Shift invariant preduals of group algebras

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Banach spaces and duality

A first course in Banach spaces (not Hilbert spaces!) will introduce the Banach spaces $\ell^1 = \ell^1(\mathbb{N})$, and $c_0 = c_0(\mathbb{N})$:

$$\ell^1 = \Big\{ (a_n) : \|(a_n)\|_1 = \sum_n |a_n| < \infty \Big\}$$

$$c_0 = \Big\{ (a_n) : \lim_n a_n = 0 \Big\} \quad \text{with} \quad \|(a_n)\|_\infty = \sup_n |a_n|.$$

Then $c_0^* = \ell^1$. To be precise, for each $f \in c_0^*$ there exists $(f_n) \in \ell^1$ such that

$$f((a_n)) = \sum_n f_n a_n \qquad ((a_n) \in c_0),$$

and with $\|f\| = \|(f_n)\|_1$.

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Other preduals of ℓ^1

Let K be a compact Hausdorff space; form $C(K)$ and $M(K)$.

Then each member of $C(K)^*$ arises from integrating against a member of $M(K)$. So we can write $C(K)^* = M(K)$.

Now suppose that K is countable– we can enumerate K as $K = \{k_n : n \in \mathbb{N}\}$ say. Then any $\mu \in M(K)$ is countably additive, and so for $f \in C(K)$,

$$\int_K f \, d\mu = \sum_n f(k_n)\mu(\{k_n\}).$$

Hence we have an isometric isomorphism $\theta : \ell^1 \to C(K)^*$ which sends $a = (a_n) \in \ell^1$ to the functional $\theta_a \in C(K)^*$ given by

$$\theta_a(f) = \sum_n f(k_n)a_n \qquad (f \in C(K)).$$

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The weak*-topology

We have a dual pairing $\ell^1 \times C(K) \to \mathbb{C}$

$$\langle a, f \rangle = \sum_n f(k_n) a_n \qquad (f \in C(K), a \in \ell^1).$$

This induces a weak*-topology on ℓ^1.

For example, as K is compact, we have non-trivial limiting sequences– say $(k_{n_i}) \to k_n$ as $i \to \infty$.

Write δ_k for the "point-mass" in ℓ^1 at k– that is, the sequence which is 0 except for a 1 in the kth place. Thus for $f \in C(K)$,

$$\lim_i \langle \delta_{k_{n_i}}, f \rangle = \lim_i f\left(k_{n_i}\right) = f(k_n) = \langle \delta_{k_n}, f \rangle,$$

and so $\delta_{k_{n_i}} \to \delta_{k_n}$ weak*. Of course, this does not hold for the "usual" weak*-topology induced by $c_0^* = \ell^1$.

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Think more abstractly about preduals

Given a Banach space E, a *predual* for E is a Banach space F together with an isomorphism (not assumed isometric) $\theta : E \to F^*$.

- Note that the map θ is very important.
- It seems reasonable to say that two preduals "are the same" if they induce the same weak*-topology on E.
- As usual, we identify F with a closed subspace of its bidual F^{**}, and so we can talk about the image of F under the adjoint map $\theta^* : F^{**} \to E^*$. Call this F_0.
- Then $F_0 \subseteq E^*$ is a closed subspace such that:
  - F_0 separates the points of E;
  - every functional $\mu \in F_0^*$ is given by some element of E.
- We call such a subspace $F_0 \subseteq E^*$ a *concrete predual*.
- It's not hard to see that two concrete preduals F_0, F_1 induce the same weak*-topology on E if and only if $F_0 = F_1$.

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Dual Banach algebras

A dual Banach algebra is a Banach algebra A which has a predual A_* such that the resulting weak*-topology on A makes the product separately weak*-continuous.

- If we identify A_* as a subspace of A^*, then we equivalently can ask that A_* is an A-submodule of A^*.
- For example, let G be a locally compact group, and let $M(G)$ be the space of regular measures on G with the convolution product. This has predual $C_0(G)$, and is a dual Banach algebra.
- When G is discrete, this example becomes $\ell^1(G)$ with the convolution product, equipped with the predual $c_0(G)$.
- It's not hard to see that a predual E of $\ell^1(G)$ makes $\ell^1(G)$ into a dual Banach algebra if and only if $E \subseteq \ell^\infty(G)$ is "shift-invariant" for the left and right actions of G on $\ell^\infty(G)$.

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Easy examples?

Let G be a countable discrete group. Can we find a dual Banach algebra predual E for $\ell^1(G)$ which differs from $c_0(G)$?

- Well, if K is compact Hausdorff and countable, then $C(K)$ is a Banach space predual for $\ell^1(K) \cong \ell^1(G)$. Equivalently, we just choose a compact Hausdorff topology on G.
- Well, G would then be a Baire Space, and hence would have some $g \in G$ with $\{g\}$ being open.
- The identification of $C(G)$ as a closed subspace of $\ell^\infty(G)$ is simply the identification of functions. So $C(G)$ will be shift-invariant if and only if the action of G on itself is continuous.
- But then, by shifting, $\{g\}$ is open for *every* g.
- So actually G has the discrete topology, and we just get back $c_0(G)$.

(Hat tip to Yemon Choi for this simple argument).

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Unique preduals

Theorem (D., Le Pham, White)

Let G be a locally compact group, and let $E \subseteq M(G)^*$ be a concrete predual for $M(G)$. Suppose that E is a subalgebra of $M(G)^* = C_0(G)^{**}$, and that $M(G)$ becomes a dual Banach algebra with respect to E. Then $E = C_0(G)$.

Theorem (Le Pham)

Let G be a compact (quantum) group, and let $E \subseteq M(G)^*$ be a concrete predual for $M(G)$, turning $M(G)$ into a dual Banach algebra. Then $E = C(G)$.

Unique preduals

Theorem (D., Le Pham, White)

Let G be a locally compact group, and let $E \subseteq M(G)^*$ be a concrete predual for $M(G)$. Suppose that E is a subalgebra of $M(G)^* = C_0(G)^{**}$, and that $M(G)$ becomes a dual Banach algebra with respect to E. Then $E = C_0(G)$.

Theorem (Le Pham)

Let G be a compact (quantum) group, and let $E \subseteq M(G)^*$ be a concrete predual for $M(G)$, turning $M(G)$ into a dual Banach algebra. Then $E = C(G)$.

For semigroups

Together with Le Pham and White, we showed that for semigroups, the situation is very different.

Theorem (D., Le Pham, White)

With $S = \mathbb{Z} \times \mathbb{Z}_+$, consider the Banach algebra $\ell^1(S)$. There are a continuum of preduals of $\ell^1(S)$ which all turn $\ell^1(S)$ into a dual Banach algebra, and which are all subalgebras of $\ell^\infty(S)$.

My intuition here is that a group is too "symmetric" so there's no place to hide a strange limit point (and if G is compact, you can't even hide the limit point in the Banach space geometry). For a semigroup, we can introduce limit points.

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A bit more general theory

For a Banach algebra A we turn A^* into a bimodule via

$$\langle a \cdot \mu, b \rangle = \langle \mu, ba \rangle, \quad \langle \mu \cdot a, b \rangle = \langle \mu, ab \rangle \qquad (a, b \in A, \mu \in A^*).$$

A functional $\mu \in A^*$ is *weakly almost periodic* if the map

$$A \to A^*; \quad a \mapsto a \cdot \mu$$

is weakly compact (we can equivalently use $\mu \cdot a$).

When $A = L^1(G)$, then $F \in L^\infty(G)$ is weakly almost periodic if the collection of functions

$$\left\{ (t \mapsto F(st)) : s \in G \right\}$$

forms a relatively weakly compact subset of $L^\infty(G)$ (and then $F \in C^b(G)$).

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Applying this

Write $\text{wap}(A)$ for the collection of weakly almost periodic functionals on A. If A is a dual Banach algebra with predual $E \subseteq A^*$, then automatically $E \subseteq \text{wap}(A)$.

Theorem (D., Le Pham, White)

Let $S = \mathbb{N}$ equipped with the semigroup product $\max$. Then $\ell^1(S)$ is a dual Banach algebra with respect to $c_0(S)$. If B is a dual Banach algebra and $\theta : \ell^1(S) \to B$ is an isomorphism which is an algebra homomorphism, then necessarily θ is weak-continuous.*

Sketch proof.

The key point is that $\text{wap}(\ell^1(S)) = c_0(S) \oplus \mathbb{C}1 \subseteq \ell^\infty(S)$. □

For a discrete group G, Chou showed that $\text{wap}(\ell^1(G))/c_0(G)$ contains an isometric copy of ℓ^∞.

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More on weakly almost periodic functions

Let G be a (discrete) group and consider again $\mathrm{wap}(\ell^1(G))$; those functions $f \in \ell^\infty(G)$ whose (left) translates form a relatively weakly compact subset of $\ell^\infty(G)$. Write $\mathrm{wap}(G)$ for this collection.

- A result of Grothendieck shows that $f \in \mathrm{wap}(G)$ if and only if

$$\lim_n \lim_m f(s_n t_m) = \lim_m \lim_n f(s_n t_m),$$

whenever (s_n) and (t_m) are sequences in G such that all the limits exist.

- It follows easily that $\mathrm{wap}(G)$ is a unital C*-subalgebra of $\ell^\infty(G)$.
- So $\mathrm{wap}(G) = C(K)$, where K is the character space of $\mathrm{wap}(G)$.
- You can lift the group product from G to K (think: Arens Products!)
- Then K becomes a semigroup, and the product is separately continuous (semitopological). Write G^{wap} for K (and then G^{wap} is the largest semitopological semigroup containing a dense homomorphic image of G).

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Characterising preduals

Let's specialise to the case when $G = \mathbb{Z}$.

Theorem (D., Haydon, Schlumprecht, White)

Let $E \subseteq \ell^\infty(\mathbb{Z})$ be a dual Banach algebra predual for $\ell^1(\mathbb{Z})$. Then there is a semitopological semigroup K containing $\mathbb{Z}$ as a dense subgroup, and a bounded projection $\Theta : M(K) \to \ell^1(\mathbb{Z})$ which is an algebra homomorphism, such that $E = {}^\perp \ker \Theta$.

As $\mathbb{Z} \subseteq K$ densely, the restriction map $C(K) \to \ell^\infty(\mathbb{Z})$ is an isomorphism onto its range. Hence we can regard

$$^\perp \ker \Theta = \{f \in C(K) : \langle \mu, f \rangle = 0 \ (\Theta(\mu) = 0)\}$$

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- Given a predual $E \subseteq \ell^\infty(\mathbb{Z})$, we consider the unital C*-algebra formed by E, which will have spectrum K.
- Arens products argument gives K a semigroup structure.
- By construction, $E \subseteq C(K)$. For $\mu \in M(K) = C(K)^*$, the restriction of μ to E forms a member of $E^* = \ell^1(\mathbb{Z})$. This gives the map $\Theta : M(K) \to \ell^1(\mathbb{Z})$.
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- The rest is a careful check.

Constructing preduals

We can reverse the argument.

Theorem

Let K be a semitopological semigroup containing $\mathbb{Z}$ as a dense subgroup, and suppose there is a bounded projection $\Theta : M(K) \rightarrow \ell^1(\mathbb{Z})$ which is an algebra homomorphism. Supposing that $\ker\Theta$ is weak-closed, the space*

$$^{\perp}\ker\Theta = \{f \in C(K) \subseteq \ell^\infty(\mathbb{Z}) : \langle \mu, f \rangle = 0 \ (\Theta(\mu) = 0)\}$$

is a Banach algebra predual for $\ell^1(\mathbb{Z})$.

Constructing K

Let $K = \mathbb{Z} \times \mathbb{Z}^+ \cup \{\infty\}$ where ∞ is a "semigroup zero"–
$x + \infty = \infty + x = \infty$ for any $x \in K$.

If $\Theta : M(K) \to \ell^1(\mathbb{Z})$ is a projection, and an algebra homomorphism, then once we fix $a_1 = \Theta(\delta_{(0,1)}) \in \ell^1(\mathbb{Z})$, we see that Θ is completely determined. Indeed, then

$$\Theta(\delta_{(n,m)}) = \Theta(\delta_{(n,0)}\delta_{(0,m)}) = \Theta(\delta_{(n,0)})\Theta(\delta_{(0,m)}) = \delta_n a_1^m.$$

Also, $\Theta(\infty)$ must be 0. If a_1 is power bounded then Θ will be bounded.

Lemma

Supposing that

$$\lim_n \|a_1^n\|_\infty = 0,$$

then $\ker\Theta$ *will be weak*-closed, for any semitopological semigroup topology on* K.

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Constructing a topology

We construct a suitable locally compact topology on $\mathbb{Z} \times \mathbb{Z}^+$, and then K will be the one-point compactification. We will now construct a topology base.

- Fix $J \subseteq \mathbb{Z}$ an infinite set.
- For $\gamma = (\gamma_0, \gamma_1) \in \mathbb{Z} \times \mathbb{Z}^+$ and $n \in \mathbb{N}$, let $V_{\gamma,n}$ be the collection of points (β_0, β_1) such that $\beta_1 \leq \gamma_1$, and
$$\beta_0 = \gamma_0 + \sum_{r=1}^{\gamma_1 - \beta_1} j_r,$$
where $(j_r) \subseteq J$ and $n < |j_1| < |j_2| < \cdots$.
- We get a suitable topology with these sets as a base if and only if, whenever $a, b \in \mathbb{Z}^+$ and $t \in \mathbb{Z}$, there is $n \in \mathbb{N}$ such that, if
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$$\beta_0 = \gamma_0 + \sum_{r=1}^{\gamma_1 - \beta_1} j_r,$$

  where $(j_r) \subseteq J$ and $n < |j_1| < |j_2| < \cdots$.
- We get a suitable topology with these sets as a base if and only if, whenever $a, b \in \mathbb{Z}^+$ and $t \in \mathbb{Z}$, there is $n \in \mathbb{N}$ such that, if

$$\sum_{r=1}^{a} j_r = t + \sum_{s=1}^{b} l_s$$

  for some $n < |j_1| < |j_2| < \cdots$ and $n < |l_1| < |l_2| < \cdots$, then $t = 0$ and $a = b$.

What's the weak* topology

Recall that $K = \mathbb{Z} \times \mathbb{Z}^+ \cup \{\infty\}$ and we have $\Theta : M(K) \to \ell^1(\mathbb{Z})$.

- The map Θ is determined by setting $a_1 = \Theta(\delta_{(0,1)}) \in \ell^1(\mathbb{Z})$.
- The topology on K is determined by the set $J \subseteq \mathbb{Z}$.
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An example

The condition on J is (roughly)

$$\sum_{r=1}^{a} j_r = t + \sum_{s=1}^{b} l_s \quad \Rightarrow ? \quad t = 0, a = b.$$

The condition on a_1 is

$$\sup_n \|a_1^n\|_1 < \infty, \qquad \lim_n \|a_1^n\|_\infty = 0.$$

- For example, take $J = \{2^n\}$ and $a_1 = \lambda\delta_0$ for some $|\lambda| < 1$.
- We have analysed this example in depth– the resulting space E is isomorphic (but not isometric) to some $C(K)$ space. Calculating the Szlenk index of K proves possible, and we conclude that E is isomorphic to c_0.
- Of course, the dual pairing between $E \cong c_0$ and $\ell^1(\mathbb{Z})$ is very strange!

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Another example

- Again take $J = \{2^n\}$, but now set $a_1 = \dfrac{1}{2}(\delta_0 + \delta_1)$.
- Now a little calculation shows that it is true that convolution powers of a_1 tend to 0 in the ∞-norm; but of course they don't in the 1-norm.
- Some general Banach space theory (Szlenk index again!) shows that the resulting predual E cannot be isomorphic to c_0.
- Now take $a_1 = 5^{-1/2}(\delta_0 + \delta_1 - \delta_2)$.
- Old work of Newman can be used to show that a_1 is power bounded in $\ell^1(\mathbb{Z})$, and a little Fourier analysis shows that $a_1^n \to 0$ in the ∞-norm.
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Open questions

- A Banach algebra predual E of $\ell^1(\mathbb{Z})$ is associated with a compact semigroup K– what K can occur?
- What Banach spaces E can occur?
- Produce "interesting" examples for other groups G (the basic theory goes through).
- For example, can we do this for $\mathbb{F}_2$ and get unusual weak*-cohomological properties for $\ell^1(\mathbb{F}_2)$? E.g. make it Connes-amenable for a new predual?

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