An introduction to non-commutative graphs

Matthew Daws

Lancaster

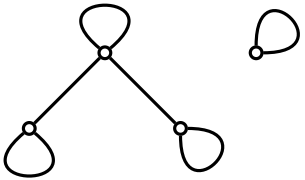
Workshop on Quantum Graphs
Saarland University, Saarbrücken

Motivation

A classical communication channel is modelled as follows:

- Finite input alphabet X and finite output alphabet Y;
- If $x \in X$ is send down the channel, then $y \in Y$ can be recieved with probability $p(y|x)$. So $0 \leqslant p(y|x) \leqslant 1$, $\sum_y p(y|x) = 1$.

So $x_1, x_2 \in X$ can be "confused" if there exists $y \in Y$ with $p(y|x_1)p(y|x_2) > 0$. This leads to the "confusability graph" of a channel: vertices X and $x_1 \sim x_2$ if they can be confused.



Example from the channel

$$\begin{pmatrix} 0.5 & 0.2 & 0 & 0 \\ 0 & 0.8 & 0.4 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{pmatrix}$$

We might care about the independence number.

Quantum channels

A *quantum state* is a "density matrix": a trace one positive-definite matrix $\rho \in \mathbb{M}_n$.

- A *pure state* is $\rho = |\psi\rangle\langle\psi|$. But states can also be "mixed".

A *quantum channel* is a linear map $T \colon \mathbb{M}_n \to \mathbb{M}_m$:

- which sends density matrices to density matrices, so is positive, and *trace-preserving*;
- can be tensored and remains positive, so is *completely positive*.

It is reasonable to say that two states ρ_1, ρ_2 are "distinguishable" when $0 = \mathrm{tr}(\rho_1 \rho_2)$.

Question

When are two states ρ_1, ρ_2 distinguishable after being sent along a quantum channel T?

Kraus representation

A nice exercise using the Stinespring dilation construction gives us the Kraus representation for a quantum channel $T\colon \mathbb{M}_n \to \mathbb{M}_m$:

$$T(x) = \sum_{i=1}^{k} x_i x x_i^* \qquad (x \in \mathbb{M}_n),$$

for some linear maps $x_i\colon \mathbb{C}^n \to \mathbb{C}^m$. That T is trace-preserving is equivalent to $\sum_i x_i^* x_i = 1$.
When is $\operatorname{tr}(T(\rho_1)T(\rho_2)) = 0$? As T is positive, equivalently

$$\operatorname{tr}\big(T(|\psi\rangle\langle\psi|)T(|\phi\rangle\langle\phi|)\big) = 0 \qquad (\psi \in \operatorname{Im}(\rho_1), \phi \in \operatorname{Im}(\rho_2)).$$

But calculate:

$$0 = \sum_{i,j} \operatorname{tr}(x_i|\psi\rangle\langle\psi|x_i^* x_j|\phi\rangle\langle\phi|x_j^*) = \sum_{i,j} |(\psi|x_i^* x_j \phi)|^2.$$

Equivalently, $(\psi|x_i^* x_j \phi) = 0$ for all i, j.

Kraus representation

Conclusion

To decide if states (given by ψ and ϕ) are distinguishable after applying T, we only need to know $(\psi|x\phi) = 0$ for each $x \in \text{lin}\{x_i^* x_j\}$.

So set $\mathcal{S} = \text{lin}\{x_i^* x_j\}$ and consider:

- $\mathcal{S}$ is a linear space, by construction;
- $x \in \mathcal{S}$ if and only if $x^* \in \mathcal{S}$;
- $1 \in \mathcal{S}$ (as $\sum_i x_i^* x_i = 1$).

So $\mathcal{S}$ is an *operator system*.

A less obvious fact is that $\mathcal{S}$ depends only on T, and not the chosen Kraus representation.

Question

Let $\mathcal{S}_1, \mathcal{S}_2$ be operator systems in $\mathbb{M}_n$. Is $\mathcal{S}_1 = \mathcal{S}_2$ implied by

$$(\psi|x\phi) = 0 \quad (x \in \mathcal{S}_1) \quad \Leftrightarrow \quad (\psi|x\phi) = 0 \quad (x \in \mathcal{S}_2)?$$

Classical case revisited

Channel from X to Y with conditional probabilities $p(y|x)$. Encode as:

- $H_X = \ell^2(X)$ with o.n. basis (e_x); same from Y.
- Define Kraus operators

$$K_{xy} \colon H_X \to H_Y; \quad e_x \mapsto p(y|x)^{1/2} e_y, \ e_z \mapsto 0 \ (z \neq x).$$

  Notice that

$$K_{xy}^* \colon H_Y \to H_X; \quad e_y \mapsto p(y|x)^{1/2} e_x, \ e_w \mapsto 0 \ (w \neq y).$$

Then the quantum channel T acts as

$$|e_c\rangle\langle e_c| \mapsto \sum_{x,y} K_{xy} |e_c\rangle\langle e_c| K_{xy}^* = \sum_y p(y|c) |e_y\rangle\langle e_y|.$$

So a pure state at c is sent to the linear combination of pure states at y, weighted by the probability of obtaining y, given that c was sent.

The associated operator system

Consider $K_{xy}^* K_{wz}$. This is the operator

$$e_w \mapsto p(z|w)^{1/2} K_{xy}^* e_z = \delta_{y,z} p(z|w)^{1/2} p(y|x)^{1/2} e_x, \quad e_t \mapsto 0 \ (t \neq w).$$

So is 0, or a scalar multiple of $|e_x\rangle\langle e_w|$ if $z = y$ and $p(y|w)p(y|x) \neq 0$. So

$$\mathcal{S} = \text{lin}\{K_{xy}^* K_{wz}\} = \text{lin}\{|e_x\rangle\langle e_w| : \exists y,\ p(y|w)p(y|x) \neq 0\}.$$

Recall that the confusability graph has $x \sim w$ exactly when $p(y|w)p(y|x) \neq 0$ for some y. Writing $e_{x,w}$ for the matrix unit with a 1 in the (x, w) entry,

$$\mathcal{S} = \text{lin}\{e_{x,w} : x \sim w\},$$

the "operator system of the confusability graph".
So... a "non-commutative graph" is an operator system in $\mathbb{M}_n$. [Stahlke]

Quantum relations

Simultaneously, and motivated more by "noncommutative geometry":

Definition (Weaver)

Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. A *quantum relation* on M is a weak*-closed subspace $S \subseteq \mathcal{B}(H)$ with $M'SM' \subseteq S$. We say that the relation is:

1. *reflexive* if $M' \subseteq S$;
2. *symmetric* if $S^* = S$ where $S^* = \{x^* : x \in S\}$;
3. *transitive* if $S^2 \subseteq S$ where $S^2 = \overline{\text{lin}}^{w^*}\{xy : x, y \in S\}$.

When $M = \ell^\infty(X) \subseteq \mathcal{B}(\ell^2(X))$ then $M' = \ell^\infty(X)$, and $M'SM' \subseteq S$ means that S is the weak*-closed linear span of the matrix units it contains. So there is a bijection between the usual meaning of "relation" on X and quantum relations on M, given by

$$x \sim y \text{ when } e_{x,y} \in S, \qquad S = \overline{\text{lin}}^{w^*}\{e_{x,y} : x \sim y\}.$$

Operator bimodules

[Optional]
The condition that $M'SM' \subseteq S$ means that S is an *operator bimodule* over M'.
(Not to be confused with Hilbert C^*-modules!)

- We assume $M \subseteq \mathcal{B}(H)$ and $S \subseteq \mathcal{B}(H)$.
- If $M \subseteq \mathcal{B}(K)$ as well, we want a $T \subseteq \mathcal{B}(K)$ corresponding to S.
- This can be found by using the structure theory for normal $*$-homomorphisms $\theta : M \to \mathcal{B}(K)$. Essentially θ is a dilation followed by a cut-down in the commutant.
- That S is a bimodule over M' is needed to get this correspondence with T.

So this notion is really independent of the choice of embedding $M \subseteq \mathcal{B}(H)$. [Weaver] gives an intrinsic notion just using M.

Quantum graphs

As a graph on a (finite) vertex set V is simply a relation, with:

- undirected graphs corresponding to symmetric relations;
- reflexive relations corresponding to having a "loop" at every vertex.

Definition (Weaver)

A *quantum graph* on a von Neumann algebra $M \subseteq \mathcal{B}(H)$ is a reflexive, symmetric quantum relation. That is, a unital, self-adjoint, weak*-closed subspace $S \subseteq \mathcal{B}(H)$, which is an M'-bimodule ($M'SM' \subseteq S$).

If $M = \mathcal{B}(H)$ with H finite-dimensional, then as $M' = \mathbb{C}$, a quantum graph is just an operator system: that is, exactly what we had before!
[Duan, Severini, Winter; Stahlke]

Adjacency matrices

Given a graph $G = (V, E)$ consider the $\{0, 1\}$-valued matrix A with

$$A_{i,j} = \begin{cases} 1 & : (i, j) \in E, \\ 0 & : \text{otherwise}, \end{cases}$$

the *adjacency matrix* of G.

- A is idempotent for the *Schur product*;
- G is undirected if and only if A is self-adjoint;
- A has 1s down the diagonal when G has a loop at every vertex.

We can think of A as an operator on $\ell^2(V)$. This is the GNS space for the C^*-algebra $\ell^\infty(V)$ for the state induced by the uniform measure.

General C^*-algebras

Let B be a finite-dimensional C^*-algebra, and let ψ be a faithful state on B, with GNS space $L^2(B)$. Thus B bijects with $L^2(B)$ as a vector space, and so we get:

- The multiplication on B induces a map $m : L^2(B) \otimes L^2(B) \longrightarrow L^2(B)$;
- the Hilbert space structure now allows us to define $m^* : L^2(B) \longrightarrow L^2(B) \otimes L^2(B)$.
- The unit in B induces a map $\eta : \mathbb{C} \longrightarrow L^2(B)$;
- similarly we obtain $\eta^* : L^2(B) \longrightarrow \mathbb{C}$, which is just φ.

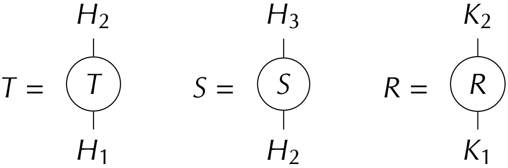
We get an analogue of the Schur product:

$$x \bullet y = m(x \otimes y)m^* \qquad (x, y \in \mathcal{B}(L^2(B))).$$

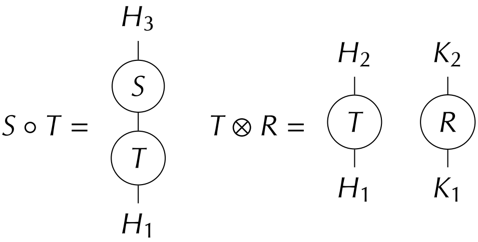
Graphical calculus

[I remain ambivalent about this!]

We use the graphical calculus / string diagrams. (See the book by Heunen and Vicary.) Read bottom-to-top. Let

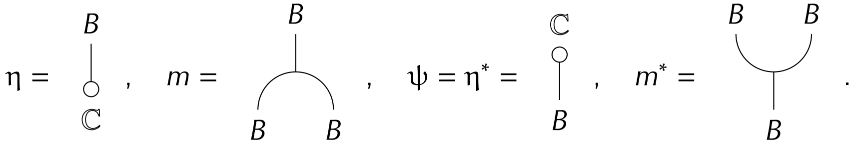


Composition is stacking; tensor product is juxtaposition.

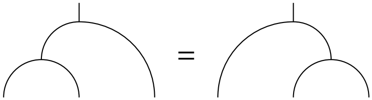


Elements of Frobenius algebras

The operators associated with (B, ψ) are



Notice we identify B with $L^2(B)$ here.
For example, that m is an associative product can be encoded in diagrams as



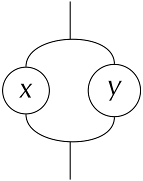
$$m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m)$$

See [Matsuda] (where I borrow some Tikz code from!)

Schur product

We can hence write the non-commutative Schur product as

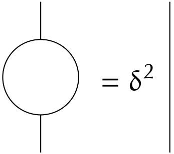
$$x \bullet y = m(x \otimes y)m^* =$$



There are various operators we can perform with diagrams– the graphical calculus– but care needs to be taken if ψ is not a trace. I won't say too much more.

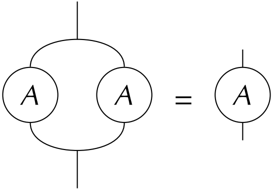
Definition

ψ is a δ-form (for a constant $\delta > 0$) if $mm^* = \delta^2 \, \mathrm{id}$.



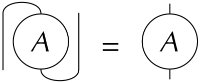
Quantum adjacency matrices

Due to [Many authors]. A quantum adjacency matrix (operator / map) is $A \in \mathcal{B}(L^2(B))$ with:

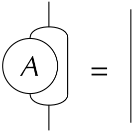


- Schur product idempotent: $m(A \otimes A)m^* = A$

- A is self-adjoint: $A^* = A$



- A is "self-transpose": $(1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^* \eta \otimes 1) = A$



- Is "reflexive" (has a loop at every vertex): $m(A \otimes 1)m^* = \mathrm{id}$

Subspaces to projections

Fix a finite-dimensional C^*-algebra (von Neumann algebra) B. A "quantum graph" is either:

- A subspace of $\mathcal{B}(H)$ (where $B \subseteq \mathcal{B}(H)$) with some properties; or
- An operator on $L^2(B)$ with some properties.

How do we move between these?

$S \subseteq \mathcal{B}(H)$ is a bimodule over B'. As H is finite-dimensional, $\mathcal{B}(H)$ is a Hilbert space (just the Hilbert–Schmidt operators) for:

$$(x|y) = \mathrm{tr}(x^* y).$$

Then $B \otimes B^{\mathrm{op}}$ is represented on $\mathcal{HS}(H)$ via

$$\pi : B \otimes B^{\mathrm{op}} \longrightarrow \mathcal{B}(\mathcal{HS}(H)); \quad \pi(x \otimes y) : T \mapsto xTy.$$

- The commutant of $\pi(B \otimes B^{\mathrm{op}})$ is $B' \otimes (B')^{\mathrm{op}}$.
- An B'-bimodule of $\mathcal{B}(H)$ corresponds to a $B' \otimes (B')^{\mathrm{op}}$-invariant subspace of the Hilbert space $\mathcal{HS}(H)$;
- which corresponds to a *projection* in $B \otimes B^{\mathrm{op}}$.

KMS states

As B is finite-dimensional, there is a positive, invertible $Q \in B$ with

$$\psi(x) = \mathrm{tr}(Qx) \qquad (x \in B),$$

where tr is your favourite faithful trace on B. Define

$$\sigma_{iz}(x) = Q^{iz} x Q^{-iz} \qquad (x \in B, z \in \mathbb{C}).$$

This is the *modular automorphism group*; we have

$$\psi(xy) = \mathrm{tr}(Qxy) = \mathrm{tr}(yQx) = \mathrm{tr}(QyQxQ^{-1}) = \psi(y\sigma_{-i}(x)) \quad (x, y \in B).$$

Operators to Algebras

Identify B with $L^2(B)$, so B has an inner-product $(x|y) = \psi(x^*y)$. As B is finite-dimensional, any operator on $L^2(B)$ is finite-rank, and so we have a linear bijection

$$\Psi'_{t,s} : \mathcal{B}(L^2(B)) \to B \otimes B^{\mathrm{op}}; \quad |b\rangle\langle a| \mapsto \sigma_{it}(b) \otimes \sigma_{is}(a)^*.$$

for any $s, t \in \mathbb{R}$. (Here $|b\rangle\langle a| : c \mapsto \psi(a^*c)b$.)

Lemma

The linear isomorphisms $\Psi'_{s,t}$ are homomorphisms for the Schur product on $\mathcal{B}(L^2(B))$ and the usual product on $B \otimes B^{\mathrm{op}}$.

So $A \in \mathcal{B}(L^2(B))$ being Schur idempotent gives that $e = \Psi'_{t,s}(A)$ is idempotent. What about the other axioms?

Operators to Algebras 2

Let $\tau\colon B \otimes B^{\mathrm{op}} \to B \otimes B^{\mathrm{op}}$ be the tensor swap map, an anti-homomorphism.

Proposition

Let $e = \Psi'_{t,s}(A)$ for some $A \in \mathcal{B}(L^2(B))$. Then:

1. $\Psi'_{t,s}(A^*) = (\sigma_{i(t-s)} \otimes \sigma_{i(t-s)})\tau(e^*)$;
2. $\Psi'_{t,s}\big((1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^*\eta \otimes 1)\big) = (\sigma_{i(s+t-1)} \otimes \sigma_{-i(s+t)})\tau(e)$.

In particular, $\Psi'_{1/2,0}$ has:

1. $\Psi'_{1/2,0}(A^*) = (\sigma_{i/2} \otimes \sigma_{i/2})\tau(e^*)$;
2. $\Psi'_{1/2,0}\big((1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^*\eta \otimes 1)\big) = (\sigma_{-i/2} \otimes \sigma_{-i/2})\tau(e)$.

So a quantum adjacency matrix A gives e with

$$\begin{aligned} & e = e^2 = (\sigma_{i/2} \otimes \sigma_{i/2})\tau(e^*) = (\sigma_{-i/2} \otimes \sigma_{-i/2})\tau(e) \\ \Leftrightarrow\quad & e = e^* = e^2 = \tau(e) \text{ and } (\sigma_z \otimes \sigma_z)(e) = e \ (z \in \mathbb{C}) \end{aligned}$$

The Proof

[Optional]

$$e = e^2 = (\sigma_{i/2} \otimes \sigma_{i/2})\tau(e^*) = (\sigma_{-i/2} \otimes \sigma_{-i/2})\tau(e)$$

Firstly use $\sigma_{i/2}(x^*) = \sigma_{-i/2}(x)^*$, so taking adjoint gives

$$e^* = (\sigma_{-i/2} \otimes \sigma_{-i/2})\tau(e) = (\sigma_{i/2} \otimes \sigma_{i/2})\tau(e^*) \quad = e.$$

Secondly use $e^* = (\sigma_{i/2} \otimes \sigma_{i/2})\tau(e^*)$ and $e = e^*$ to get

$$\begin{aligned} (\sigma_{i/2}\otimes\sigma_{i/2})\tau(e) &= e = (\sigma_{-i/2} \otimes \sigma_{-i/2})\tau(e) \\ \implies \quad \tau(e) &= (\sigma_i \otimes \sigma_i)\tau(e) \quad \Leftrightarrow \quad e = (\sigma_i \otimes \sigma_i)(e). \end{aligned}$$

This uses that $\sigma_{i/2} \circ \sigma_{i/2} = \sigma_i$.
So $e = (Q^{-1} \otimes Q)e(Q \otimes Q^{-1})$ in $B \otimes B^{\mathrm{op}}$, and so e commutes with any power of $(Q \otimes Q^{-1})$, that is, $(\sigma_z \otimes \sigma_z)(e) = e$.

Conclusions

We obtain bijections between:

1. Quantum adjacency matrices A;
2. projections $e \in B \otimes B^{op}$ with $\tau(e) = e$, $(\sigma_z \otimes \sigma_z)(e) = e$ for each $z \in \mathbb{C}$;
3. subspaces $\mathcal{S} \subseteq \mathcal{B}(L^2(B))$ which are B'-operator bimodules, self-adjoint, and satisfy $Q\mathcal{S}Q^{-1} = \mathcal{S}$.

That A is also reflexive corresponds to $\mathcal{S}$ containing $Q^{-1/2}$.

If ψ is a *trace* then we obtain bijections between:

1. reflexive Quantum adjacency matrices A;
2. projections $e \in B \otimes B^{op}$ with $\tau(e) = e$ and $m(e) = 1$;
3. operator systems $\mathcal{S} \subseteq \mathcal{B}(L^2(B))$ which are B'-operator bimodules.

Completely positive maps

Before we used $\Psi'_{1/2,0} \colon \mathcal{B}(L^2(B)) \to B \otimes B^{\mathrm{op}}$.

Theorem (BHINW, Matsuda)

The map $\Psi'_{0,1/2} : \mathcal{B}(L^2(B)) \to B \otimes B^{\mathrm{op}}$ gives a bijection between maps $A \colon B \to B$ which are completely positive, and positive elements $e \in B \otimes B^{\mathrm{op}}$. It restricts to a bijection between:

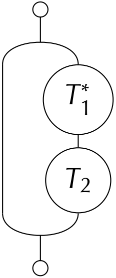
1. *projections $e \in B \otimes B^{\mathrm{op}}$;*
2. *CP maps $A \colon B \to B$ with $m(A \otimes A)m^* = A$;*
3. *linear maps $A \colon B \to B$ with $m(A \otimes A)m^* = A$ which are "real", meaning $A(x^*) = A(x)^*$ for $x \in A$.*

The map $\Psi'_{0,1/2}$ is a generalisation of the Choi Matrix construction. This result views A as a map on B, and not really on $\mathcal{B}(L^2(B))$. It also suggests that the "real" condition is perhaps more natural than asking that A be self-adjoint and "self-transpose".

Projections from diagrams

We follow the presentation of [Yamashita]. Introduce an inner-product on $\mathcal{B}(L^2(B))$ by

$(T_1|T_2) =$



$(T_1, T_2 \in \mathcal{B}(L^2(B))).$

This is not the usual (tracial) inner-product, unless ψ is a trace.
Then the identification of $\mathcal{B}(L^2(B))$ with $\mathcal{HS}(L^2(B))$ becomes "twisted".
Let $B \otimes B^{\mathrm{op}}$ and $B' \otimes (B')^{\mathrm{op}}$ act on $\mathcal{B}(L^2(B))$ in the usual (operator composition) way. Then the action on $\mathcal{HS}(L^2(B))$ also becomes twisted.

Projections from diagrams (cont.)

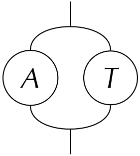
We maintain a bijection between:

1. B'-bimodules $\mathcal{S} \subseteq \mathcal{B}(L^2(B))$;
2. $B' \otimes (B')^{\mathrm{op}}$-invariant subspaces $V \subseteq \mathcal{HS}(L^2(B))$;
3. projections $\mathrm{e} \in B \otimes B^{\mathrm{op}}$.

However, now $\mathcal{S}$ and V are related in a way which involves the modular conjugation.

If we use $\Psi'_{0,1/2}$ to link a quantum adjacency matrix A with $\mathrm{e} \in B \otimes B^{\mathrm{op}}$, then e is the projection onto V, and so we obtain an induced idempotent map from $\mathcal{B}(L^2(B))$ onto $\mathcal{S}$. This is simply

$$\theta_A : \mathcal{B}(L^2(B)) \to \mathcal{B}(L^2(B)); \qquad T \mapsto$$



Conclude: $\Psi'_{0,1/2}$ seems the "natural" map; so maybe complete positivity / "reality" is the "best" starting axiom?

Selected bibliography

This very conference has a very good bibliography; here is the very partial list of the papers I've directed mentioned in this talk.

- M. Brannan, M. Hamidi, L. Ismert, B. Nelson, and M. Wasilewski Quantum edge correspondences and quantum Cuntz-Krieger algebras, J. Lond. Math. Soc. (2) **107** (2023), no. 3, 886–913.
- A. Chirvăsitu and M. Wasilewski, Random quantum graphs, Trans. Amer. Math. Soc. **375** (2022), no. 5, 3061–3087.
- M. Daws, Quantum graphs: different perspectives, homomorphisms and quantum automorphisms, Commun. Am. Math. Soc. **4** (2024), 117–181.
- R. Duan, S. Severini and A. J. Winter, Zero-error communication via quantum channels, noncommutative graphs, and a quantum Lovász number, IEEE Trans. Inform. Theory **59** (2013), no. 2, 1164–1174.
- J. Matsuda, Classification of quantum graphs on M_2 and their quantum automorphism groups, J. Math. Phys. **63** (2022), no. 9, Paper No. 092201, 34 pp.
- D. Stahlke, Quantum zero-error source-channel coding and non-commutative graph theory, IEEE Trans. Inform. Theory **62** (2016), no. 1, 554–577.
- M. Wasilewski, On quantum Cayley graphs, Doc. Math. **29** (2024), no. 6, 1281–1317.
- N. Weaver, Quantum relations, Mem. Amer. Math. Soc. **215** (2012), no. 1010, v–vi, 81–140.

See `https://github.com/MatthewDaws/Mathematics/tree/master/Quantum-Graphs` for some notes about the final 2 slides.