# Almost periodic functionals

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# Dual Banach algebras; personal history

A Banach algebra $A$

- Banach space, algebra, $\|ab\| \leq \|a\|\|b\|$.

which is a dual space $A = E^*$, with *separate* continuity of the product.

- von Neumann algebras.
- $G$ a locally compact group, $M(G) = C_0(G)^*$ the measure algebra.
- $B(G) = C^*(G)^*$ or $B_r(G) = C_r^*(G)^*$ or $M_{cb}A(G) = Q_{cb}(G)^*$.
- $E$ a reflexive Banach space, $\mathcal{B}(E) = (E \widehat{\otimes} E^*)^*$.

**Theorem (Daws '07, after Young, Kaiser)**

*Every dual Banach algebra is isometrically a weak*-closed subalgebra of $\mathcal{B}(E)$ for a reflexive $E$.*

# Weakly almost periodic functionals

Let $A$ be a Banach algebra and turn $A^*$ into an $A$-bimodule

$$\langle a \cdot \mu, b \rangle = \langle \mu, ba \rangle, \quad \langle \mu \cdot a, b \rangle = \langle \mu, ab \rangle \qquad (a, b \in A, \mu \in A^*).$$

Then consider the "orbit maps"

$$L_\mu : A \to A^*; a \mapsto a \cdot \mu, \quad R_\mu : A \to A^*; a \mapsto \mu \cdot a.$$

## Definition

Say that $\mu \in A^*$ is weakly almost periodic, $\mu \in \mathrm{wap}(A)$ if $L_\mu$ is a weakly compact operator (i.e. $\{a \cdot \mu : \|a\| \leq 1\}$ is relatively weakly compact in $A^*$). (Equivalently can use $R_\mu$.)

# Turning into an algebra

Choose bounded nets $(a_i), (b_i)$ converging to $\Phi, \Psi \in A^{**}$ respectively. Then we have two choices for a product:

$$\Phi \Box \Psi = \lim_i \lim_j a_i b_j, \quad \Phi \Diamond \Psi = \lim_j \lim_i a_i b_j,$$

the limits in the weak*-topology on $A^{**}$. These are the Arens products. For example, given $\mu \in A^*$,

$$\begin{aligned} \lim_i \lim_j \langle \mu, a_i b_j \rangle &= \lim_i \lim_j \langle \mu \cdot a_i, b_j \rangle = \lim_i \langle \Psi, \mu \cdot a_i \rangle \\ &= \lim_i \langle \Psi \cdot \mu, a_i \rangle = \langle \Phi, \Psi \cdot \mu \rangle. \end{aligned}$$

**Theorem (Hennefeld, '68)**

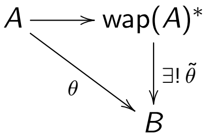
*For $\mu \in A^*$, we have that $\langle \Phi \Box \Psi, \mu \rangle = \langle \Phi \Diamond \Psi, \mu \rangle$ for all $\Phi, \Psi$ if and only if $\mu \in \mathrm{wap}(A)$.*

# Universal property; link with dual Banach algebras

**Theorem (Lau, Loy, Runde, ...?)**

*By separate weak$^*$-continuity, the product on $A$ extends to* $\mathrm{wap}(A)^*$*, turning* $\mathrm{wap}(A)^*$ *into a dual Banach algebra. This is the universal object for dual Banach algebras.*

That is, if $B$ is a dual Banach algebra and $\theta : A \to B$ a bounded homomorphism, then there is a unique $\tilde{\theta} : \mathrm{wap}(A)^* \to B$, a weak$^*$-continuous homomorphism, with:



# What if I want joint continuity?

**Definition**

$\mu \in A^*$ is almost periodic, $\mu \in \mathrm{ap}(A)$, if $L_\mu$ (equivalently $R_\mu$) is a compact operator.

**Theorem (Lau)**

$\mathrm{ap}(A)^*$ *is a dual Banach algebra such that the product is jointly weak*$^*$*-continuous, on bounded sets.*

It's then easy to adapt the argument before, and show that $\mathrm{ap}(A)^*$ is universal for dual Banach algebras where the product is jointly weak$^*$-continuous, on bounded sets.

# Example: C*-algebras

Quigg [1985] studied almost periodic functionals on C*-algebras.

- Let $A$ be a C*-algebra, and set $M = A^{**}$ a von Neumann algebra.
- Then $\mu \in A^*$ is almost periodic if and only if $M \to M_* = A^*; x \mapsto x \cdot \mu$ is compact, say $\mu \in \mathrm{ap}_*(M)$.
- If $N \subseteq M$ is a $\sigma$-weakly closed ideal, then it has a support projection, and so there is another ideal $N'$ with $M = N \oplus N'$, also $M_* = N_* \oplus N'_*$.
- If $M_{\mathrm{ap}}$ is the largest ideal of $M$ which is a direct sum of matrix algebras, then $\mathrm{ap}_*(M) = (M_{\mathrm{ap}})_*$.
- So $\mathrm{ap}(A)^* = M_{\mathrm{ap}}$.

# For (semi)groups

Let $S$ be a discrete semigroup and consider $A = \ell^1(S)$:

$$a = \sum_{s \in S} a_s \delta_s, \quad \|a\| = \sum_{s \in S} |a_s|, \quad \delta_s \delta_t = \delta_{st}.$$

- $f \in \ell^\infty(S)$ will be almost periodic if and only if the shifts $\{\delta_s \cdot f : s \in S\}$ form a relatively compact set (as taking the convex hull doesn't change compactness).
- Easy to see that $\mathrm{ap}(A) \subseteq \ell^\infty(S)$ will be a unital (commutative) $C^*$-algebra.
- Let $S^{\mathrm{ap}}$ be the (compact, Hausdorff) character space, so $\mathrm{ap}(A) = C(S^{\mathrm{ap}})$.
- Thus $M(S^{\mathrm{ap}})$ becomes a dual Banach algebra with joint continuity on bounded sets.

# The semigroup $S^{\mathrm{ap}}$

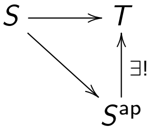
$$\mathrm{ap}(\ell^1(S)) = C(S^{\mathrm{ap}}), \qquad \mathrm{ap}(\ell^1(S))^* = M(S^{\mathrm{ap}}).$$

- The map $\ell^1(S) \to \mathrm{ap}(A)^* = M(S^{\mathrm{ap}})$ determines the product on $M(S^{\mathrm{ap}})$.
- Point-evaluation gives a map $S \to S^{\mathrm{ap}}$.
- These maps are compatible if we identify $S^{\mathrm{ap}}$ with point-masses in $M(S^{\mathrm{ap}})$. Then $S$ is dense in $S^{\mathrm{ap}}$.
- So if $u \in S^{\mathrm{ap}}$ there is a net $(s_i) \subseteq S$ with $s_i \to u$; similarly let $t_i \to v$. Then
$$\delta_u \delta_v = \lim_i \delta_{s_i} \delta_{t_i} = \lim_i \delta_{s_i t_i}.$$
- So $S^{\mathrm{ap}}$ is a semigroup. Can show that the product is jointly continuous, and that the product on $M(S^{\mathrm{ap}})$ is convolution.

# Universal property

Using some Banach algebra techniques, we started with a semigroup $S$, and formed a compact (jointly continuous) semigroup $S^{\mathrm{ap}}$.

- Call such semigroups "compact topological semigroups".
- $S^{\mathrm{ap}}$ is universal in the sense that if $T$ is any compact topological semigroup then have:



# For groups

If we start with a locally compact group $G$, form $A = L^1(G)$, then similarly we find $G^{\text{ap}}$ with $\text{ap}(A) = C(G^{\text{ap}})$, and have exactly the same universal property.

- By joint continuity, as $G^{\text{ap}}$ contains a dense subgroup (the image of $G$) it follows that $G^{\text{ap}}$ is a compact group.
- This is the Bohr compactification of $G$.
- As $G^{\text{ap}}$ is a compact group, Peter-Weyl tells us that functions of the form

$$G^{\text{ap}} \to \mathbb{C}; \quad s \mapsto (\pi(s)\xi|\eta),$$

  are dense in $C(G^{\text{ap}})$. Here $\pi : G^{\text{ap}} \to U(n)$ is a finite-dimensional unitary representation, and $\xi, \eta \in \mathbb{C}^n$.
- Such $\pi$ are in 1-1 correspondence with finite-dimensional unitary representations of $G$. So such continuous functions are dense in $\text{ap}(L^1(G))$. Not clear how to see this directly...

# Non-commutative world

Let $G$ be a locally compact group, and let $\pi : G \to \mathcal{U}(H)$ be the universal, strongly continuous, unitary representation of $G$ (direct sum over "all" such representations).

- We can "integrate" this to a map $\pi : L^1(G) \to \mathcal{B}(H)$

$$\pi(f)\xi = \int_G f(s)\pi(s)\xi \, ds \qquad (f \in L^1(G), \xi \in H).$$

- This is a $*$-homomorphism of $L^1(G)$; it's the universal one.
- The closure of $\pi(L^1(G))$ is $C^*(G)$ the universal group C$^*$-algebra.
- If we replace $\pi$ by $\lambda$ the left-regular representation on $L^2(G)$, we get the reduced group C$^*$-algebra $C^*_r(G)$.
- $C^*(G) \to C^*_r(G)$ is an isomorphism precisely when $G$ is amenable.
- Finally, define $VN(G) = C^*_r(G)''$ the group von Neumann algebra.

# Fourier theory

If $G$ is an abelian group, then we have the Pontryagin dual $\widehat{G}$. The Fourier transform gives a unitary map

$$\mathcal{F} : L^2(G) \to L^2(\widehat{G}).$$

- Let $C_0(G)$ act on $L^2(G)$ by multiplication, say $f \leftrightarrow M_f$.
- The conjugation map $M_f \mapsto \mathcal{F} M_f \mathcal{F}^{-1}$ gives a $*$-isomorphism $C_0(G) \to C_r^*(\widehat{G})$.
- It also gives a normal $*$-isomorphism $L^\infty(G) \to VN(\widehat{G})$.
- So the predual $VN(\widehat{G})_*$ is isomorphic to the *algebra* $L^1(G)$.
- By biduality, $VN(G)_* \cong L^1(\widehat{G})$.
- What happens when $G$ is not abelian?

# Hopf von Neumann algebras

There is a normal $*$-homomorphism

$$\Delta : VN(G) \to VN(G)\overline{\otimes}VN(G); \quad \lambda(s) \mapsto \lambda(s) \otimes \lambda(s).$$

- That this exists is most easily seen by finding a unitary operator $W$ on $L^2(G \times G)$ with $\Delta(x) = W^*(1 \otimes x)W$.
- $\Delta$ is coassociative: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.
- Let the predual of $VN(G)$ by $A(G)$; the predual of $\Delta$ gives an associative product

$$\Delta_* : A(G) \times A(G) \to A(G).$$

- This is the Fourier algebra; the map $A(G) \to C_0(G); \omega \mapsto (\langle \lambda(s), \omega \rangle)$ is a contractive algebra homomorphism.
- So $A(G)$ is a commutative Banach algebra; it is semisimple (and regular, Tauberian) with character space $G$.

# Almost periodic for the Fourier algebra

**Question**

What is $\mathrm{ap}(A(G))$? What relation does it have to a "compactification"?

Let $C^*_\delta(G)$ be the C$^*$-algebra generated by $\{\lambda(s) : s \in G\}$ inside $VN(G)$.

- If $G$ is discrete, then $C^*_\delta(G) = C^*_r(G)$.
- If $G$ is discrete and amenable, or $G$ is abelian, then $\mathrm{ap}(A(G)) = C^*_\delta(G)$.

# Compact quantum groups

Unital C$^*$-algebra $A$ and coassociative $\Delta : A \to A \otimes A$ with "cancellation":

$$\text{lin}\{\Delta(a)(1 \otimes b) : a, b \in A\}, \quad \text{lin}\{\Delta(a)(b \otimes 1) : a, b \in A\}$$

are dense in $A \otimes A$.

- $G$ compact gives $C(G)$ with $\Delta(f)(s,t) = f(st)$;
- $G$ discrete gives $C_r^*(G)$ with $\Delta$ as before.

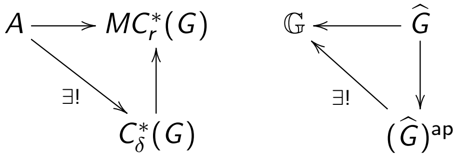
Natural morphisms are the "Hopf $*$-homomorphisms"; a morphism $(A, \Delta_A)$ to $(B, \Delta_B)$ is a $*$-homomorphism $\theta : B \to A$ with $\Delta_A \circ \theta = (\theta \otimes \theta) \circ \Delta_B$.

- If $\phi : G \to H$ is a continuous group homomorphism, then may define $\theta : C(H) \to C(G)$ by $\theta(f) = f \circ \phi$.

Can extend this to the non-compact world by considering multiplier algebras.

# Quantum Bohr compactification

Sołtan (2005) considered "compactifications" in this category. In particular, $C_\delta^*(G)$ is the universal object for $C_r^*(G)$. For any compact quantum group $(A, \Delta_A)$, have:




This gives a justification for looking at $C_\delta^*(G)$.

# More on the category LCQG??

- Let $G$ be a discrete, non-amenable group; let $\{e\}$ be the trivial group.
- The trivial homomorphism $G \to \{e\}$ induces a Hopf $*$-homomorphism $\mathbb{C} = C(\{e\}) \to C^b(G) = MC_0(G)$.
- By duality, there "should" be a Hopf $*$-homomorphism $C_r^*(G) \to C_r^*(\{e\}) = \mathbb{C}$.
- But such a map existing is equivalent to $G$ being amenable.

Work of Ng, Kustermans, and [Meyer, Roy, Woronowicz] resolves this by presenting various different, equivalent notions of a "morphism" (one being to work with $C^*(G)$ instead of $C_r^*(G)$).
I checked that Sołtan's ideas do give a "compactification" in this category— the resulting compact quantum group is quite mysterious at the C$^*$-algebraic level, but the underlying Hopf $*$-algebra is unique.

# Counter-example

It's easy to see that always $C^*_\delta(G) \subseteq \mathrm{ap}(A(G))$.

**Theorem (Chou ('90), Rindler ('92))**

*There are compact (connected, if you wish) groups $G$ such that $\mathrm{ap}(A(G)) \neq C^*_\delta(G)$.*

As $G$ is compact, the constant functions are members of $L^2(G)$. Let $E$ be the orthogonal projection onto the constants; then $E = \lambda(1_G) \in MC^*_r(G)$.

- $E \in \mathrm{ap}(A(G))$ if and only if $G$ is tall.
- $E \in C^*_\delta(G)$ if and only if $G$ does not have the weak-mean-zero containment property: there is a net of unit vectors $(\xi_i)$ in $\ker E$ with $\|\lambda(s)\xi_i - \xi_i\|_2 \to 0$ for each $s \in G$.
- [Rindler] Clever choice of $G$...

# Stronger forms of "compact"

$VN(G)$ is naturally an *operator space*: we have a family of norms on $M_n(VN(G))$. Then $A(G)$ is also an operator space. The natural morphisms are the *completely bounded* maps: those whose matrix dilations are uniformly bounded.

- There are various notions of being "completely compact"; they do not interact well with taking adjoints.
- [Runde, 2011] defined $x \in A(G)^*$ to be "completely almost periodic" if both orbit maps $L_x$ and $R_x$ are completely compact.
- If $G$ is amenable, or connected, then

  $$\operatorname{cap}(A(G)) = \{x \in VN(G) : \Delta(x) \in VN(G) \otimes VN(G)\},$$

  here $\otimes$ is the C$^*$-spatial product.
- So $x \in \operatorname{cap}(A(G))$ if and only if $\Delta(x)$ can be norm approximated by a *finite* sum $\sum_{i=1}^n a_i \otimes b_i$.

# Stronger forms of "compact" cont.

**Theorem (D.)**

*Let $G$ be discrete. Then $\Delta(x) \in VN(G) \otimes VN(G)$ if and only if $x \in C^*_\delta(G)$.*

**Theorem (D.)**

*Let $G$ be a [SIN] group (compact, discrete, abelian. . . ). Then $\Delta^2(x) \in VN(G) \otimes VN(G) \otimes VN(G)$ if and only if $x \in C^*_\delta(G)$.*

**Theorem (Woronowicz, 92)**

*Let $\mathbb{G}$ be quantum $E(2)$ (for $\mu \in (0,1)$). Then $\Delta(C_0(\mathbb{G})) \subseteq C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$, and so $C_0(\mathbb{G}) \subseteq \cap(L^\infty(\mathbb{G}))$; but $\mathbb{G}$ is not compact!*

# Even stronger forms of "compact"

**Definition**

Say that $\mu \in A^*$ is "periodic" if $L_\mu : A \to A^*$ is a finite-rank operator. Say that $\mu$ is "strongly almost periodic" if $L_\mu$ can be cb-norm approximated by operators of the form $L_{\mu'}$ with $\mu'$ periodic.

For $A(G)$, equivalently, $x$ is strongly almost periodic if $\Delta(x - x')$ can be made arbitrarily small with $\Delta(x')$ finite-rank.

**Theorem (D.)**

*$x \in VN(G)$ is strongly almost periodic if and only if $x \in C^*_\delta(G)$.*

- An analogous result holds for all Kac algebras.
- For a locally compact quantum group, also need to assume things are in $D(S) \cap D(S^*)$, which is rather messy...