectors, so that $\langle i|j\rangle = \delta_{i,j}$.

Quantum Mechanics

Definition

A *state* is a unit vector $|\psi\rangle$ in a (finite dim) Hilbert space H.

Multiplying a state by a unit modulus complex number doesn't change the physics. One way to deal with this is to identify a state with the rank-one projection $|\psi\rangle\langle\psi|$.

Definition

A *density* is a positive, trace one operator $\rho \in \mathcal{B}(H)$.

- So a rank-one density is a state; we call a general density a *mixed* state.
- Mathematically, using trace-duality, a density is nothing but a (normal) state on the C^*-algebra $\mathcal{B}(H)$.

Quantum channels

Definition

A *(quantum) channel* is a trace-preserving, completely positive (CPTP) map $\mathcal{B}(H_A) \to \mathcal{B}(H_B)$.

- positive and trace-preserving so it maps densities to densities;
- completely positive so you can tensor with another system and still have positivity.

Theorem (Stinespring)

A linear map $\theta : A \to \mathcal{B}(H)$, from a C^*-algebra A, is completely positive if and only if it admits a dilation of the form

$$\theta(a) = V^* \pi(a) V \qquad (a \in A)$$

for $\pi : A \to \mathcal{B}(K)$ a $*$-homomorphism, and $V : H \to K$ a bounded linear map.

Stinespring and Kraus

Any CP map $\mathcal{E} : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ has the form

$$\mathcal{E}(x) = V^* \pi(x) V \qquad (x \in \mathcal{B}(H_A)),$$

where $V : H_B \to K$, and $\pi : \mathcal{B}(H_A) \to \mathcal{B}(K)$ is a $*$-representation.

- Any such π is of the form $\pi(x) = x \otimes 1$ where $K \cong H_A \otimes K'$.
- Take an o.n. basis (e_i) for K' so $V(\xi) = \sum_i K_i^*(\xi) \otimes e_i$ for some operators $K_i : H_A \to H_B$.

We arrive at the *Kraus form*:

$$\mathcal{E}(x) = \sum_i K_i x K_i^* \qquad (x \in \mathcal{B}(H_A)).$$

Trace-preserving if and only if $\sum_i K_i^* K_i = 1$.

Quantum zero-error

We turn $\mathcal{B}(H)$ into a Hilbert space using the trace: $(T|S) = \mathrm{tr}(T^*S)$. A sensible notion of when densities ρ, σ are *distinguishable* is when they are orthogonal.

Let $\mathcal{E}(x) = \sum_i K_i x K_i^*$ be a quantum channel. We wish to consider when $\mathcal{E}(\rho) \perp \mathcal{E}(\sigma)$. As $\mathcal{E}$ is positive, this is equivalent to

$$\mathcal{E}(|\psi\rangle\langle\psi|) \perp \mathcal{E}(|\phi\rangle\langle\phi|) \qquad (\psi \in \mathrm{Image}\,\rho, \phi \in \mathrm{Image}\,\sigma).$$

Equivalently

$$\begin{aligned} 0 = \mathrm{tr}\left(\mathcal{E}(|\psi\rangle\langle\psi|)\mathcal{E}(|\phi\rangle\langle\phi|)\right) &= \sum_{i,j} \mathrm{tr}\left(K_i|\psi\rangle\langle\psi|K_i^* K_j|\phi\rangle\langle\phi|K_j^*\right) \\ &= \sum_{i,j} |\langle\psi|K_i^* K_j|\phi\rangle|^2 \end{aligned}$$

which is equivalent to $\langle\psi|K_i^* K_j|\phi\rangle = 0$ for each i, j.

To operator systems

So ψ, ϕ are distinguishable after applying $\mathcal{E}$ when

$$\langle \psi | T | \phi \rangle = 0 \quad \text{for each} \quad T \in \operatorname{lin}\{K_i^* K_j\}.$$

Set $\mathcal{S} = \operatorname{lin}\{K_i^* K_j\}$ which has the properties:

- $\mathcal{S}$ is a linear subspace;
- $T \in \mathcal{S}$ if and only if $T^* \in \mathcal{S}$;
- $1 \in \mathcal{S}$ (as $\sum_i K_i^* K_i = 1$ as $\mathcal{E}$ is CPTP).

That is, $\mathcal{S}$ is an *operator system*, which depends only on $\mathcal{E}$ and not the choice of (K_i).

Theorem (Duan)

For any operator system $\mathcal{S} \subseteq \mathcal{B}(H_A)$ there is some quantum channel $\mathcal{E} : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ giving rise to $\mathcal{S}$.

In the classical case

Given a classical channel from A to B with probabilities $p(b|a)$, we encode this as follows:

- Let $H_A = \ell^2(A)$ with o.n. basis $\{|a\rangle : a \in A\}$; and the same for B.
- Define Kraus operators

$$K_{ab} = p(b|a)^{1/2}|b\rangle\langle a| : H_A \to H_B.$$

Then $\mathcal{E} : \rho \mapsto \sum_{a,b} K_{ab}\rho K_{ab}^*$ sends a pure state $|c\rangle\langle c|$ to

$$\sum_{ab} K_{ab}|c\rangle\langle c|K_{ab}^* = \sum_{ab} p(b|a)|b\rangle\langle a|c\rangle\langle c|a\rangle\langle b| = \sum_{b} p(b|c)|b\rangle\langle b|.$$

That is, the combination of pure states which can be received, given that message c was sent.

- We could consider the subalgebra of diagonal matrices in $\mathcal{B}(H_A) \cong \mathbb{M}_A$ which is $\ell^\infty(A)$.
- A CPTP map is "really" acting on the predual, so we obtain a map $\ell^1(A) \to \ell^1(B)$ which maps the point mass at c to $\sum_b p(b|c)\delta_b$.
- That is, a (left) stochastic matrix.

The associated operator system

The Kraus operators are

$$K_{ab} = p(b|a)^{1/2}|b\rangle\langle a| : H_A \to H_B.$$

Hence

$$\begin{aligned}\mathcal{S} &= \mathrm{lin}\{K_{ab}^* K_{cd}\} = \mathrm{lin}\{p(b|a)^{1/2}p(d|c)^{1/2}|a\rangle\langle b|d\rangle\langle c|\} \\ &= \mathrm{lin}\{p(b|a)^{1/2}p(b|c)^{1/2}|a\rangle\langle c|\} \\ &= \mathrm{lin}\{|a\rangle\langle c| : a \sim c\},\end{aligned}$$

where $a \sim c$ exactly when $p(b|a)p(b|c) > 0$ for some b.
Thus $\mathcal{S}$ is directly linked to the confusability graph of the channel: it is the span of the matrix units e_{ac} for each edge (a, c) in the graph.
(Notice here our "graphs" and finite, simple, but we allow (single, unoriented) loops at vertices.)

Quantum relations

Simultaneously, and motivated more by "noncommutative geometry":

Definition (Weaver)

Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. A *quantum relation* on M is a weak*-closed subspace $S \subseteq \mathcal{B}(H)$ with $M'SM' \subseteq S$. We say that the relation is:

1. *reflexive* if $M' \subseteq S$;
2. *symmetric* if $S^* = S$ where $S^* = \{x^* : x \in S\}$;
3. *transitive* if $S^2 \subseteq S$ where $S^2 = \overline{\mathrm{lin}}^{w^*}\{xy : x, y \in S\}$.

When $M = \ell^\infty(X) \subseteq \mathcal{B}(\ell^2(X))$ there is a bijection between the usual meaning of "relation" on X and quantum relations on M, given by

$$x \sim y \text{ when } e_{x,y} \in S, \qquad S = \overline{\mathrm{lin}}^{w^*}\{e_{x,y} : x \sim y\}.$$

Operator bimodules

The condition that $M'SM' \subseteq S$ means that S is an *operator bimodule* over M'.
(Not to be confused with Hilbert C^*-modules!)

- We assume $M \subseteq \mathcal{B}(H)$ and $S \subseteq \mathcal{B}(H)$.
- If $M \subseteq \mathcal{B}(K)$ as well, we of course want a $T \subseteq \mathcal{B}(K)$ corresponding to S.
- This can be found by using the structure theory for normal $*$-homomorphisms $\theta : M \to \mathcal{B}(K)$. Essentially θ is a dilation followed by a cut-down in the commutant.
- That S is a bimodule over M' is needed to get this correspondence with T.

So this notion is really independent of the choice of embedding $M \subseteq \mathcal{B}(H)$. [Weaver] gives an intrinsic notion just using M.

Quantum graphs

As a graph on a (finite) vertex set V is simply a relation, and as:

- undirected graphs corresponds to symmetric relations;
- a reflexive relation corresponds to having a "loop" at every vertex.

Definition (Weaver)

A *quantum graph* on a von Neumann algebra $M \subseteq \mathcal{B}(H)$ is a reflexive, symmetric quantum relation. That is, a unital, self-adjoint, weak*-closed subspace $S \subseteq \mathcal{B}(H)$, which is an M'-bimodule ($M'SM' \subseteq S$).

If $M = \mathcal{B}(H)$ with H finite-dimensional, then as $M' = \mathbb{C}$, a quantum graph is just an operator system: that is, exactly what we had before! [Duan, Severini, Winter; Stahlke]

Adjacency matrices

Given a graph $G = (V, E)$ consider the $\{0,1\}$-valued matrix A with

$$A_{i,j} = \begin{cases} 1 & : (i,j) \in E, \\ 0 & : \text{otherwise}, \end{cases}$$

the *adjacency matrix* of G.

- A is idempotent for the *Schur product*;
- G is undirected if and only if A is self-adjoint;
- A has 1s down the diagonal when G has a loop at every vertex.

We can think of A as an operator on $\ell^2(V)$. This is the GNS space for the C^*-algebra $\ell^\infty(V)$ for the state induced by the uniform measure.

General C^*-algebras

Let B be a finite-dimensional C^*-algebra, and let φ be a faithful state on B, with GNS space $L^2(B)$. Thus B bijects with $L^2(B)$ as a vector space, and so we get:

- The multiplication on B induces a map $m : L^2(B) \otimes L^2(B) \to L^2(B)$;
- The unit in B induces a map $\eta : \mathbb{C} \to L^2(B)$.

We get an analogue of the Schur product:

$$x \bullet y = m(x \otimes y)m^* \qquad (x, y \in \mathcal{B}(L^2(B))).$$

Quantum adjacency matrix

Definition (Many authors)

A *quantum adjacency matrix* is a self-adjoint $A \in \mathcal{B}(L^2(B))$ with:

- $m(A \otimes A)m^* = A$ (so Schur product idempotent);
- $(1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^* \eta \otimes 1) = A$;
- $m(A \otimes 1)m^* = \mathrm{id}$ (a "loop at every vertex");

The middle axiom is a little mysterious: it roughly corresponds to "undirected".

I want to sketch why this definition is equivalent to the previous notion of a "quantum graph".

Subspaces to projections

Fix a finite-dimensional C^*-algebra (von Neumann algebra) M. A "quantum graph" is either:

- A subspace of $\mathcal{B}(H)$ (where $M \subseteq \mathcal{B}(H)$) with some properties; or
- An operator on $L^2(M)$ with some properties.

How do we move between these?

$S \subseteq \mathcal{B}(H)$ is a bimodule over M'. As H is finite-dimensional, $\mathcal{B}(H)$ is a Hilbert space for

$$(x|y) = \mathsf{tr}(x^*y).$$

Then $M \otimes M^{\mathsf{op}}$ is represented on $\mathcal{B}(H)$ via

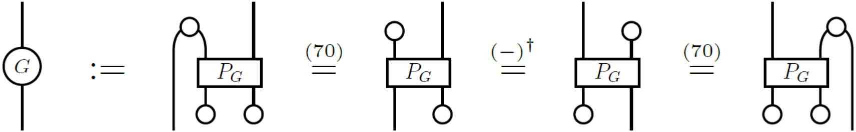
$$\pi : M \otimes M^{\mathsf{op}} \to \mathcal{B}(\mathcal{B}(H)); \quad \pi(x \otimes y) : T \mapsto xTy.$$

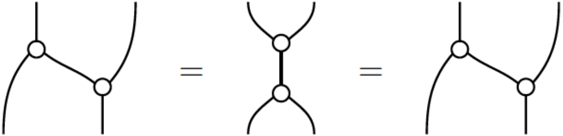
- The commutant of $\pi(M \otimes M^{\mathsf{op}})$ is $M' \otimes (M')^{\mathsf{op}}$.
- An M'-bimodule of $\mathcal{B}(H)$ corresponds to an $M' \otimes (M')^{\mathsf{op}}$-invariant subspace of the Hilbert space $\mathcal{B}(H)$;
- which corresponds to a *projection* in $M \otimes M^{\mathsf{op}}$.

Operators to algebras

So how can we relate:

- Operators $A \in \mathcal{B}(L^2(M))$;
- Projections in $M \otimes M^{\text{op}}$?





[Musto, Reutter, Verdon]

Operators to algebras 2

Recall the GNS construction for a *tracial* state ψ on M:

$$\Lambda : M \to L^2(M); \quad (\Lambda(x)|\Lambda(y)) = \psi(x^*y).$$

As $L^2(M)$ is finite-dimensional, every operator on $L^2(M)$ is a linear combination of rank-one operators. So we may define a bijection

$$\Psi : \mathcal{B}(L^2(M)) \to M \otimes M^{\mathsf{op}}; \quad |\Lambda(b)\rangle\langle\Lambda(a)| \mapsto b \otimes a^*,$$

and extend by linearity!

Operators to algebras 3

$$\Psi : \mathcal{B}(L^2(M)) \to M \otimes M^{\mathrm{op}}; \quad |\Lambda(b)\rangle\langle\Lambda(a)| \mapsto b \otimes a^*,$$

- Ψ is a homomorphism for the "Schur product" $A_1 \bullet A_2 = m(A_1 \otimes A_2)m^*$;
- $A \mapsto (1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^* \eta \otimes 1)$ corresponds to the anti-homomorphism $\sigma : a \otimes b \mapsto b \otimes a$ on $M \otimes M^{\mathrm{op}}$;
- $A \mapsto A^*$ corresponds to $e \mapsto \sigma(e)^*$.

Conclude: A quantum adjacency matrix corresponds to a projection $e \in M \otimes M^{\mathrm{op}}$ with $\sigma(e) = e$.

But: There is no clean one-to-one correspondence between the axioms.

KMS States

Any faithful state ψ is KMS: there is an automorphism σ' of M with

$$\psi(ab) = \psi(b\sigma'(a)) \qquad (a, b \in M).$$

Indeed, there is $Q \in M$ positive and invertible with

$$\psi(a) = \mathrm{tr}(Qa) \qquad \sigma'(a) = QaQ^{-1}.$$

Theorem (D.)

Twisting our bijection Ψ using σ' allows us to establish a bijection between:

- *Quantum adjacency operators $A \in \mathcal{B}(L^2(M))$;*
- *projections $e \in M \otimes M^{\mathrm{op}}$ with $e = \sigma(e)$ and $(\sigma' \otimes \sigma')(e) = e$;*
- *self-adjoint M'-bimodules $S \subseteq \mathcal{B}(H)$ with $QSQ^{-1} = S$.*

So this is *more restrictive* than the tracial case.

Invariance under the modular automorphism

Why do we end up with $(\sigma' \otimes \sigma')(e) = e$?

- The "middle axiom" is a bit mysterious: we already assume that A is self-adjoint, and shouldn't this alone correspond to the graph being undirected? (Both conditions together is a bit strong.)
- [Matsuda] looked at a different condition, that of A being "real" which means that $A : L^2(B) \to L^2(B)$, thought of as a map $B \to B$, is $*$-preserving.
- [D.] showed that replacing "self-adjoint and axiom (2)" with "real" gives a simple bijection with projections.
- [Wasilewski] has recently shown that looking at "KMS inner-products" not "GNS inner-products" is a nice framework to view this in.

(However, we are stuck with the existing literature.)

Towards homomorphisms

Let B_1, B_2 be finite-dimensional C^*-algebras (maybe just $B_i = \mathcal{B}(H_i)$), and let $\theta : B_1 \to B_2$ be a CPTP map with Kraus form

$$\theta(x) = \sum_{i=1}^{n} b_i x b_i^*.$$

For $i = 1, 2$ let $B_i \subseteq \mathcal{B}(H_i)$ and let $S_i \subseteq \mathcal{B}(H_i)$ be a quantum graph/relation over B_i.

Definition (Weaver)

The *pushforward* of S_1 is

$$\overrightarrow{S_1} = B_2'\text{-bimodule } \{b_i x b_j^* : x \in S_1, 1 \leqslant i, j \leqslant n\}.$$

The *pullback* of S_2 is

$$\overleftarrow{S_2} = B_1'\text{-bimodule } \{b_i^* y b_j : y \in S_2, 1 \leqslant i, j \leqslant n\}.$$

Motivation

Let $G = (V_G, E_G)$, $H = (V_H, E_H)$ be graphs.

- For $f : V_G \to V_H$ a map, define $\theta : \ell^\infty(V_H) \to \ell^\infty(V_G)$ in the usual way, $\theta(a)(u) = a(f(u))$ for $u \in V_G, a \in \ell^\infty(V_H)$.
- So θ is a $*$-homomorphism, in particular, a UCP map.

We find a Kraus form for θ. Define $b_i : \ell^2(V_G) \to \ell^2(V_H)$ by

$$b_i(\delta_u) = \delta_x \qquad \text{if } u \text{ is the } i\text{th vertex with } f(u) = x.$$

Then indeed

$$\sum_i b_i^* a b_i(\delta_u) = a(f(u)) = \theta(a)(\delta_u) \qquad (a \in \ell^\infty(V_H), u \in V_G).$$

CPTP maps

These (b_i) satisfy the pleasing fact that

$$\sum_i b_i e_u b_i^* = e_{f(u)} \qquad (u \in V_G),$$

where $e_u \in \ell^\infty(V_G)$ is the minimal projection. So we obtain a TPCP map $\hat{\theta} : \ell^\infty(V_G) \to \ell^\infty(V_H)$.

- *Really* what's going on is that we have the natural positive map $\hat{\theta} : \ell^1(V_G) \to \ell^1(V_H)$.

The operator system associated to G is

$$S_G = \mathrm{lin}\{e_{u,v} : (u,v) \in E_G\} \subseteq \mathbb{M}_{V_G}.$$

Then, using $\hat{\theta}$,

$$\overrightarrow{S_G} = \mathrm{lin}\{e_{f(u),f(v)} : (u,v) \in E_G\}.$$

Similarly, given S_H, we find that

$$\overleftarrow{S_H} = \mathrm{lin}\{e_{u,v} : (f(u),f(v)) \in E_H\}.$$

Homomorphisms

$$\overrightarrow{S_G} = \mathrm{lin}\{e_{f(u),f(v)} : (u,v) \in E_G\}.$$

So $\overrightarrow{S_G} \subseteq S_H$ means exactly that

$$(u,v) \in E_G \quad \implies \quad (f(u),f(v)) \in E_H.$$

That is, $f : V_G \to V_H$ induces a graph homomorphism.

- For general quantum graphs, and general TPCP maps, Stahlke takes this as the definition of a *homomorphism*.
- Weaver calls these *CP morphisms*; tentatively suggests we should start with a $*$-homomorphism if we want a "homomorphism".

Pullbacks

[We "reverse the arrows" and use UCP maps not TPCP maps.]

Let $\theta : M \to N$ be a normal CP map between von Neumann algebras $M \subseteq \mathcal{B}(H_M)$ and $N \subseteq \mathcal{B}(H_N)$. The Stinespring dilation takes a special form:

- there is a Hilbert space K and $U : H_N \to H_M \otimes K$;
- $\theta(x) = U^*(x \otimes 1)U$ for $x \in M \subseteq \mathcal{B}(H_M)$;
- there is a normal $*$-homomorphism $\rho : N' \to H_M \otimes K$ with $Ux' = \rho(x')U$ for $x' \in N'$.

Proposition (D.)

The pullback satisfies

$$\overleftarrow{S} = \textit{weak}^*\textit{-closure}\{U^*xU : x \in S\overline{\otimes}\mathcal{B}(K)\},$$

independent of choice of U. In particular, this is already an N'-bimodule.

Duality

Let B_1, B_2 be finite-dimensional with faithful traces φ_i. Given a UCP map $\theta : B_2 \to B_1$ there is a TPCP map $\hat{\theta} : B_1 \to B_2$ satisfying/defined by

$$\varphi_1(a\theta(b)) = \varphi_2(\hat{\theta}(a)b) \qquad (a \in B_1, b \in B_2).$$

("Accardi–Cecchini adjoint".)

Proposition (D.)

Let φ_i be the "Markov Traces", and given θ for $\hat{\theta}$. Then a pushforward of a quantum relation using θ is the same as the pullback using $\hat{\theta}$.

(We saw this for our maps on ℓ^∞ and ℓ^1. The general case is more complicated, but follows roughly the same idea.)

Homomorphisms

Recall that $\theta : M \to N$ is a *homomorphism* / *CP-morphism* $S_1 \to S_2$ when $\overrightarrow{S_2} \subseteq S_1$.

Theorem (Stahlke)

Let $\theta : C(V_H) \to C(V_G)$ be a UCP map giving a homomorphism G to H (that is, with $\overrightarrow{S_G} \subseteq S_H$). Then there is some map $f : V_G \to V_H$ which is a (classical) graph homomorphism.

- In general θ need not be directly related to f.
- However, often we just care about the *existence* of a homomorphism.
- E.g. a k-colouring of G corresponds to some homomorphism $G \to K_k$, the complete graph.

Further

- For a “homomorphism” do we really want our UCP map to be a $*$-homomorphism?
- It turns out some ideas from “quantum games” [Brannan et al.] naturally separate out the conditions on a “CP-morphism”, and these actually force a $*$-homomorphism.
- Also related to trying to “ignore loops”.

Possible future things:

- What are the “correct axioms”? E.g. self-adjointness or “reality”? Applications which might motivate this?
- Is there some sort of infinite-dimensional theory?