Functional Analytic modules

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Algebras

An *algebra* is a vector space A (over $\mathbb{C}$) with an associative, bilinear product.

- For example, $\mathbb{M}_n = M_n(\mathbb{C})$
- ...and subalgebras.
- More generally, $\mathcal{L}(V)$ the linear maps $V \longrightarrow V$ for a vector space V
- ...and subalgebras.

Each $a \in A$ defines a linear map

$$L_a \colon A \longrightarrow A; \quad b \mapsto ab,$$

and the map $L \colon A \longrightarrow \mathcal{L}(A); a \mapsto L_a$ is a homomorphism: $L_a \circ L_b = L_{ab}$.

- If A is unital, then L is injective, and so A is a subalgebra of $\mathcal{L}(A)$.

Unital?

For a Banach space E, we can consider the algebra $\mathcal{B}(E)$ of bounded linear maps, and also $\mathcal{K}(E)$ the algebra of compact operators.

- Let $\mathcal{A}(E)$ be the closure of the finite-rank operators: the *approximable operators*.
- When is $\mathcal{A}(E) = \mathcal{K}(E)$? True if E is "nice".

Natural to consider A to be non-unital:

- $\mathcal{K}(E)$ for E is infinite-dimensional
- ...also *ideals* in algebras.

Some "non-degenerate" conditions:

- A has a *faithful product* when $aA = \{ab : b \in A\} = \{0\}$ implies $a = 0$ and $Aa = \{0\}$ implies $a = 0$.
- An ideal I is *essential* if $aI = \{0\}$ implies $a = 0$, and $Ia = \{0\}$ implies $a = 0$, for any $a \in A$. If I has faithful product, then equivalently, I has non-zero intersection with any non-zero ideal in A.

Unitisations

A *unitisation* of A is a unital algebra B which contains A as an (essential) ideal. (This stops B being "too large".)

- Let $A^+ = A \oplus \mathbb{C}$;
- with product $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$.
- Then $1 = (0, 1)$ is the unit of A^+, and $A \trianglelefteq A^+$ is essential (supposing that A is not itself unital, and A has faithful product).

Are there larger unitisations?

- If $A \trianglelefteq B$ then any $x \in B$ gives rise to maps $L_x : A \longrightarrow A$ and $R_x : A \longrightarrow A$,

$$L_x(a) = xa, \qquad R_x(a) = ax.$$

These satisfy $L_x(ab) = L_x(a)b$, $R_x(ab) = aR_x(b)$ and $aL_x(b) = R_x(a)b$.

Centralisers

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad aL(b) = R(a)b \qquad (a, b \in A).$$

- A "centraliser" is a pair of maps L, R on A with these properties.
- With composition $(L_1, R_1) \cdot (L_2, R_2) = (L_1 L_2, R_2 R_1)$ and unit element $(\mathrm{id}_A, \mathrm{id}_A)$ this becomes a unital algebra $C(A)$.
- There is an embedding $A \longrightarrow C(A); a \mapsto (L_a, R_a)$; assuming A faithful.
- We have $(L, R) \cdot (L_a, R_a) = (L_{L(a)}, R_{L(a)})$ because

  $$LL_a(b) = L(ab) = L(a)b = L_{L(a)}(b); \; R_a R(b) = R(b)a = bL(a) = R_{L(a)}(b),$$

  and similarly $(L_a, R_a) \cdot (L, R) = (L_{R(a)}, R_{R(a)})$. So $A \trianglelefteq C(A)$.
- A is even essential in $C(A)$.

So $C(A)$ is the largest unitisation. Often write $M(A)$ for "multipliers".

Banach algebras are often non-unital

A Banach algebra is of course a Banach space which is an algebra, and with bounded product. We can (and will) renorm to get a contractive product:

$$\|ab\| \leqslant \|a\|\|b\| \qquad (a, b \in A).$$

- $\mathcal{B}(E), \mathcal{K}(E), \mathcal{A}(E)$ or more “exotic” ideals in $\mathcal{B}(E)$.

Let G be a locally compact group. Pick a left Haar measure. Give $L^1(G)$ the convolution product

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)\, dt.$$

There is a pairing between $L^1(G)$ and, say, $C_0(G)$, given by integrating functions. Then, for $F \in C_0(G)$,

$$\langle F, f * g \rangle = \int_G \int_G F(s)f(t)g(t^{-1}s)\, dt\, ds = \int_G \int_G F(ts)f(t)g(s)\, dt\, ds.$$

Measure algebra; psuedo-functions

- The dual space to $C_0(G)$ is $M(G)$ the space of (finite regular Borel) measures on G.
- We get a product on $M(G)$:

$$\langle F, \mu_1 * \mu_2 \rangle = \int_G \int_G F(ts)\, d\mu_1(t)\, d\mu_2(s) \qquad (F \in C_0(G)).$$

- Then $L^1(G) \trianglelefteq M(G)$ is an essential ideal.
- In fact, $M(G)$ is isometric to the centralisers of $L^1(G)$.

We can represent $L^1(G)$ on $L^p(G)$ via left convolution. The closure of the image is an algebra $PF_p(G)$ or $F_p^\lambda(G)$.

- It's rather hard to compute the norms of operators in $PF_p(G)$, and so the closure is a bit mysterious.
- $FP_p(G)$ for $p \neq 2$, and $L^1(G)$, "remember" G.

Not unital, but approximately so

$A = L^1(G)$ is only unital when G is discrete, but it always has a *contractive approximate identity*, a net (e_α) with $\|e_\alpha\| \leqslant 1$ and

$$\lim_\alpha \|a - ae_\alpha\| = \lim_\alpha \|a - e_\alpha a\| = 0 \qquad (a \in A).$$

In many ways, this is "good enough".

Definition

A *left module* E over an algebra A is a linear space E with a homomorphism $\pi\colon A \to \mathcal{L}(E)$. We write

$$a \cdot x = \pi(a)(x) \qquad (a \in A, x \in E).$$

So $a \cdot (b \cdot x) = ab \cdot x$ and so forth. Similarly *right modules* (given by an anti-homomorphism) and *bimodules* (with commuting actions).

For a Banach algebra, we want E to be a Banach space, with contractive actions: $\|a \cdot x\| \leqslant \|a\|\|x\|$.

Factorisation

A module is *essential* when (we've overloaded "essential")

$$E = \overline{\mathrm{lin}}\{a \cdot x : a \in A, x \in E\}.$$

Theorem (Cohen–Hewitt)

Let E be essential, and let A have a bounded approximable identity. Every $x \in E$ is equal to $a \cdot y$ for some a, y (and we can choose y close to x).

This reduces analytic problems to algebraic ones.

- For example, we can let $M(A)$ act on E by defining

$$m \cdot x = m \cdot (a \cdot y) = ma \cdot y \qquad (m \in C(A), x = a \cdot y \in E).$$

- This is well-defined as if $x = a \cdot y = b \cdot z$ then for any $c \in A$,

$$c \cdot (ma \cdot y) = (cm)a \cdot y = (cm) \cdot x = (cm)b \cdot z = c \cdot (mb \cdot z),$$

  and letting c run through the approximable identity shows $ma \cdot y = mb \cdot z$.

Group algebras and group representations

We have a bijection between:

- Essential modules E over $L^1(G)$;
- strongly continuous group representations of G on E (by isometries).

The "essential" condition both ensures that the action of the group unit is the identity map on E (and not a proper idempotent) and that we get strong continuity.

- So we see that modules over $L^1(G)$ correspond to a natural class of objects, namely group representations.
- That our algebras aren't unital isn't so much of an issue.

Application: representations on reflexive spaces

As $M(L^1(G)) \cong M(G)$ is a *dual Banach algebra*, work of [Young] shows that there is a reflexive Banach space E and an isometric, weak*-weak*-continuous representation $M(G) \to \mathcal{B}(E)$.

- Can we find a "nice" E?
- Let $L^1(G) \to \mathcal{B}(L^p(G))$ be (generalised) left-regular representation, for any $1 < p < \infty$.
- These are essential modules…so we have an extension $M(G) \to \mathcal{B}(L^p(G))$.
- This turns out to be weak*-weak*-continuous.
- Let (p_n) be a sequence tending to 1 from above.
- The resulting representation

$$M(G) \to \mathcal{B}\Big(\bigoplus_n L^{p_n}(G) \Big)$$

is isometric.

A question

Suppose A is not faithful.

- We can still construct $C(A)$ of course, but…
- It seems to have little to do with a "unitisation" of A.
- For example, $C(A)$ might not contain a copy of A.

Question

Is there a "unitisation" theory for rather general (not faithful) algebras?

Suppose A is a faithful Banach algebra, but $A \to C(A)$ is not bounded below.

Question

Is there a "maximal unitisation" of A which contains A at least as a *closed* ideal?

The Fourier algebra

The Fourier transform converts convolution into the pointwise product.

- For example, $\mathcal{F} : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is a homomorphism.
- Write $A(\mathbb{Z})$ for the image, a non-closed subalgebra of $c_0(\mathbb{Z})$.
- Give $A(\mathbb{Z})$ the norm so that $\mathcal{F}$ is an isometry.

Suppose we take an approximable identity in $L^1(\mathbb{T})$: perhaps

$$f_n(e^{i\theta}) = \tfrac{n}{2}[-1/n < \theta < 1/n].$$

Then $\mathcal{F}(f_n)$ is an approximate identity for $A(\mathbb{Z})$; e.g.

$$\hat{f}_n(k) = \frac{n}{k} \sin\left(\tfrac{k}{n}\right) \qquad (k \in \mathbb{Z}).$$

This approximates the constant 1 function.

The Fourier algebra, in general

Consider $\lambda_2 \colon G \to \mathcal{B}(L^2(G))$ the left-translation representation. A *coefficient functional* is a function $G \to \mathbb{C}$ of the form

$$\omega(s) = \omega_{\xi,\eta}(s) = (\xi|\lambda_2(s)\eta) = \int_G \overline{\xi(t)}\eta(s^{-1}t)\, dt,$$

for some choice $\xi, \eta \in L^2(G)$.

Theorem (Eymard)

Let $A(G)$ be the collection of all coefficient functionals. Then $A(G) \subseteq C_0(G)$ is a (non-closed) subalgebra. The norm

$$\|\omega\| = \inf\{\|\xi\|\|\eta\| : \omega = \omega_{\xi,\eta}\}$$

turns $A(G)$ into a (commutative) Banach algebra.

For $p \neq 2$ we get an analogous algebra $A_p(G)$ (though now you need to take closed linear spans, in some sense).

When is the Fourier algebra approximately unital?

When G is compact, $1 = \omega_{1,1}$ is the unit of $A(G)$.

Definition

A locally compact group G is *amenable* if it has a Følner net: a net (U_i) of subsets of G of finite measure, such that

$$\frac{|U_i \triangle g U_i|}{|U_i|} \to 0 \qquad (g \in G).$$

So, approximately, the U_i are shift-invariant.

Theorem (Leptin)

$A(G)$ has a bounded approximate identity if and only if G is amenable.

More bad news

Theorem (Losert)

The following are equivalent:

1. *the map* $A(G) \longrightarrow M(A(G))$ *is bounded below;*
2. *the regular-representation of* $A(G)$ *on itself is bounded below;*
3. G *is amenable.*

Matrix norms

Given a C^*-algebra A, we have unique norms on matrix algebras $M_n(A)$.

$$A \subseteq \mathcal{B}(H) \quad \implies \quad M_n(A) \subseteq \mathcal{B}(H^n).$$

A map $T\colon A \to B$ is *completely bounded* when the matrix amplifications

$$(T)_n\colon M_n(A) \to M_n(B)$$

are uniformly bounded: $\sup_n \|(T)_n\| < \infty$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)$$

(Pre)duals

The dual space A^* also has matrix norms:

$$M_n(A^*) \cong \mathcal{CB}(A, M_n)$$

the linear maps from A to M_n, but with the completely bounded norm.

- $A(G)$ is the predual of the group von Neumann algebra $VN(G)$;
- upshot: each matrix level $M_n(A(G))$ has a natural norm.
- Terminology: "Operator Space" (Effros, Ruan, Pisier, Paulsen).

Similarly a bilinear map $T\colon A_1 \times A_2 \longrightarrow B$ gives matrix amplifications

$$M_n(A_1) \times M_m(A_2) \longrightarrow M_{n\times m}(B); \quad (a_{ij}), (b_{kl}) \mapsto (T(a_{ij}, b_{kl}))_{ik,jl}.$$

There is a notion of tensor product which linearises such maps: the *operator space projective tensor product* $\widehat{\otimes}$.

Completely Contractive Banach Algebras

Theorem (Ruan)

The algebra product $A(G)\widehat{\otimes}A(G) \to A(G)$ is a complete contraction.

- Similarly get the notion of a completely contractive module.
- Completely bounded centralisers / multipliers $M_{cb}A(G)$ as pairs of maps in $\mathcal{CB}(A)$.

In many ways $A(G)$ behaves "better" in this category.

- Many more groups than amenable groups have that $A(G)$ has an approximate identity which is bounded in $M_{cb}A(G)$: the "weakly amenable groups" e.g. $\mathbb{F}_2$.
- However, some groups (e.g. $SL_3(\mathbb{Z})$) do not even have the very weak "AP".

Application: representations of the cb-multiplier algebra

- Somehow, $L^p(G)$ "lives between" $L^1(G)$ and $L^\infty(G)$.
- Interpolation spaces make this idea precise.
- We can similarly interpolate between $A(G)$ and $VN(G)$ to get the *non-commutative* L^p spaces $L^p(VN(G))$.
- These become completely contractive $A(G)$ modules.
- This action extends to $M_{cb}A(G)$.
- Letting $p_n \downarrow 1$, we again find

$$M_{cb}A(G) \to \mathcal{CB}\Big(\bigoplus_n L^{p_n}(VN(G)) \Big)$$

  is completely isometric, and weak*-weak*-continuous.
- The image is the "multiplier algebra" of $A(G)$

$$\{ T \in \mathcal{CB}(E) : Ta, aT \in A(G) \ (a \in A(G)) \}.$$

Self-induced algebras

Given an algebra A and a left-module E, set

$$N_E = \operatorname{lin}\{ab \otimes x - a \otimes b \cdot x : a, b \in A, x \in E\}.$$

The *balanced tensor product* is $A \otimes_A E = (A \otimes E)/N_E$.

- The multiplication map $A \otimes E \longrightarrow E; a \otimes x \mapsto a \cdot x$ annihilates the quotient, and so we get a well-defined map $m_E : A \otimes_A E \longrightarrow E$.
- E is *self-induced* when this is a bijection.
- Similarly an algebra is self-induced. (Also “firm ring”.)

For a Banach algebra, we want our tensor products to linearise bounded bilinear maps, and this leads to the *projective tensor product* $\widehat{\otimes}$. We let N_E be the *closed* linear span now.

Definition (Grønbaek)

A Banach algebra is self-induced when $A \widehat{\otimes}_A A \longrightarrow A$ is an isomorphism.

Actions of centralisers

Let $m = (L, R) \in C(A) = M(A)$, so

$$L(ab) = L(a)b \implies L(ab \otimes x - a \otimes b \cdot x) = L(a)b \otimes x - L(a) \otimes b \cdot x \in N_E.$$

Hence $L \otimes \mathrm{id}_E$ maps N_E to N_E.

- So $L \otimes \mathrm{id}_E$ is well-defined on $A \otimes_A E$.
- Suppose E is self-induced. Then define an action of m on E by

$$E \cong A \otimes_A E \xrightarrow{L \otimes \mathrm{id}} A \otimes_A E \cong E.$$

- This turns E into a left $M(A)$ module.

Slight wrinkle: don't really need $A \otimes_A E \to E$ a bijection; that this were onto, and A had a faithful product, would be enough.

Fourier algebra is self-induced

Consider $A(G)$ in the category of Operator Spaces.

Theorem

$A(G)$ is self-induced as a completely contractive Banach algebra.

- $A(G)$ often fails to have an approximate identity bounded in any reasonable sense.
- (Open question: does $A(G)$ have an unbounded approximate identity?)
- However, we always have the property of being self-induced.

Question

What do self-induced (completely contractive) modules over $A(G)$ look like?