Analysis in “non-commutative” mathematics

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Outline

C*-algebras as non-commutative spaces

Compact quantum groups

Moving on

C*-algebras

A *C^*-algebra* is a complex algebra with:

- An *involution*, $(ab)^* = b^* a^*$ and $(ta)^* = \bar{t} a^*$.
- A complete norm with:
  - $\|ab\| \leq \|a\|\|b\|$;
  - $\|a^* a\| = \|a\|^2$.

In this talk, I'll mostly stick to unital algebras.

Let X be a compact Hausdorff space, and consider $C(X)$, the space of complex-valued continuous functions on X, made into an algebra with pointwise operations, given an involution by taking pointwise complex conjugation, and given the supremum norm:

$$\|f\| = \sup_{x \in X} |f(x)|.$$

This gives a commutative C*-algebra.

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Gelfand Theory

In fact, every commutative C*-algebra is of this form!

Recall that a *character* on an algebra A is a (unital) homomorphism $\varphi : A \to \mathbb{C}$. If A is a Banach algebra, then characters are always contractive maps.

Theorem (Gelfand)

Let A be a unital commutative C^*-algebra, and let Φ_A be the collection of characters on A, given the relative weak*-topology. Then Φ_A is a compact Hausdorff space, and the map

$$\mathcal{G} : A \to C(\Phi_A); \quad \mathcal{G}(a)(\varphi) = \varphi(a),$$

is an isometric isomorphism.

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$*$-homomorphisms

- The natural maps between (unital) C*-algebras are bounded algebra homomorphisms, which preserve the involution (so are $*$-*homomorphisms*) and which are unital.
- In fact, C*-algebras are such rigid objects that any $*$-homomorphism is automatically bounded; in fact, automatically contractive (and if injective, is automatically an isometry).
- Given $T : A \to B$ a $*$-homomorphism, the "adjoint" or "dual" operator T^* sends characters to characters, and so induces a continuous map $\Phi_B \to \Phi_A$.
- Conversely, given a continuous map $\phi : X \to Y$, the map $T : C(Y) \to C(X); f \mapsto f \circ \phi$ is a unital $*$-homomorphism.
- These processes are mutual inverses.

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A little “dictionary”

Algebras	Spaces
$A, C(X)$	Φ_A, X
$*$-homomorphisms $\leftrightarrow$ continuous map	
injection	surjection
surjection	injection
automorphism	homeomorphism
direct sum	disjoint union
tensor product	Cartesian product
closed ideal	closed subspace
linear functional	finite Borel measure
state	probability measure
separable	metrisable

Rough philosophy: non-commutative topology

A non-commutative unital C^*-algebra can be thought of as the algebra of continuous functions on some "non-commutative" space (which does not really exist!)

- This is a *formal analogy*: we wish to use intuition and ideas from, and the language of, spaces to study non-commutative algebras.
- Alain Connes popularised the notion of "non-commtative geometry". But there you are interested in genuine "geometry"– so some notion of a differentiable manifold structure; end up looking at cohomology theories.
- I'm more interested in generalities; more interested in topological spaces than manifolds; more interested in all compact groups rather than Lie groups etc. One might call this "non-commutative topology".

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What is a non-commutative C*-algebra anyway?

Recap: algebra over $\mathbb{C}$, with involution $(ab)^* = b^* a^*$, and C*-condition: $\|a^* a\| = \|a\|^2$.

Let H be a Hilbert space, and let $\mathcal{B}(H)$ be the algebra of all bounded linear maps on H. Then taking the "adjoint" of an operator defines an involution on $\mathcal{B}(H)$; and this involution satisfies the C*-condition.

$$(T(\xi)|\eta) = (\xi|T^*(\eta)).$$

In fact, every C*-algebra arises as a norm closed, involution closed, subalgebra of $\mathcal{B}(H)$ for a suitable H.
In this talk, it will be better to think of abstract algebras.

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Compact groups

A compact group is a group G which is also a compact Hausdorff space, such that the group operations

$$G \times G \to G; (s,t) \mapsto st; \quad G \to G; s \mapsto s^{-1}$$

are continuous.

- All finite groups.
- The circle group $\mathbb{T} = \{e^{i\theta} : \theta \in \mathbb{R}\}$ under multiplication; $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$.
- Orthogonal and unitary groups.
- Disconnected groups, such as $\prod_I \mathbb{Z}/2\mathbb{Z}$.

As C*-algebras

Let G be a compact group. So we can consider the algebra $A = C(G)$. How do we capture the group operations using A?

- Identify $C(G \times G)$ with $A \otimes A$.
- We always use the minimal, or spacial, tensor product.
- So the product map $G \times G \to G$ induces a $*$-homomorphism $\Delta : A \to A \otimes A$.
- That the product map is associative corresponds to Δ being *coassociative*: $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.
- For now, we ignore the inverse and group identity.

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Cancellation properties

- Suppose we just have a commutative C*-algebra $A = C(S)$, and a coassociative map $\Delta : A \to A \otimes A$.
- This means that S is a compact semigroup.
- The Stone-Weierstrauss theorem shows that the subspaces

$$\text{lin}\{(a \otimes 1)\Delta(b) : a, b \in A\}, \quad \text{lin}\{(1 \otimes a)\Delta(b) : a, b \in A\}$$

are dense in $A \otimes A = C(S \times S)$, if and only if we have the "cancellation conditions"

$$st = st' \implies t = t', \quad st = s't \implies s' = s.$$

- A fun exercise is to show that a compact semigroup has cancellation if and only if it is a group. (Easier is to show this for a finite semigroup).

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Compact quantum groups

The following definition is due to Woronowicz:

Definition

A *compact quantum group* is a unital C*-algebra A together with a coassociative $*$-homomorphism $\Delta : A \to A \otimes A$, such that the sets

$$\{(a \otimes 1)\Delta(b) : a, b \in A\}, \quad \{(1 \otimes a)\Delta(b) : a, b \in A\},$$

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We've seen that if $A = C(G)$ is commutative, then G is a compact group, and Δ comes from the group product.

"Quantum" $\cong$ "Non-commutative"!

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First example

- Let Γ be a discrete group (i.e. Γ is any group; ignore topology).
- Consider the Hilbert space $\ell^2(\Gamma)$ with canonical orthonormal basis $(e_g)_{g\in\Gamma}$.
- For each $g \in \Gamma$, let $\lambda(g)$ be the "left-translation map" $e_h \mapsto e_{gh}$.
- We have $\lambda(g)\lambda(h) = \lambda(gh)$ and $\lambda(g^{-1}) = \lambda(g)^*$.
- Let $C_r^*(\Gamma)$ be the closed linear span of $\{\lambda(g) : g \in \Gamma\}$. This is a C*-algebra. The "r" stands for "reduced".
- There is a $*$-homomorphism $\Delta : C_r^*(\Gamma) \to C_r^*(\Gamma) \otimes C_r^*(\Gamma) \cong C_r^*(\Gamma \times \Gamma)$ given by $\lambda(g) \mapsto \lambda(g) \otimes \lambda(g)$. Clearly Δ is coassociative.

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First example (cont.)

- We see that

$$\begin{aligned}\operatorname{lin}\{(a \otimes 1)\Delta(b) : & a, b \in C_r^*(\Gamma)\} \\ &= \operatorname{lin}\{\lambda(gh) \otimes \lambda(h) : g, h \in \Gamma\} \\ &= \operatorname{lin}\{\lambda(g) \otimes \lambda(h) : g, h \in \Gamma\}\end{aligned}$$

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Fourier transform

Consider $\Gamma = \mathbb{Z}$. The Fourier transform is the unitary map

$$\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T}); \quad e_n \mapsto \left(e^{in\theta}\right).$$

- We give $\mathbb{T}$ the Lebesgue measure– a rotationally invariant probability measure.
- We can think of $C(\mathbb{T})$ as being an algebra acting on $L^2(\mathbb{T})$ by multiplication of functions.
- Then the map

$$\text{lin}\{\lambda(n) : n \in \mathbb{Z}\} \to C(\mathbb{T}); \quad \lambda(n) \mapsto \mathcal{F}\lambda(n)\mathcal{F}^{-1}$$

  extends continuously to an isometric $*$-isomorphism between $C^*_r(\mathbb{Z})$ and $C(\mathbb{T})$, say $\mathcal{F}_0$.
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Towards a genuinely quantum example

Let's think about $SU(2)$: these are 2×2 complex matrices which are unitary, with determinant 1. That is,

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} : \alpha, \gamma \in \mathbb{C}, |\alpha|^2 + |\gamma|^2 = 1 \right\}.$$

- Let $a, c \in C(SU(2))$ be the evaluation maps $a(g) = \alpha$ and $c(g) = \gamma$. Thus $a^*a + c^*c = 1$.
- Then $C(SU(2))$ is the commutative unital C*-algebra generated by elements a, c with the relation that $a^*a + c^*c = 1$.
- Equivalently, $C(SU(2))$ is the (commutative) unital C*-algebra generated by elements a, c such that the matrix

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Twisted $SU(2)$

Let $SU_\mu(2)$ be the universal unital C*-algebra generated by elements a, c such that the matrix

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is unitary; here $\mu \in [-1, 1] \setminus \{0\}$.

Here *universal* means that if A is any other C*-algebra containing elements a', c' satisfying the same conditions, then there is a $*$-homomorphism $SU_\mu(2) \to A$ which maps $a \mapsto a'$ and $c \mapsto c'$.

Unpacking this, we get the conditions:

$$a^*a + c^*c = 1, \quad aa^* + \mu^2 c^*c = 1,$$
$$c^*c = cc^*, \quad ac = \mu ca, \quad ac^* = \mu c^*a$$

Notice that if $\mu = 1$ then $SU_\mu(2)$ must be commutative, and so must actually be $C(SU(2))$.

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Twisted $SU(2)$ cont

Define Δ by

$$\Delta(a) = a \otimes a - \mu c^* \otimes c, \quad \Delta(c) = c \otimes a + a^* \otimes c.$$

We can do this because if

$$a' = a \otimes a - \mu c^* \otimes c, \quad c' = c \otimes a + a^* \otimes c,$$

then in the algebra of 2×2 matrices over $SU_\mu(2) \otimes SU_\mu(2)$, we find that

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is unitary. So by the universal property of $SU_\mu(2)$, the $*$-homomorphism Δ does exist.

Then $(SU_\mu(2), \Delta)$ is a compact quantum group (that the "cancellation" properties hold requires a bit of theory, or some messing about with generators).

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Haar measure

Every compact group G admits a unique shift-invariant probability measure, called the *Haar measure*:

$$\int_G f(st)\,dt = \int_G f(t)\,dt.$$

- This measure induces a state h on $C(G)$.
  - An element of a C*-algebra is *positive* if it's of the form a^*a.
  - Then a *state* is a linear functional $h : A \to \mathbb{C}$ with $h(1) = 1$ and $h(a^*a) \geq 0$ for all a.
  - Always have Cauchy-Schwarz: $|h(a^*b)| \leq h(a^*a)h(b^*b)$.
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Theorem (Woronowicz, Van Daele)

Let (A, Δ) be a compact quantum group. There is a unique state h on A with $(h \otimes \iota)\Delta(a) = (\iota \otimes h)\Delta(a) = h(a)1$ for all $a \in A$.

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- For $C_r^*(\Gamma)$, the Haar state is

$$h(a) = \big(a(e_{e_\Gamma})\big|e_{e_\Gamma}\big).$$

  This means that $h(\lambda(g)) = 1$ for $g = e_{e_\Gamma}$, and 0 otherwise.
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Representations

A *unitary representation* of a (compact) group G is a continuous group homomorphism π from G to the unitary matrices $U(n)$ for some n.

- $U(n)$ is nothing but the collection of unitary operators on a n-dimensional Hilbert space.
- Let the (i, j)th matrix entry of $\pi(g)$ be $U_{ij}(g)$.
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$\pi : G \to U(n)$ corresponds to $U = (U_{ij}) \in \mathbb{M}_n(C(G))$.

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Definition

A *corepresentation* of (A, Δ) is a unitary $U \in \mathbb{M}_n(A)$ such that $\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}$.

Corepresentations

$\pi : G \to U(n)$ corresponds to $U = (U_{ij}) \in \mathbb{M}_n(C(G))$.

- That $\pi(gh) = \pi(g)\pi(h)$ means that

$$U_{ij}(gh) = \Delta(U_{ij})(g,h) = \sum_k U_{ik}(g)U_{kj}(h).$$

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Intertwiners, irreducibles etc.

Just as for representations, we can define:

- Intertwining maps between two corepresentations;
- Isomorphisms between corepresentations;
- Invariant subspaces for corepresentations;
- What an irreducible corepresentation is.

We can also (with more work!) define infinite-dimensional corepresentations.

Then every corepresentation of a compact quantum group splits as a direct sum of *irreducible*, *finite-dimensional* unitary corepresentations.

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Matrix coefficients

Given a unitary corepresentation $U = (U_{ij})$, the *matrix coefficients* of U is simply the linear span of the elements U_{ij} in A.

Take all the irreducible corepresentations, take all their matrix coefficients, and let $\mathcal{A}$ be the linear span.

- This turns out to be a $*$-algebra.
  - The product comes from the tensor product of corepresentations;
  - That it is $*$-closed is more mysterious.
- Δ restricts to a map $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (because $\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}$.
- h is a *faithful* state on $\mathcal{A}$ (so if $h(a^* a) = 0$ then $a = 0$).
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Hopf algebra

We have a *counit*, a character $\epsilon : \mathcal{A} \to \mathbb{C}$, playing the role of the group identity

$$(\epsilon \otimes \iota)\Delta(a) = a = (\iota \otimes \epsilon)\Delta(a).$$

This might not be bounded, so might not extend to A. (Already happens for $C^*_r(\Gamma)$, when Γ not *amenable*).
We have an *anitpode*, playing the role of the group inverse

$$m(\kappa \otimes \iota)\Delta = \epsilon = m(\iota \otimes \kappa)\Delta.$$

Here $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is multiplication.
Again, κ may fail to be bounded. In general,

$$\kappa(ab) = \kappa(b)\kappa(a), \quad \kappa\big(\kappa(a)^*\big)^* = a.$$

We already see this behaviour for $SU_\mu(2)$.

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Modular properties of Haar state

There is an automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ with

$$h(ab) = h(\sigma(b)a) \qquad (a, b \in \mathcal{A}).$$

- So h is "nearly" a trace.
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Summary

So we did something quite unpromising– we encoded the group product of a compact group G into a C^*-algebra, abstracted the "density conditions", and then deleted the word "commutative".

- Amazingly, this works!
- We can construct a Haar state.
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Algebra

- The data $(\mathcal{A}, \Delta, \epsilon, \kappa)$ is a Hopf $*$-algebra.
- You can characterise which Hopf $*$-algebras arise from compact quantum groups by looking at their corepresentations.
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Multiplier algebras

- If G is a locally compact, but not compact, group (e.g. $\mathbb{R}$) then natural to look at $C_0(G)$, the algebra of continuous functions which vanish at ∞.
- The *multiplier algebra* of $A = C_0(\mathbb{R})$ is $MA = C^b(\mathbb{R})$:
  - so if $F \in MA, a \in A$ then $aF, Fa \in A$.
  - MA is not "too large": if $F \in MA$ with $Fa = 0 = aF$ for all $a \in A$, then $F = 0$.
- Notice that if $a \in C_0(G)$ then $\Delta(a)(g,h) = a(gh)$ will only be in $C^b(G \times G)$ (as $\Delta(a)(g, g^{-1}h) = a(h)$ for any g).
- At the level of algebra, looking at non-unital $\mathcal{A}$ together with a coassociative $\Delta : \mathcal{A} \to M(\mathcal{A} \otimes \mathcal{A})$ (where Δ must have "nice" cancellation properties) we get van Daele's notion of a "multiplier Hopf algebra".
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More analysis

Similarly, we can work with non-unital C*-algebras A and a coassociative $\Delta : A \to M(A \otimes A)$.

- Again, we now need to *assume* the existence of Haar *weights* (which will be unbounded– Haar measure is not finite unless G is compact).
- This gives the notion of a *locally compact quantum group* (lcqg).
- Of interest is that to every lcqg (A, Δ) we find a “dual” $(\hat{A}, \hat{\Delta})$. If we form the bidual, we get back to (A, Δ).
- If $A = C_0(G)$ then $\hat{A} = C_r^*(G)$; so

  $$A = C_0(\mathbb{Z}), \quad \hat{A} = C_r^*(\mathbb{Z}) \cong C(\mathbb{T}),$$

  and we have generalised the Fourier transform!
- In fact, this was the original motivation– what's an operator algebraic setting in which to study non-abelian Fourier transforms?

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Convolution algebras

Let (A, Δ) be a compact quantum group. Then A becomes a pre-inner-product space for the sesquilinear form $(a|b) = h(b^* a)$. (GNS construction).

- Complete to get a Hilbert space, $L^2(A)$.
- Then A acts on $L^2(A)$ by left mutliplication. This realises (a quotient of) A as a subalgebra of $\mathcal{B}(L^2(A))$.
- Given $\xi, \eta \in L^2(A)$, we get a linear functional

$$\omega_{\xi,\eta} : A \to \mathbb{C}; \quad a \mapsto (a(\xi)|\eta).$$

  Let $L^1(A)$ be the closed linear span of such functionals.
- We turn A^* into a Banach algebra for the product

$$(\mu\lambda)(a) = (\mu \otimes \lambda)\Delta(a).$$

  Then $L^1(A)$ becomes an ideal in A^*.
- If $A = C_0(G)$, then $L^1(A)$ is just $L^1(G)$ with the convolution product. If $A = C_r^*(\Gamma)$, then $L^1(A)$ is just $A(\Gamma)$, the *Fourier algebra* of Γ.

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