Quantum graphs and homomorphisms

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Graphs

A graph consists of a (finite) set of *vertices* V and a collection of *edges* $E \subseteq V \times V$.

- A graph is *undirected* if $(x, y) \in E \Leftrightarrow (y, x) \in E$.
- We allow *self-loops*, that is, allow $(x, x) \in E$.

Notice that a graph $G = (V, E)$ is exactly a *relation* on the set V. An undirected graph gives a symmetric relation; having a loop on each vertex gives a reflexive relation.

Quantum relations, a la Weaver, Kuperberg

Definition

Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. A *quantum relation* on M is a weak*-closed subspace $S \subseteq \mathcal{B}(H)$ with $M'SM' \subseteq S$.

The relation is:

1. *reflexive* if $M' \subseteq S$ $\quad (\Leftrightarrow 1 \in S)$;
2. *symmetric* if $S^* = S$ where $S^* = \{x^* : x \in S\}$;
3. *transitive* if $S^2 \subseteq S$ where $S^2 = \overline{\text{lin}}^{w^*}\{xy : x, y \in S\}$.

- Why a bimodule over M' and not M?
- There is a dependence on the embedding $M \subseteq \mathcal{B}(H)$ ...
- but as S is a bimodule over M', given a new embedding $M \subseteq \mathcal{B}(H_0)$ we get a canonical order preserving bijection between quantum relations in $\mathcal{B}(H)$ and those in $\mathcal{B}(H_0)$.

Weaver also has an “intrinsic” characterisation.

Quantum relations over a commutative algebra

Definition

A *quantum relation* on M is a weak*-closed subspace $S \subseteq \mathcal{B}(H)$ with $M' S M' \subseteq S$.

Take $M = \ell^\infty(X) \subseteq \mathcal{B}(\ell^2(X))$ so $M' = M$.

- Think of $\mathcal{B}(\ell^2(X))$ as $X \times X$ matrices.
- Any $\ell^\infty(X)$ bimodule is spanned (weak*) by the matrix units it contains.

So we obtain a bijection between the usual meaning of "relation" on X and quantum relations on M, given by

$$S = \overline{\text{lin}}^{w^*}\{e_{x,y} : x \sim y\},$$

$$\{(x,y) : x \sim y\} = \{(x,y) : e_{x,y} \in S\}$$

Quantum graphs

As a graph on a (finite) vertex set V is simply a relation, and

- undirected graph corresponds to a symmetric relation;
- a reflexive relation corresponds to having a "loop" at every vertex.

Definition (Weaver)

A *quantum graph* on a von Neumann algebra $M \subseteq \mathcal{B}(H)$ is a reflexive, symmetric quantum relation. That is, a unital, self-adjoint, weak*-closed subspace $S \subseteq \mathcal{B}(H)$, which is an M'-bimodule ($M'SM' \subseteq S$).

If $M = \mathcal{B}(H)$ with H finite-dimensional, then as $M' = \mathbb{C}$, a quantum graph is just an operator system: this was also explored by [Duan, Severini, Winter; Stahlke].

Adjacency matrices

Given a graph $G = (V, E)$ consider the $\{0,1\}$-valued matrix A with

$$A_{i,j} = \begin{cases} 1 & : (i,j) \in E, \\ 0 & : \text{otherwise}, \end{cases}$$

the *adjacency matrix* of G.

- A is idempotent for the *Schur product*;
- G is undirected if and only if A is self-adjoint;
- A has 1s down the diagonal when G has a loop at every vertex.

We can think of A as an operator on $\ell^2(V)$. This is the GNS space for the C^*-algebra $\ell^\infty(V)$ for the state induced by the uniform measure.

General C^*-algebras

Let B be a finite-dimensional C^*-algebra, and let φ be a faithful state on B, with GNS space $L^2(B)$. Thus B bijects with $L^2(B)$ as a vector space, and so we get:

- The multiplication on B induces a map $m : L^2(B) \otimes L^2(B) \to L^2(B)$;
- Using the inner product on $L^2(B)$ we can form m^*, and then interpret this as a map $B \to B \otimes B$;
- The unit in B induces a map $\eta : \mathbb{C} \to L^2(B)$;
- Again form η^*, but notice this is just $\varphi : B \to \mathbb{C}$.

We get an analogue of the Schur product:

$$x \bullet y = m(x \otimes y)m^* \qquad (x, y \in \mathcal{B}(L^2(B))).$$

Quantum adjacency matrix

Definition (Many authors)

A *quantum adjacency matrix* is a self-adjoint $A \in \mathcal{B}(L^2(B))$ with:

1. $m(A \otimes A)m^* = A$ (so Schur product idempotent);
2. $(1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^* \eta \otimes 1) = A$;
3. $m(A \otimes 1)m^* = \mathrm{id}$ (a "loop at every vertex");

The middle axiom is a little mysterious: it roughly corresponds to "undirected".

I want to sketch why this definition is equivalent to the previous notion of a "quantum graph".

Subspaces to projections

Fix a finite-dimensional C^*-algebra (von Neumann algebra) M. Start with $S \subseteq \mathcal{B}(H)$ is a bimodule over M'. As H is finite-dimensional, $\mathcal{B}(H)$ is a Hilbert space for

$$(x|y) = \mathrm{tr}(x^* y).$$

Then $M \otimes M^{\mathrm{op}}$ is represented on $\mathcal{B}(H)$ via

$$\pi : M \otimes M^{\mathrm{op}} \to \mathcal{B}(\mathcal{B}(H)); \quad \pi(x \otimes y) : T \mapsto xTy.$$

- The commutant of $\pi(M \otimes M^{\mathrm{op}})$ is naturally $M' \otimes (M')^{\mathrm{op}}$.
- So an M'-bimodule of $\mathcal{B}(H)$ corresponds to an $M' \otimes (M')^{\mathrm{op}}$-invariant subspace of the Hilbert space $\mathcal{B}(H)$;
- Which corresponds to a *projection* in $M \otimes M^{\mathrm{op}}$.

Operators to algebras

So how can we relate:

$$\text{Operators } A \in \mathcal{B}(L^2(M)) \qquad \text{with} \qquad \text{Projections in } M \otimes M^{\text{op}}?$$

Recall the GNS construction for a (faithful) *tracial* state ψ on M:

$$\Lambda : M \to L^2(M); \quad (\Lambda(x)|\Lambda(y)) = \psi(x^* y).$$

As $L^2(M)$ is finite-dimensional, Λ is bijective, and every operator on $L^2(M)$ is a linear combination of rank-one operators of the form

$$\theta_{\Lambda(a),\Lambda(b)} : \xi \mapsto (\Lambda(a)|\xi)\Lambda(b) \qquad (\xi \in L^2(M)).$$

Define a bijection

$$\Psi : \mathcal{B}(L^2(M)) \to M \otimes M^{\text{op}}; \quad \theta_{\Lambda(a),\Lambda(b)} = b \otimes a^*,$$

and extend by linearity!

Operators to algebras 2

$$\Psi : \mathcal{B}(L^2(M)) \to M \otimes M^{\text{op}}; \quad \theta_{\Lambda(a),\Lambda(b)} = b \otimes a^*,$$

- Ψ is a homomorphism for the "Schur product" on $\mathcal{B}(L^2(M))$, recall $A_1 \bullet A_2 = m(A_1 \otimes A_2)m^*$;
- $A \mapsto (1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^* \eta \otimes 1)$ transformed by Ψ to the anti-homomorphism $\sigma : a \otimes b \mapsto b \otimes a$;
- $A \mapsto A^*$ corresponds to $e \mapsto \sigma(e)^*$.

Let A be a quantum adjacency matrix, and set $e = \Psi(A)$. Then:

$$e^2 = e, \qquad \sigma(e) = e, \qquad e = \sigma(e)^*$$

So e is a projection with $e = \sigma(e)$. BUT: There is no clean one-to-one correspondence between the axioms.

Non-tracial case

Some partial references: [Musto, Reutter, Verdon], [Gromada], [Chirvasitu, Wasilewski], [Matsuda], [BCEHPSM].
If the functional ψ on M is not tracial, then this correspondence fails. (But see [Matsuda].)
However:

Theorem (D.)

There is a bijection between:

- *"Schur idempotent", self-adjoint operators A on $L^2(M)$;*
- *$e \in M \otimes M^{\mathrm{op}}$ with $e^2 = e$ and $e = \sigma(e)^*$;*
- *self-adjoint M'-bimodules $S \subseteq \mathcal{B}(H)$ such that there is another self-adjoint M'-bimodule S_0 with $S \oplus S_0 = \mathcal{B}(H)$*

KMS States

Any faithful state ψ is KMS: there is an automorphism σ' of M with

$$\psi(ab) = \psi(b\sigma'(a)) \qquad (a, b \in M).$$

Indeed, there is $Q \in M$ positive and invertible with

$$\psi(a) = \mathrm{tr}(Qa) \qquad \sigma'(a) = QaQ^{-1}.$$

Theorem (D.)

Twisting our bijection Ψ using σ' allows us to establish a bijection between:

- $A \in \mathcal{B}(L^2(M))$ *self-adjoint with axioms (1) and (2);*
- *projections* $e \in M \otimes M^{\mathrm{op}}$ *with* $e = \sigma(e)$ *and* $(\sigma' \otimes \sigma')(e) = e$*;*
- *self-adjoint* M'*-bimodules* $S \subseteq \mathcal{B}(H)$ *with* $QSQ^{-1} = S$*.*

So this is *more restrictive* than the tracial case.

Complete positivity and reality

Following [Chirvasitu, Wasilewski].

Definition (Matsuda)

Let $A \in \mathcal{B}(L^2(M))$ be interpretted as the linear map $A_0 : M \to M$. We say that A is *real* when $A_0(x^*) = A_0(x)^*$ for $x \in M$.

Theorem (D.)

A bijection similar to Ψ, again twisting by KMS $\frac{1}{2}$-automorphism, gives a bijection between:

- *A_0 being completely positive with $m(A \otimes A)m^* = A$;*
- *A being real with $m(A \otimes A)m^* = A$.*

Similarly, we can look a A being self-adjoint and with axiom (2). Arguably, this "reality" condition is more natural than being self-adjoint and satisfying axiom (2).

Pullbacks

Let $\theta : M \to N$ be a normal CP map between von Neumann algebras $M \subseteq \mathcal{B}(H_M)$ and $N \subseteq \mathcal{B}(H_N)$. The Stinespring dilation tales a special form:

- there is K and $U : H_N \to H_M \otimes K$;
- $\theta(x) = U^*(x \otimes 1)U$ for $x \in M \subseteq \mathcal{B}(H_M)$;
- there is a normal $*$-homomorphism $\rho : N' \to H_M \otimes K$ with $Ux' = \rho(x')U$ for $x' \in N'$.

Given $S \subseteq \mathcal{B}(H_M)$ a Quantum (Graph/Relation) over M, define

$$\overleftarrow{S} = \text{weak}^*\text{-closure}\{U^*xU : x \in S\overline{\otimes}\mathcal{B}(K)\}.$$

Use of ρ shows that $\overleftarrow{S}$ is a Quantum (Graph/Relation) over N, the "pullback". [Weaver; D.]

Pullbacks: Kraus forms; Pushfowards

When M, N are finite-dimensional, $\theta : M \to N$ has a Kraus form

$$\theta(x) = \sum_{i=1}^{n} b_i^* x b_i.$$

(Notice I have swapped to considering UCP maps, not TPCP maps.)

Then we recover Weaver's original definition $S \subseteq \mathcal{B}(H_M)$

$$\overleftarrow{S} = \mathrm{lin}\{b_i^* x b_j : x \in S_1\}.$$

Given $S_2 \subseteq \mathcal{B}(H_N)$ a quantum relation over N, also

$$\overrightarrow{S_2} = \mathrm{lin}\{b_i x b_j^* : x \in S_2\}$$

is a quantum relation over M, the "pushforward".

Homomorphisms

[Stahkle] defines $\theta : M \to N$ to be a *homomorphism* between S_1 and S_2 when $\overrightarrow{S_2} \subseteq S_1$. [Weaver] calls this a *CP-morphism*.

Theorem (Stahkle)

Let $\theta : C(V_H) \to C(V_G)$ be a UCP map giving a homomorphism G to H (that is, with $\overrightarrow{S_G} \subseteq S_H$). Then there is some map $f : V_G \to V_H$ which is a (classical) homomorphism.

- In general θ need not be directly related to f.
- However, often we just care about the *existence* of a homomorphism.
- E.g. a k-colouring of G corresponds to some homomorphism $G \to K_k$, the complete graph.

Questions

Take $S = M'$ and $\theta : M \to N$ and form the pullback $\overleftarrow{S}$, a quantum graph over N.

- Which quantum graphs can so arise?
- [Duan] shows that for $N = \mathbb{M}_n$ all quantum graphs arise in this way.

[Brannan, Ganesan, Harris] consider a "quantum to classical" game which ends up with a stronger notion of "homomorphism".

Here we have worked exclusively with the operator bimodule picture of Quantum Graphs.

- Can we say something useful about homomorphisms and "adjacency matrices"?

M. Daws, "Quantum graphs: different perspectives, homomorphisms and quantum automorphisms", arXiv:2203.08716 [math.OA].